In Search of Optimal Disjunctions in Mixed Integer Programming

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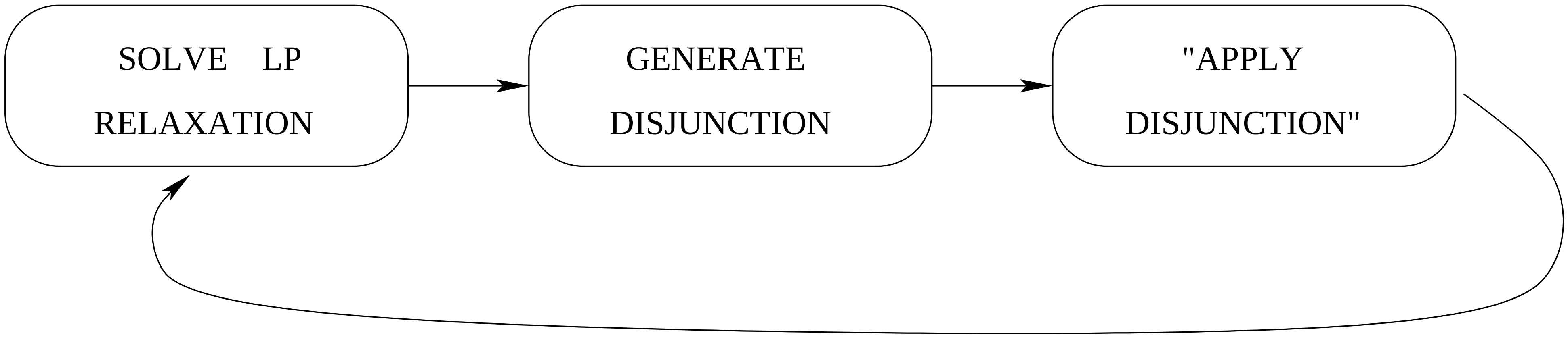
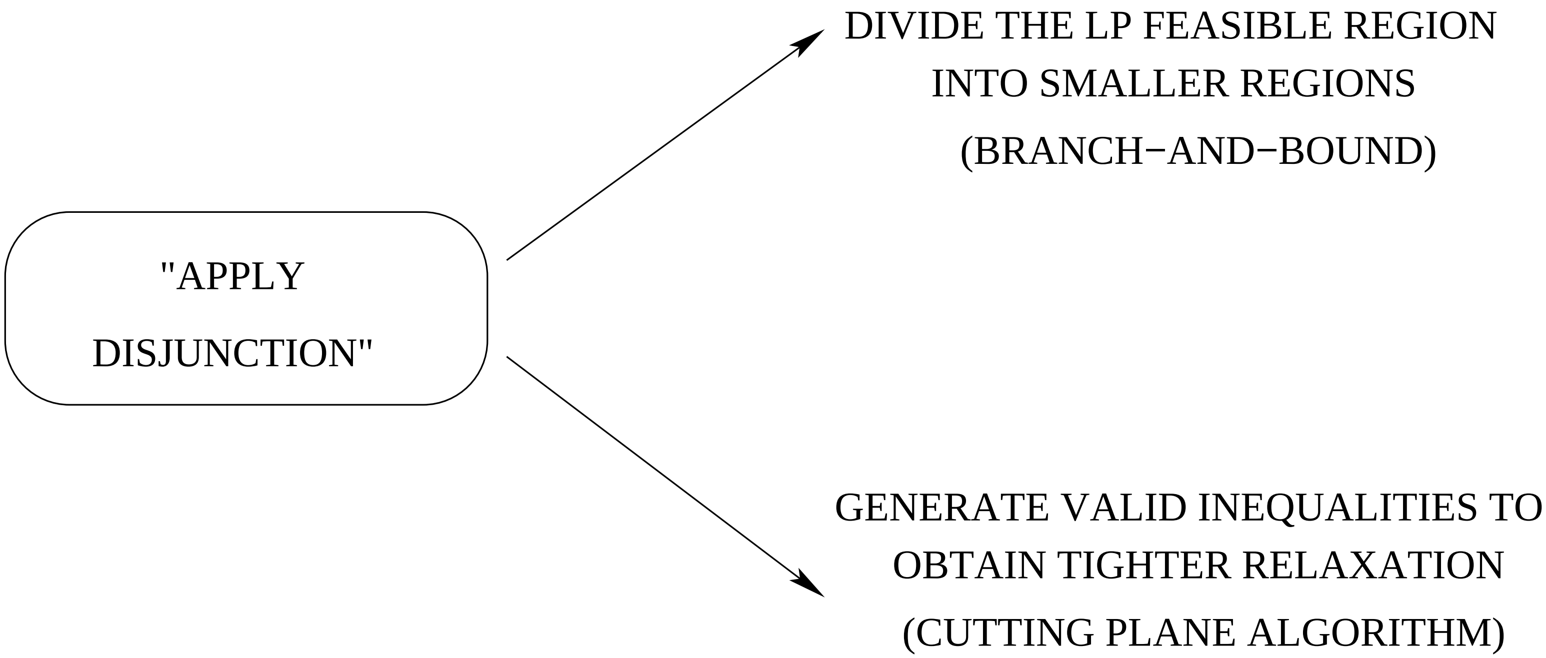
Solving MIPs Using Disjunctions





