

We reformulate 3 types of nonlinear constraints in such a way that they become convex (Second Order Cones) after branching on specific disjunctions.

## 1. Quadratic constraints with one negative eigenvalue

Consider a quadratic constraint:  $x^T A x + c^T x + d \leq 0$ . (QC)

Eigenvectors of symmetric matrix  $A$   $\Rightarrow$   $A = Q D Q^T$   
 $Q$  is Orthogonal  
 $D$  is a Diagonal Matrix  $\Rightarrow$   $A = Q R E R Q^T$   
 $E, R$  Diagonal Matrices  
 $e_{ii} \in \{-1, 0, 1\}$   
 $r_{ii} = \sqrt{|d_{ii}|}$  if  $|d_{ii}| > 0, 1$  otherwise

Let  $\begin{cases} I_0 = \{i \mid e_{ii} = 0\}, \\ I_+ = \{i \mid e_{ii} = 1\}, \\ I_- = \{i \mid e_{ii} = -1\}. \end{cases}$  and let  $\begin{cases} b = R^{-1} Q^T c, \\ z = \sum_{i \in I_0} b_i y_i + d, \\ y = R Q^T x. \end{cases}$

$$(QC) \Leftrightarrow \sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + z + \frac{\sum_{i \in I_0} b_i^2 - \sum_{i \in I_+} b_i^2}{4} \leq \sum_{j \in I_-} \left(y_j - \frac{b_j}{2}\right)^2 \quad (S)$$

If  $|I_-| = 0$ ,

$$S \Leftrightarrow \sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + z + K \leq 0$$

$\Rightarrow$  Convex!

Both cases can be solved using an NLP solver with at most 2 nodes!

If  $|I_0| = 0, I_- = \{0\}, K = d - \frac{\sum_{i \in I_+} b_i^2}{4} \geq 0$ , (C1)

$$S \begin{cases} \sqrt{\sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + K} \leq \left(y_0 - \frac{b_0}{2}\right) \\ \sqrt{\sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + K} \leq \left(-y_0 + \frac{b_0}{2}\right) \end{cases}$$

## 2. More general constraints with quadratic functions

Consider the general constraint:  $x^T A x + c^T x + d + \sum_{i=1}^t g_i(x) \leq 0$  (NC)

$$S' \begin{cases} \sqrt{\sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + K + g_i(x)} \leq \left(y_0 - \frac{b_0}{2}\right) \\ \sqrt{\sum_{i \in I_+} \left(y_i + \frac{b_i}{2}\right)^2 + K + g_i(x)} \leq \left(-y_0 + \frac{b_0}{2}\right) \end{cases}$$

Under conditions (C1) both nodes are convex if:

