SOC Avatars for Solving MINLPs (in 3-D)

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We reformulate 3 types of nonlinear constraints in such a way that they become convex (Second Order Cones) after branching on specific disjunctions.

1. Quadratic constraints with one negative eigenvalue

Consider a quadratic constraint: $x^T Ax + c^T x + d \le 0$. (QC)

Eigenvectors of symmetric matrix A ⇒ $A = QDQ^T$, Q is Orthogonal, D is a Diagonal Matrix ⇒ $A = QRERQ^T$, E, R Diagonal Matrices, $e_{ii} \in \{-1, 0, 1\}$, $r_{ii} = \sqrt{|d_{ii}|}$ if $|d_{ii}| > 0$, 1 otherwise

Let $\begin{cases} I_0 = \{i \mid e_{ii} = 0\}, \\ I_+ = \{i \mid e_{ii} = 1\}, \\ I_- = \{i \mid e_{ii} = -1\}. \end{cases}$ and let $\begin{cases} b = R^{-1}Q^T c, \\ z = \Sigma_{i\in I_0} b_i y_i + d, \\ y = RQ^T x. \end{cases}$

$$(QC) \Leftrightarrow \sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + z + \frac{\Sigma_{i\in I_0} b_i^2 - \Sigma_{i\in I_+} b_i^2}{4} \le \sum_{j\in I_-}\left(y_j - \frac{b_j}{2}\right)^2 \quad \text{(S)}$$

If $|I_-| = 0$,

$$S \Leftrightarrow \sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + z + K \le 0$$

⇒ Convex!

If $|I_0| = 0, I_- = \{0\}, K = d - \frac{\Sigma_{i\in I_+} b_i^2}{4} \ge 0$, (C1)

S branches into:

$$\sqrt{\sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + K} \le \left(y_0 - \frac{b_0}{2}\right)$$

$$\sqrt{\sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + K} \le \left(-y_0 + \frac{b_0}{2}\right)$$

Both cases can be solved using an NLP solver with at most 2 nodes!

2. More general constraints with quadratic functions

Consider the general constraint: $x^T Ax + c^T x + d + \sum_{i=1}^{t} g_i(x) \le 0$ (NC)

S' branches into:

$$\sqrt{\sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + K + g_i(x)} \le \left(y_0 - \frac{b_0}{2}\right)$$

$$\sqrt{\sum_{i\in I_+}\left(y_i + \frac{b_i}{2}\right)^2 + K + g_i(x)} \le \left(-y_0 + \frac{b_0}{2}\right)$$

Under conditions (C1) both nodes are convex if:

1. $g_i(x) = x^p, p \in \{2, 4, 6, \ldots\}$
2. $g_i(x) = \alpha^x, \alpha \ge 0$
3. $g_i(x) = \ldots$ (Future work)

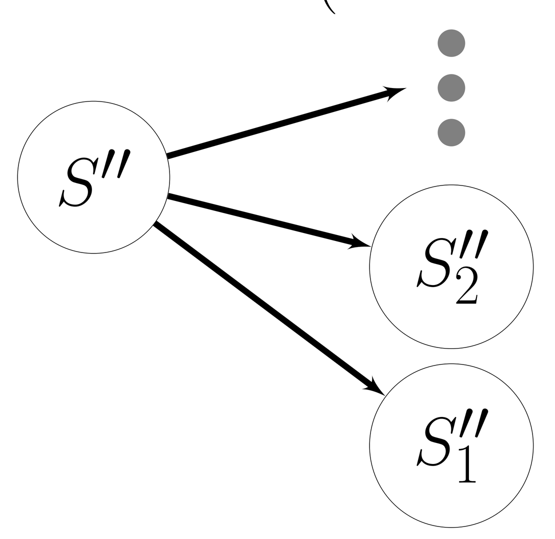
3. Factorizable polynomial (geometric) constraints

Consider a constraint: $P(x) \ge K$, where P is a polynomial function of degree t, $P : \mathbb{R}^n \to \mathbb{R}, K > 0$. Suppose if P can be factorized into t linear factors, $P(x) = \prod_{i=1}^{t}(a_i^T x + b_i)$.

Then the feasible set can be described by $S'' = \left\{(x, y) \mid \prod_{i=1}^{t} y_i \ge K, y_i = (a_i^T x + b_i)\right\}$.

Branch on each $y_i \ge 0 \lor y_i \le 0$

(2^t nodes)



At each node:

- Let $I_- = \{i \mid y_i \le 0\}$
- If $|I_-|$ is odd, prune the node.
- Draw a pair (y_i, y_j), from I_-
- $y_i y_j \ge w_{ij}^2, w_{ij} \ge 0$
- $I_+ = \{i \mid y_i \ge 0\}$.
- Draw a pair (y_k, y_l), from I_+
- $y_k y_l \ge v_{ij}^2, v_{ij} \ge 0$

⇒ Write as SOCs:

t new constraints and variables at each node.

