Intersection Cuts for Quadratic Constraints

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1. Introduction

Consider the problem of optimizing a linear function subject to quadratic constraints:

$$\begin{aligned} &\min p^T x \\ \text{subject to: } & x^T H_i x + \sum_{j=1}^{n} a_{ij} x_j + d_i \le 0, \quad i = 1,2,\dots m \qquad \text{(QCQP)} \\ & l \le x \le u. \end{aligned}$$

Three Approaches

Branch-and-Bound
Linear relaxations,
or
SDP Relaxations.

Cutting Plane Method
- Based upon the work of Tuy [1964].
- Uses intersection cuts.

Others
- See Ritter [1966]; Zwart [1974]; Konno [1976].
- Quadratic objective and linear constraints.

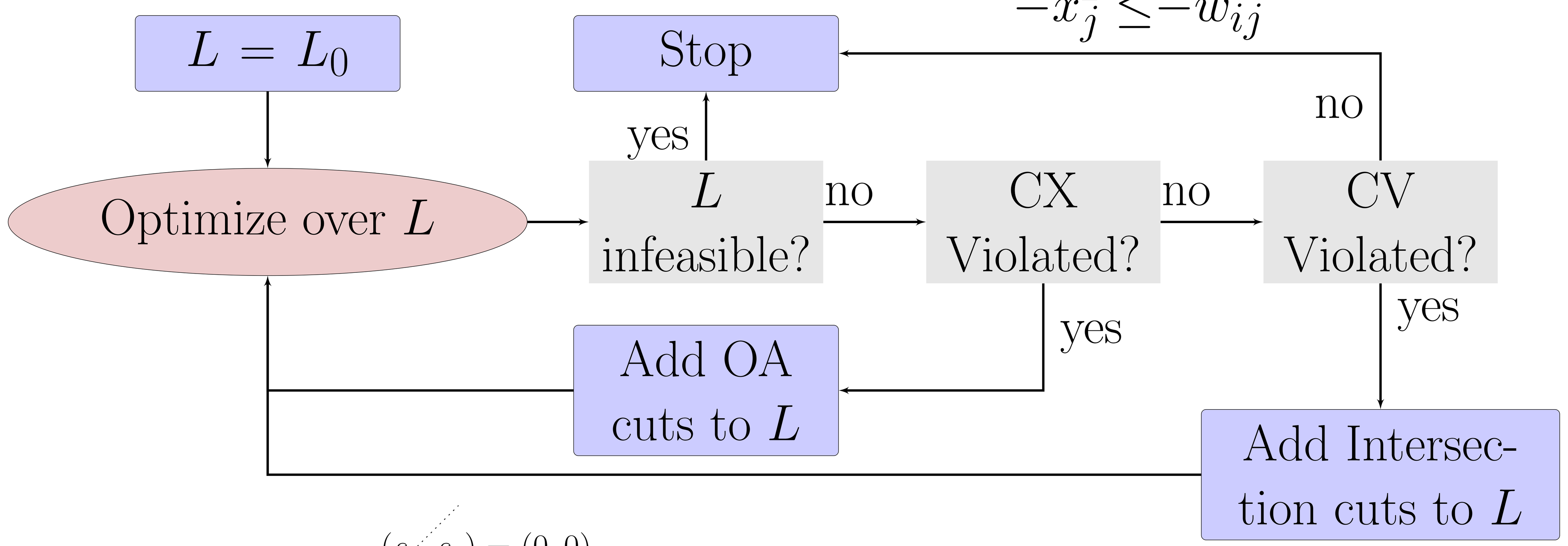
2. A Cutting Plane Method

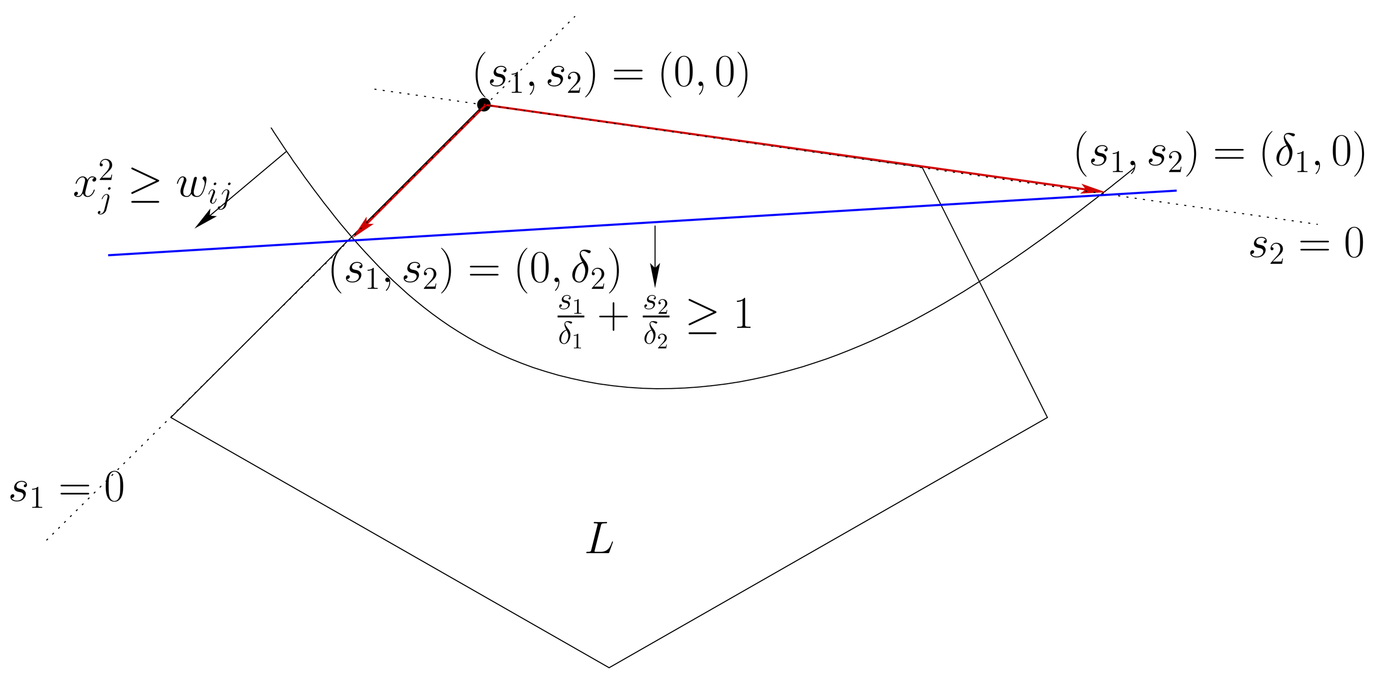
Assume for now (i) **non-diagonal entries** in all H_i are zero, (ii) l, u are finite. We rewrite each constraint of QCQP as:

$$h^i_{jj} \sum_{j \in \mathcal{I}_{i+}} w_j - h^i_{jj} \sum_{j \in \mathcal{I}_{i-}} v_j + \sum_{j=1}^{n} a_{ij} + d_i \le \quad 0, \quad i = 1,2,\dots,m \qquad (L_0)$$

$$x_j^2 \le \quad w_{ij} \qquad \text{(CX)}$$

$$-x_j^2 \le -w_{ij} \qquad \text{(CV)}$$





Pros and Cons:

- Inexpensive cuts, no CGLP.
- No branching.
- Needs only simplex based solver. But also needs *BInvACol()*.
- May not converge!

General Remarks

- If a secant inequality for $-x_j^2 \le -w_{ij}$ is included in L, and if (x_B, w_B) is a basic feasible solution of L, then x_j is non-basic. w_{ij} may or may not be non-basic at such points.
