Convex Sets

Lecture 05

Optimization Techniques, IE 601

Industrial Engineering and Operations Research
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Combining Vectors

Given vectors $a^1, \ldots, a^k$ in $\mathbb{E}^n$, a vector b defined as

$$b = \sum_{j=1}^{k} \lambda_j a^j$$

is said to be a linear combination of $a^1, \ldots, a^k$. Further, b is

- a positive combination of $a^1, \ldots, a^k$, if $\lambda_1, \ldots, \lambda_k \geq 0$,
- an affine combination of $a^1, \ldots, a^k$, if $\sum_{j=1}^{k} \lambda_k = 1$,
- a convex combination of $a^1, \ldots, a^k$, if $\sum_{j=1}^{k} \lambda_k = 1, \lambda_1, \ldots, \lambda_k \geq 0$,

- Can you check by solving a system of equations whether a given b is a linear combination of given vectors $a^1, \ldots, a^k$?
- Can you also check if it is their affine combination?
- How about positive and convex combinations?

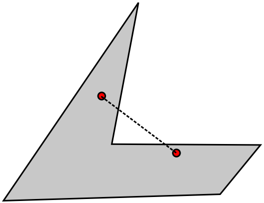
Question

Let $a^1 = [1 \;\; 2]^\top, a^2 = [2 \;\; 1]^\top$. Draw all vectors which are

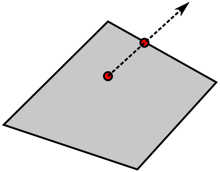
1. linear combinations of a^1, a^2
2. positive combinations of a^1, a^2
3. affine combinations of a^1, a^2
4. convex combinations of a^1, a^2

Convex Sets

- A set X in $\mathbb{E}^n$ is a convex set if given any two points $x^1, x^2 \in X$, their all convex combinations also lie in X
- Line joining x^1, x^2 should lie in X for any two points x^1, x^2 in X.
- $\lambda x^1 + (1 - \lambda)x^2 \in X$ for each $\lambda \geq 0$.



A nonconvex set



A convex set

- If we move in a given direction starting from a point in a convex set and hit the boundary, then there can not be any more feasible points further in the direction.
- Quite useful when we are searching for good points

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Extreme Points

- A point x in a convex set, X, is its extreme point if it can not be represented as a convex combination of any other (distinct from itself) points of X. It is equivalent to saying, there are no points x^1, x^2 in X such that $x = \lambda x^1 + (1-\lambda)x^2, x \neq x^1, x \neq x^2, \lambda \in [0,1]$.
- A nonempty convex set may have no extreme points, a finite number of extreme points, countably infinite, or uncountably infinite number of extreme points.
- Can you draw some examples of each the above cases?

Hyperplanes

- Given a vector p in $\mathbb{E}^n$ and a scalar k, the set $\{x : p^\intercal x = k\}$ is called a hyperplane.
- Every point on a hyperplane must satisfy the linear equation $px = k$.
- There can not be n linearly independent points on a hyperplane. (why?)
- There are $(n-1)$ linearly independent points on every hyperplane. Can you find such points?
- Conversely, there is a unique hyperplane passing through $(n-1)$ linearly independent points.
- Let x^1, x^2 be any two points in $\{x : p^\intercal x = k\}$, then $p^\intercal(x^1 - x^2) = 0$. p is thus normal or orthogonal to the hyperplane, and so is $-p$.
- Can you show that every hyperplane is a convex set?

Halfspaces

- Every hyperplane $\{x : p^\intercal x = k\}$ divides E^n into two half-spaces:

$$\{x : p^\intercal x \leq k\}, \text{ and } \{x : p^\intercal x \geq k\}.$$

- The intersection of the above two half-spaces is the hyperplane that generated it.
- Every half-space has n linearly independent points in it. Can you find one such set?
- Can you show that every halfspace is a convex set?
- How many extreme points does it have?

Rays

- A ray is a set of the form $\{x \in \mathbb{E}^n : x = x^0 + \lambda d, \lambda \geq 0\}$, where x^0, d are given vectors in $\mathbb{E}^n$.
- It is a one dimensional set.
- x^0 is called the vertex, and d the direction of the ray.
- Can you show that it is a convex set?
- How many extreme points does it have?
- If we scale d by multiplying by a positive scalar, do we get a different set?

Recession Direction

- A nonzero vector d in $\mathbb{E}^n$ is a recession direction of a convex set X if for each x^0 in X, the ray $\{x \in \mathbb{E}^n : x^0 + \lambda d, \lambda \geq 0\}$ is contained in X.
- A convex set may have zero, finite or infinite number of recession directions.
- If d^1, d^2 are recession directions of a convex set X, then $\lambda_1 d^1 + \lambda_2 d^2$, $\lambda_1, \lambda_2 > 0$, is also a recession direction.
- If a convex set X contains a ray starting at a point x^0 in X, then X also contains rays starting at every point in X along the same direction.