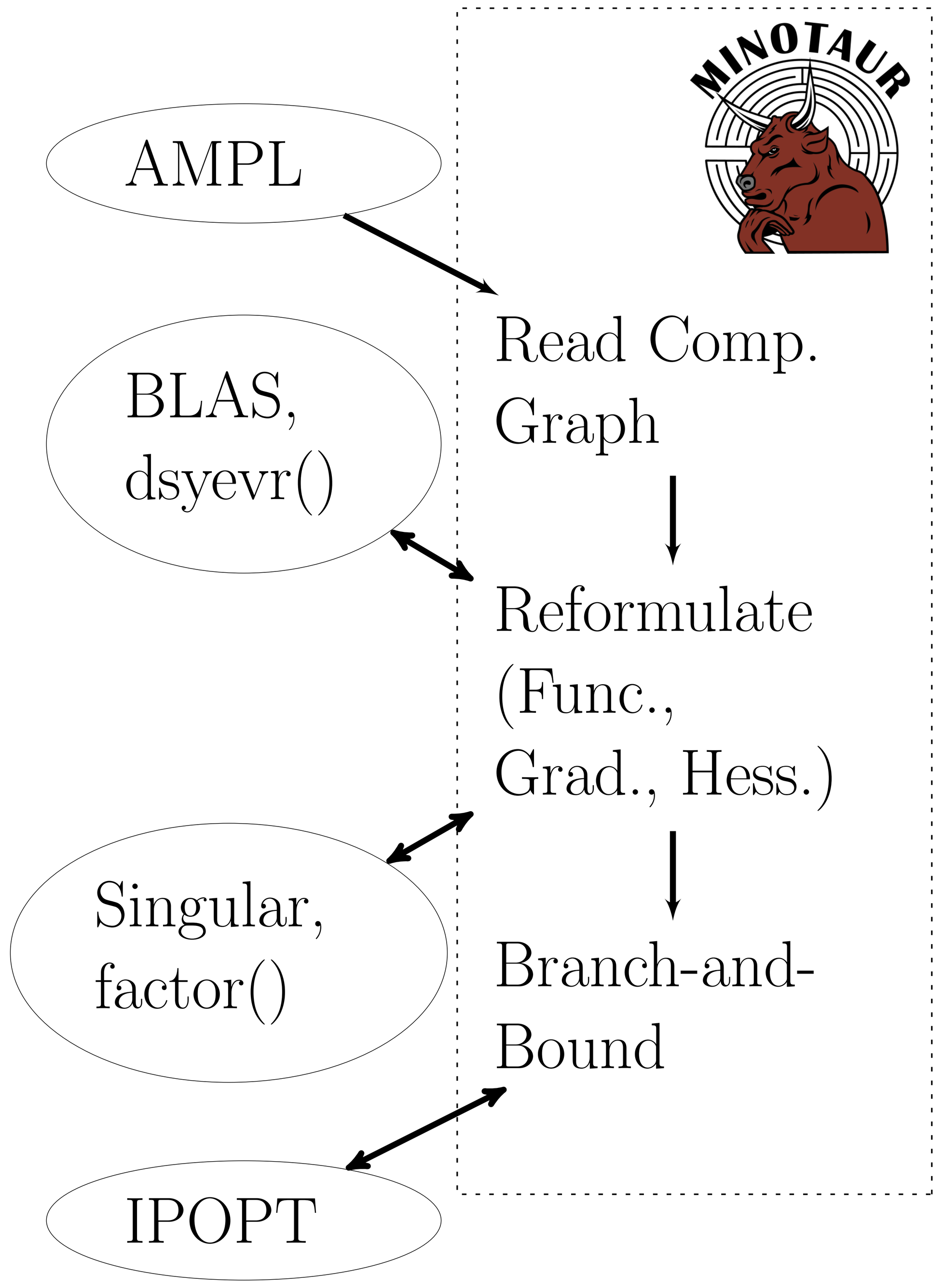
$$\sqrt{4w_{ij}^2 + (y_i - y_j)^2} \le -y_i - y_j$$

$$\sqrt{4v_{kl}^2 + (y_k - y_l)^2} \le y_k + y_l$$

$$\prod v_{kl} \times \prod w_{ij} \ge \sqrt{K}$$

Treat recursively.

Computational Experiments



Instance			# Nodes		
Name	Vars	Cons	BARON	Couenne	soc
q2d6	6	4	107505	3868500	4
q3d9	9	6	>1250100	>1844800	8
q5d10	10	10	1532839	3125701	32
q5d10b	10	10	>1033800	>2818700	32
q6d12	12	12	>1358100	>3377600	64
p4d12	12	12	1061	13720	31
p5d10	10	10	>927400	>2070800	11
p5d10e	10	10	>939900	>184100	11
p5d15e	15	10	322745	>160000	63
p6d18	18	12	234687	>910500	91
fb1d3	3	2	1425	98501	4
fb2d6	6	4	366201	>2308900	16
fb3d9	9	6	>525924	>1400691	64