Writing Linear Optimization Problems

Lecture 02
Optimization Techniques, IE 601

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Linear Optimization

- Sections 1.1 and 1.2 of [BJS]
- Linear Optimization is also called Linear Programming (LP)
- Can be written in different forms – all equivalent to each other
- Provide different perspectives and insights
- 'Among all feasible vectors, find one that minimizes the objective function'
- Not all vectors are allowed – there are linear constraints
- Objective function is linear

General Form

$$
\begin{aligned}
\min_{x}\; & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to: } & g_1 \geq a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1, \\
& g_2 \geq a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2, \\
& \vdots \quad \vdots \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad \vdots \\
& g_m \geq a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m, \\
& u_1 \geq x_1 \geq l_1, \\
& \vdots \quad \vdots \quad \vdots \\
& u_n \geq x_n \geq l_n,
\end{aligned}
$$

- where $c_1, c_2, \ldots c_n, g_1, \ldots, g_m, b_1, \ldots, b_m, l_1, \ldots, l_n, u_1, \ldots, u_n, a_{11}, \ldots, a_{mn}$, are inputs (parameters).
- Output is the optimal value which is a scalar, and
- the optimal solution which is a vector (x^*).

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Properties of the General Form

- A vector that satisfies all constraints is called a feasible solution to the problem
- An optimal solution must be feasible, and
- no other feasible solution should have value lower than its value
- The number of constraints, m and variables, n must be finite
- Some of the u_j and g_i can be ∞
- Some of the l_j and b_i can be $-\infty$
- The constraints $u_j \geq x_j \geq l_j$ are also called bound constraints
- For computations, all inputs can be assumed to be rational
- Strict inequalities ($<$ and $>$) are not allowed (why?)

Example

The model

$$\begin{aligned} \min_x \; & -4x_1 + x_2 \\ \text{subject to: } & 2x_1 + x_2 \leq 10, \\ & -x_1 + 2x_2 \leq 10, \\ & 0 \leq x_1 \leq 4, \\ & -x_1 + x_2 \geq -3, \\ & 0 \leq x_2. \end{aligned}$$

is same as the following model in general form:

$$\begin{aligned} \min_x \; & -4x_1 + x_2 \\ \text{subject to: } & 10 \geq 2x_1 + x_2 \geq -\infty, \\ & 10 \geq -x_1 + 2x_2 \geq -\infty, \\ & \infty \geq -x_1 + x_2 \geq -3, \\ & 4 \geq x_1 \geq 0, \\ & \infty \geq x_2 \geq 0. \end{aligned}$$

Standard Form

$$
\begin{aligned}
\min_{x}\ & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to: } & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\
& \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \\
& x_1 \geq 0, \\
& \vdots \;\; \vdots \;\; \vdots \\
& x_n \geq 0,
\end{aligned}
$$

- where $c_1, c_2, \ldots c_n, b_1, \ldots, b_m, a_{11}, \ldots, a_{mn}$, are inputs (parameters).
- It has the properties of the general form described earlier.
- Easier to analyze and develop intuition

Standard Form

The model

$$\begin{aligned} \min_{x} \; & -4x_1 + x_2 \\ \text{subject to: } & 2x_1 + x_2 \leq 10, \\ & -x_1 + 2x_2 \leq 10, \\ & 0 \leq x_1 \leq 4, \\ & -x_1 + x_2 \geq -3, \\ & 0 \leq x_2. \end{aligned}$$

is equivalent, but not the same, as the following model in standard form:

$$\begin{aligned} \min_{x} \; & -4x_1 + x_2 \\ \text{subject to: } & 2x_1 + x_2 + x_3 = 10, \\ & -x_1 + 2x_2 + x_4 = 10, \\ & x_1 + x_5 = 4, \\ & -x_1 + x_2 - x_6 = -3, \\ & x_1, \ldots, x_6 \geq 0. \end{aligned}$$

Canonical Form

$$
\begin{aligned}
\min_{x}\ & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to: } & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1, \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2, \\
& \quad \vdots \qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m, \\
& x_1 \geq 0, \\
& \vdots \ \ \vdots \ \ \vdots \\
& x_n \geq 0,
\end{aligned}
$$

- where $c_1, c_2, \ldots c_n$, $b_1, \ldots, b_m$, $a_{11}, \ldots, a_{mn}$, are inputs (parameters).
- It has the properties of the general form described earlier.
- Nice geometric interpretation and intuitive appeal

Commonly Used Transformations

- $f(x) = g(x) \iff f(x) \leq g(x), f(x) \geq g(x)$
- $\min_x f(x) \iff -\max_x -f(x)$
- $f(x) \leq g(x) \iff -f(x) \geq -g(x)$
- $f(x) \leq g(x) \iff f(x) + s = g(x), s \geq 0$

- Transforming a standard form or canonical form to general form of LP is straightforward.
- Every LP in general form can be transformed into an equivalent LP in standard form.
- One can develop an algorithm for any one form – it will work for all others.
- Differences in theoretical properties arise, one must be careful in analysis.

Vector Notation

General Form

$$\min_x c^\intercal x$$
$$\text{subject to: } g \geq Ax \geq b,$$
$$u \geq x \geq l,$$

Standard Form

$$\min_x c^\intercal x$$
$$\text{subject to: } Ax = b,$$
$$x \geq 0,$$

Canonical Form

$$\min_x c^\intercal x$$
$$\text{subject to: } Ax \geq b,$$
$$x \geq 0,$$

where A is an $m \times n$ matrix, b, g are vectors of size m and c, l, u of size n.