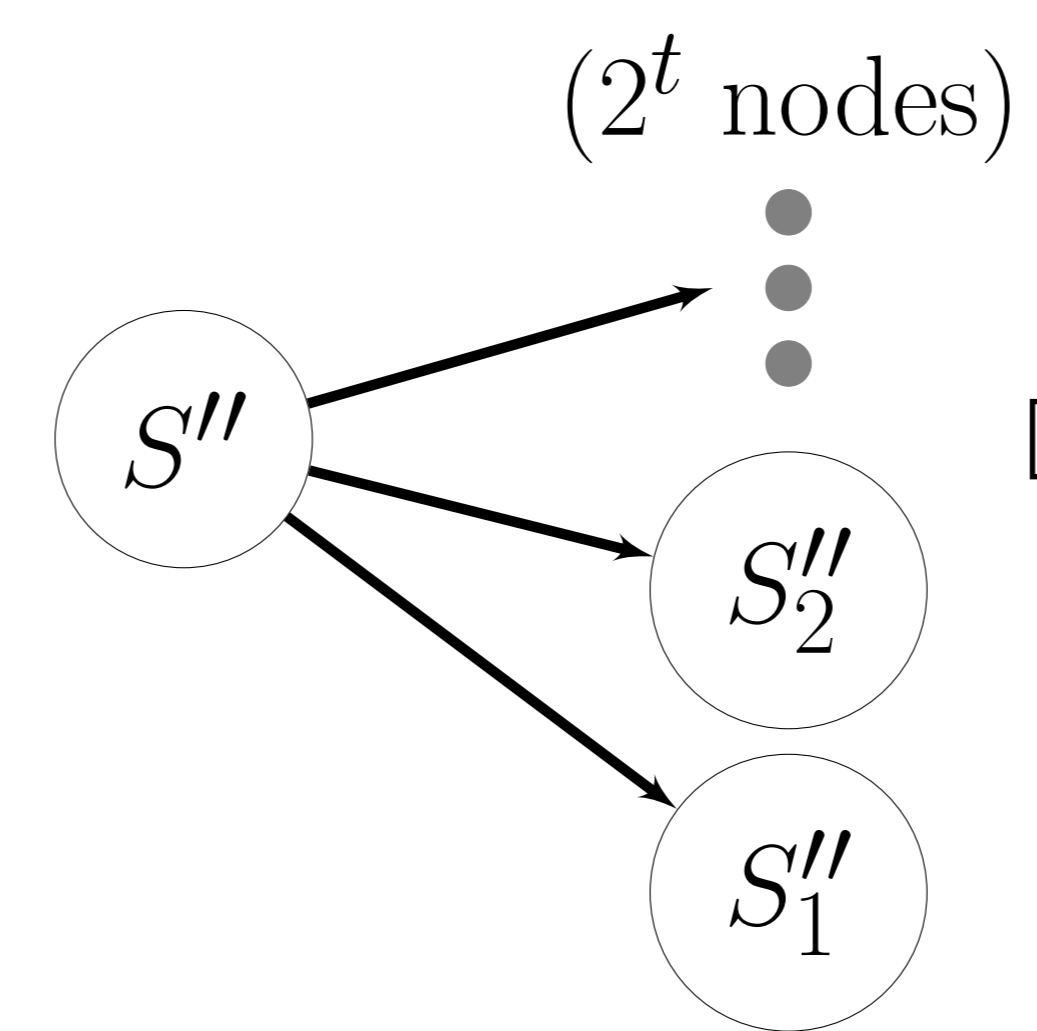
1.  $g_i(x) = x^p, p \in \{2, 4, 6, \dots\}$
2.  $g_i(x) = \alpha^x, \alpha \geq 0$
3.  $g_i(x) = \dots$  (Future work)

## 3. Factorizable polynomial (geometric) constraints

Consider a constraint:  $P(x) \geq K$ , where  $P$  is a polynomial function of degree  $t$ ,  $P: \mathbb{R}^n \rightarrow \mathbb{R}, K > 0$ . Suppose if  $P$  can be factorized into  $t$  linear factors,  $P(x) = \prod_{i=1}^t (a_i^T x + b_i)$ . Then the feasible set can be described by  $S'' = \left\{ (x, y) \mid \prod_{i=1}^t y_i \geq K, y_i = (a_i^T x + b_i) \right\}$ .

Branch on each  $y_i \geq 0 \vee y_i \leq 0$

At each node:



- Let  $I_- = \{i \mid y_i \leq 0\}$
- If  $|I_-|$  is odd, prune the node.
- Draw a pair  $(y_i, y_j)$ , from  $I_-$
- $y_i y_j \geq w_{ij}^2, w_{ij} \geq 0$
- $I_+ = \{i \mid y_i \geq 0\}$ .
- Draw a pair  $(y_k, y_l)$ , from  $I_+$
- $y_k y_l \geq v_{kl}^2, v_{kl} \geq 0$

Write as SOCs:

$t$  new constraints and variables at each node.

