Vectors and Matrices

Lecture 04
Optimization Techniques, IE 601

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August 02, 2019

Vectors

- Recall definition and basic properties of vectors in Euclidean space ($\mathbb{E}^n$)
- A vector $b \in \mathbb{E}^n$ is a linear combination of vectors $a^1, \ldots, a^k$ in $\mathbb{E}^n$ if we can find scalars $\lambda_1, \ldots, \lambda_k$ such that

$$b = \sum_{j=1}^{k} \lambda_j a^j$$

- Vectors $a^1, \ldots, a^k$ are linearly independent if none of them can be written as a linear combination of other vectors, i.e., the system

$$\sum_{j=1}^{k} \lambda_j a^j = 0,$$

has a unique solution $\lambda_j = 0, j = 1, \ldots, k$.

Basis

- Vectors $a^1, \ldots, a^k$ from $\mathbb{E}^n$ span $\mathbb{E}^n$ if every vector b in $\mathbb{E}^n$ can be written as a linear combination of $a^1, \ldots, a^k$.
- e.g. the vectors $[0\ 1\ 1]^T, [1\ 0\ 1]^T, [1\ 1\ 0]^T, [1\ 1\ 1]^T$ span $\mathbb{E}^3$
- Vectors $a^1, \ldots, a^k$ from $\mathbb{E}^n$ form a basis of $\mathbb{E}^n$ if
  1. $a^1, \ldots, a^k$ span $\mathbb{E}^n$, and
  2. No other subset of $\{a^1, \ldots, a^k\}$ spans $\mathbb{E}^n$.
- The vectors $[0\ 1\ 1]^T, [1\ 0\ 1]^T, [1\ 1\ 0]^T, [1\ 1\ 1]^T$ do not constitute a basis of $\mathbb{E}^3$

Useful Results

- A set of vectors is a basis of $\mathbb{E}^n$ if any only if it has exactly n linearly independent vectors.
- Let $a^1, \ldots, a^k$ be a basis of E^n and b be any vector of $\mathbb{E}^n$. Then

$$b = \sum_{j=1}^{k} \lambda_j a^j,$$

  where λ is unique.
- In the above system, b can replace the vector a^j to yield a new basis if and only if $\lambda_j \neq 0$.
- Can you argue why?

Matrices

- Recall definition and basic properties of a matrix
- Elementary row operations on a matrix:
  1. Row i of the matrix is multiplied by a nonzero scalar, say $k \neq 0$
  2. Row i of the matrix is replaced by the sum of row i and a multiple k of row j
  3. Rows i and j are interchanged
- These are useful when solving a system of equations or inverting a matrix
- Each elementary operation is equivalent to pre-multiplying the matrix by a square matrix
- e.g. multiplying 2nd row of a matrix A is same as:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} A$$

- Can you find the multipliers for other two cases?

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Matrices and Systems of Equations

- Rank of matrix is the maximum number of linearly independent columns of A
- It is also the maximum number of linearly independent rows of A
- Given a matrix square matrix of size $m \times m$, A, its inverse A^{-1} exists iff $det(A) \neq 0$
- A is invertible iff $rank(A) = m$
- A is invertible iff $Ax = b$ has a unique solution for every b
- If A^{-1} exists, it can be obtained by elementary row operations on A.
- Equivalently, A^{-1} is a product of several matrices each of which corresponds to the three operations
- Can you recall the relation between solving $Ax = b$ and finding A^{-1}?

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