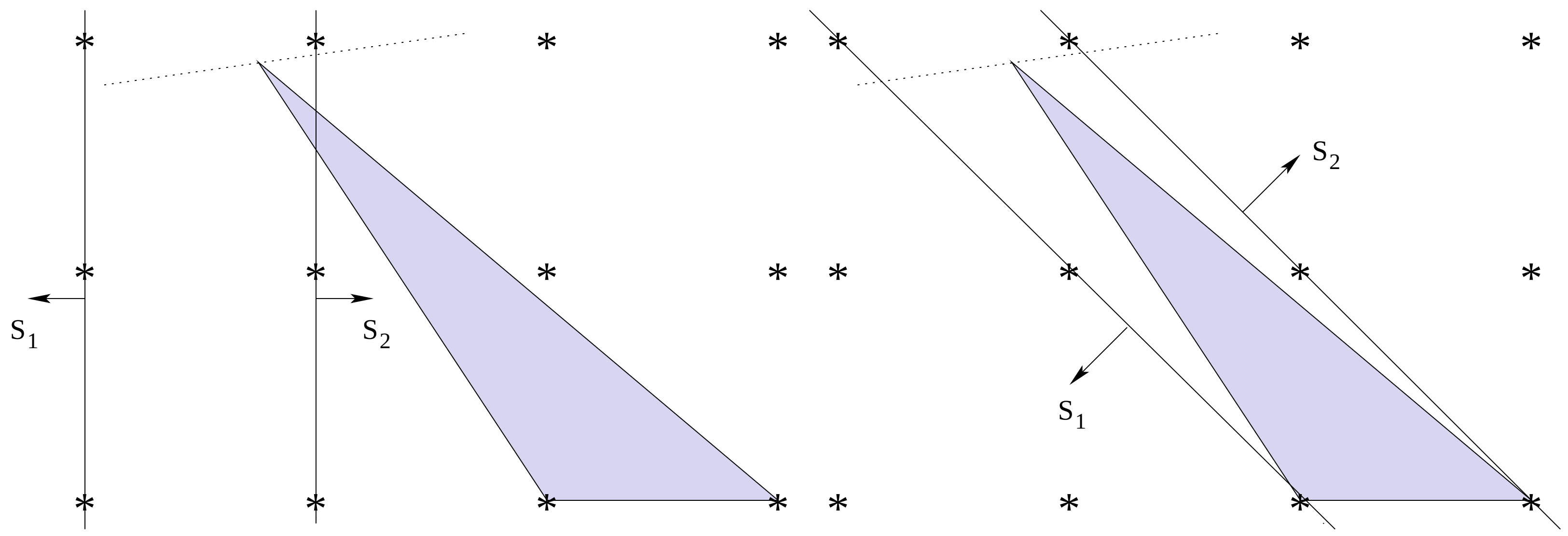
How to select disjunctions for branching or generating valid inequalities?