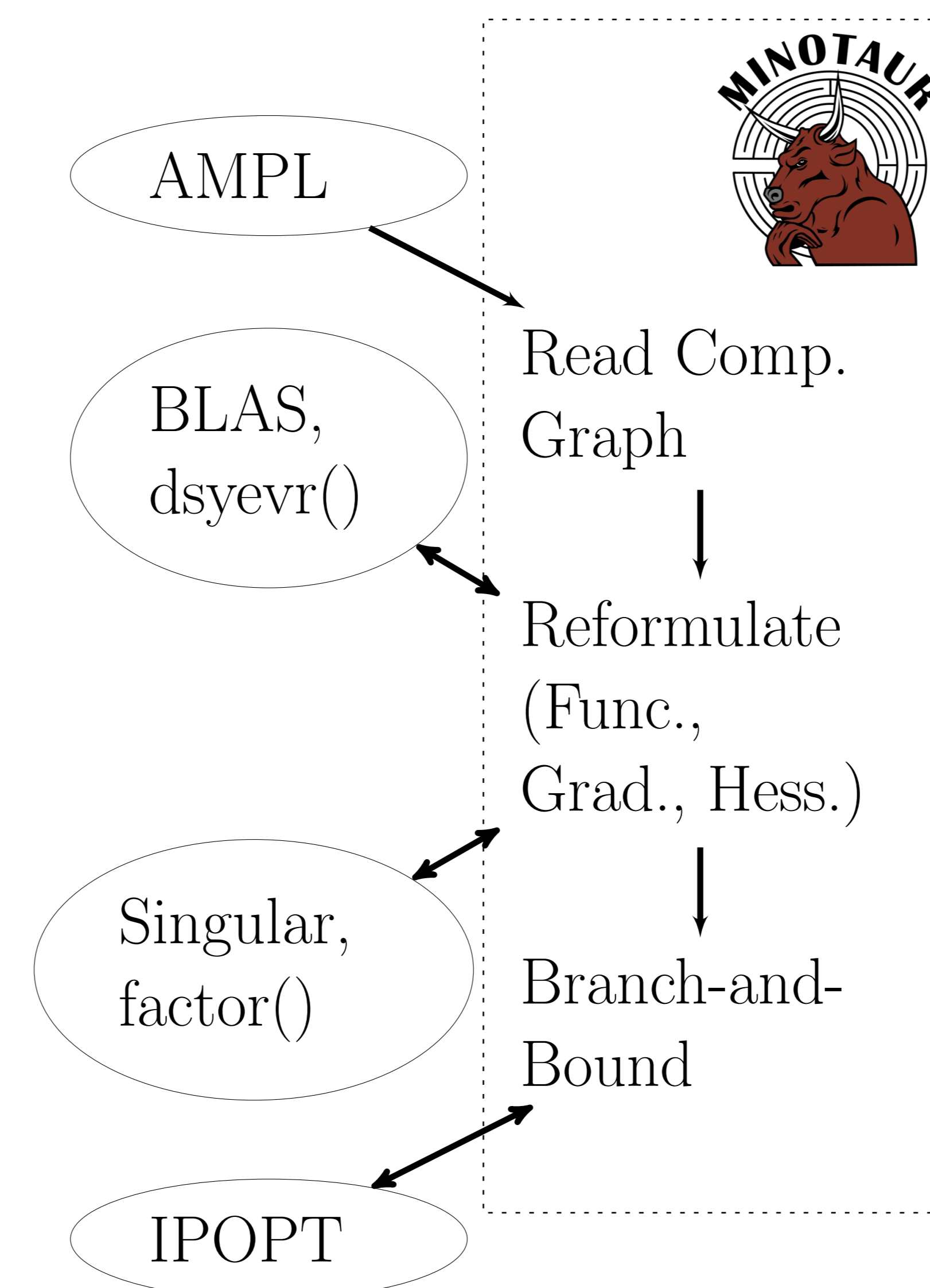
$$\sqrt{4w_{ij}^2 + (y_i - y_j)^2} \leq -y_i - y_j$$

$$\sqrt{4v_{kl}^2 + (y_k - y_l)^2} \leq y_k + y_l$$

$$\prod v_{kl} \times \prod w_{ij} \geq \sqrt{K}$$

Treat recursively.

## Computational Experiments



Instance			# Nodes		
Name	Vars	Cons	BARON	Couenne	soc
q2d6	6	4	107505	3868500	4
q3d9	9	6	>1250100	>1844800	8
q5d10	10	10	1532839	3125701	32
q5d10b	10	10	>1033800	>2818700	32
q6d12	12	12	>1358100	>3377600	64
p4d12	12	12	1061	13720	31
p5d10	10	10	>927400	>2070800	11
p5d10e	10	10	>939900	>184100	11
p5d15e	15	10	322745	>160000	63
p6d18	18	12	234687	>910500	91
fb1d3	3	2	1425	98501	4
fb2d6	6	4	366201	>2308900	16
fb3d9	9	6	>525924	>1400691	64