- In general, for each cut, we may need to find roots of up to n different quadratic equations in one variable.

3. Transforming QCQP to Remove Bilinear Terms

Drop the subscript i and consider a quadratic constraint: $x^T H x + a^T x + d \le 0$. (QC)

Eigenvectors of symmetric matrix H ⇨ $H = QDQ^T$, Q is Orthogonal, D is a Diagonal Matrix ⇨ $H = QRERQ^T$, E, R Diagonal Matrices, $e_{ii} \in \{-1, 0, 1\}$, $r_{ii} = \sqrt{|d_{ii}|}$ if $|d_{ii}| > 0$, 1 otherwise

$$\text{Let } \begin{cases} I_0 = \{i \mid e_{ii} = 0\}, \\ I_+ = \{i \mid e_{ii} = 1\}, \\ I_- = \{i \mid e_{ii} = -1\}. \end{cases} \quad \text{and let } \begin{cases} b = R^{-1} Q^T a, \\ z = \Sigma_{i \in I_0} b_i y_i + d, \\ y = RQ^T x. \end{cases}$$

$$(QC) \Leftrightarrow \sum_{i \in I_+} \left(y_i + \frac{b_i}{2} \right)^2 + z + \frac{\Sigma_{i \in I_0} b_i^2 - \Sigma_{i \in I_+} b_i^2}{4} \le \sum_{j \in I_-} \left(y_j - \frac{b_j}{2} \right)^2 \qquad \text{(S)}$$

4. Future Work and Research Directions

1. Report computational results!
2. Improve our implementation of the above methods. Handle degeneracy, free variables, unbounded LPs, …
3. Strengthen these inequalities by using existing MILP techniques: using different basic solutions for multiple cuts, exploiting structure, …

References

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