Polyhedral Sets

Lecture 06
Optimization Techniques, IE 601

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Convex Cone

- A convex set C is a convex cone if it has the following property. If $x \in C$, then $\lambda x \in C$ for every $\lambda \geq 0$.
- Any vector in a cone can be scaled by a non-negative scalar to obtain another vector in the cone.
- Origin is always contained in a cone
- Is the set $\{x \in \mathbb{E}^n : Ax = 0, x \geq 0\}$ a convex cone?

- The set $\{x \in \mathbb{E}^n : x = \sum_{j=1}^{k} \lambda_j a^j, \lambda_j \geq 0, j = 1, \ldots, k\}$ is a convex cone
- It is said to be generated by the vectors $a^1, \ldots, a^k$.

Polyhedron

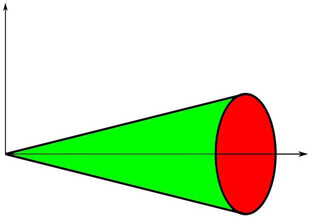
- A polyhedron is an intersection of a finite number of half-spaces
- Every polyhedron can be described using a system of finitely many linear inequalities $Ax \leq b$
- It is a convex set
- A polyhedron P is bounded if there exists a positive scalar k such that every point in P satisfies $||x|| \leq k$.
- Can you prove: If P has a recession direction, then it is unbounded?
- ...and the converse: If P does not have any recession directions, then it is bounded?
- Feasible region of a linear program is a polyhedron.

Redundant Constraints

- Suppose a polyhedral region is described by constraints $Ax \leq b$.
- A constraint, say $\sum_j a_{ij}x \leq b_i$ of this syste is redundant if the feasible region does not change on removing this constraint.
- If two or more redundant constraints are removed, the feasible region may change. (Think of an example)
- A redundant constraint can be identified by solving an LP (How?).

Polyhedral Cone

- A convex cone C is called a polyhedral cone if it is a polyhedron.
- Not all convex cones are polyhedral
- $C = \{x \in \mathbb{R}^3 : \sqrt{x_2^2 + x_3^2} \leq x_1, x_1 \geq 0\}$, also called a second order cone or the icecream cone



- What are its recession directions?
- A cone generated by a finite number of points is a polyhedral cone (why?)

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Active Constraints of a Polyhedral Set

- A constraint $\alpha x \leq \beta$ (or $= \beta$, or $\geq \beta$) is said to be active or binding or tight at a given point $\hat{x}$ if $\alpha\hat{x} = \beta$.
- An equality constraint is active at all feasible points
- A point is said to be in the interior of a polyhedral set if no constraint is active at it.
- If a point is in the interior, we can take a small step in any direction and still be feasible.
- Same is not true if the point is not in the interior
- A point in the interior can not be an extreme point (why?)

Extreme Points of a Polyhedral Set

Result: Let $\bar{x}$ be a point in a polyhedron $X = \{x \in \mathbb{E}^n : Ax \geq b, x \geq 0\}$. Suppose one of the constraints (including the bound constraints), say $\alpha x \geq \beta$ is active at $\bar{x}$. Let $x^1, x^2 \in X$ be two points so that $\bar{x}$ is their convex combination. Then the constraint $\alpha x \leq \beta$ is also active at both x^1 and x^2

- Can you prove the above result?
- A feasible point of polyhedral set X is called its corner or vertex if n linearly independent constraints of X are active at that point.
- Using the above result one can show that a feasible point of a polyhedron X is its vertex if and only if it is its extreme point.
- A polyhedral set may also have faces and edges (See book).

Extreme Directions of a Polyhedral Set

- Recall that for a polyhedral set $X = \{x \in \mathbb{E}^n : Ax \leq b, x \geq 0\}$, a recession direction satisfies $Ad \leq 0, d \geq 0$.
- The set of all recession directions is a cone
- It is called a recession cone
- Recession cone is a polyhedron (Why?)
- What are its extreme points?
- To find its extreme rays, one can add a 'normalization constraint' $\sum_j d_j = 1$ to the cone of recession cones and find its extreme points.
- Why does the above method work?

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