What is a Disjunction?

Given a MIP of the form

$$\begin{aligned} &\min cx && \text{(objective function)} \\ &\text{such that} \quad Ax \geq b && \text{(constraints)} \\ &x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}. && \text{(integrality)} \end{aligned}$$

- *Variable Disjunction*: $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1, i \in \{1, 2, \ldots, d\}$
- *General Disjunction*: $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$, $(\pi, \pi_0) \in \mathbb{Z}^d \times \{0\}^{n-d} \times \mathbb{Z}$.



Problem of Maximizing Lower Bound

Criterion for selecting general disjunction: Maximization of the lower bound obtained after applying the disjunction.

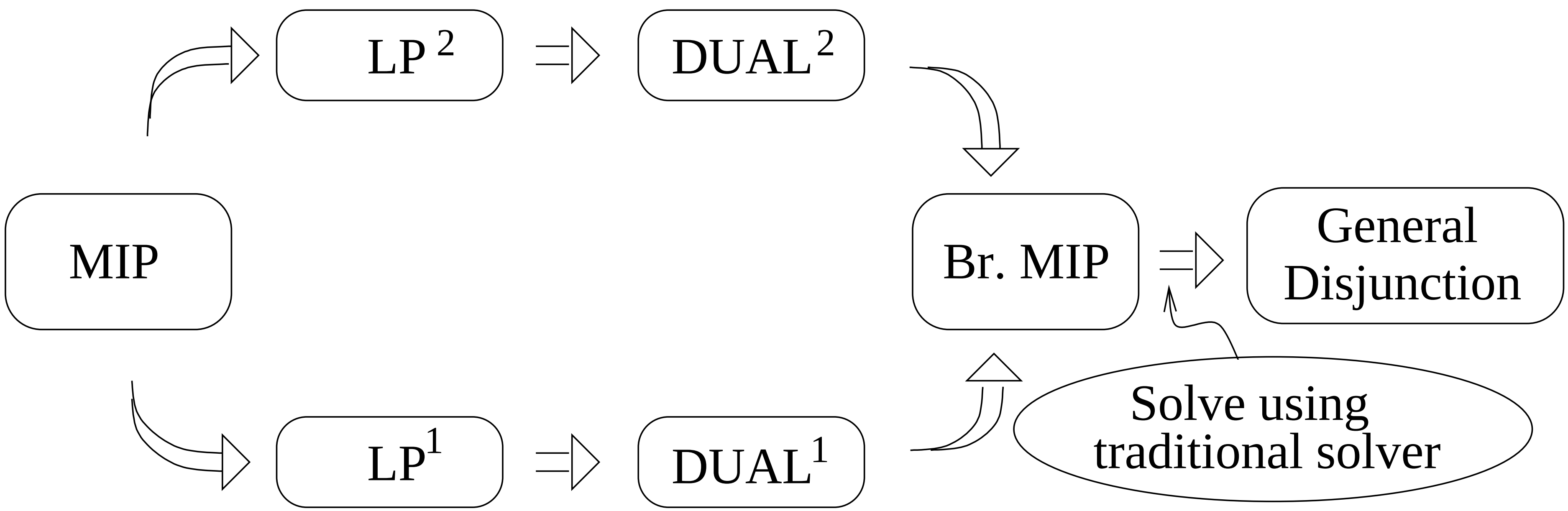
Problem

Find $(\hat{\pi}, \hat{\pi}_0) \in \mathbb{Z}$ such that objective function value is at least (some given) K in both subproblems.

$$\begin{aligned} Ax &\geq b \\ cx &\leq K \\ \pi x &\leq \pi_0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi} x &\geq \hat{\pi}_0 + 1 \end{aligned} \tag{1}$$

should both be infeasible.

Formulation Technique



The Formulation

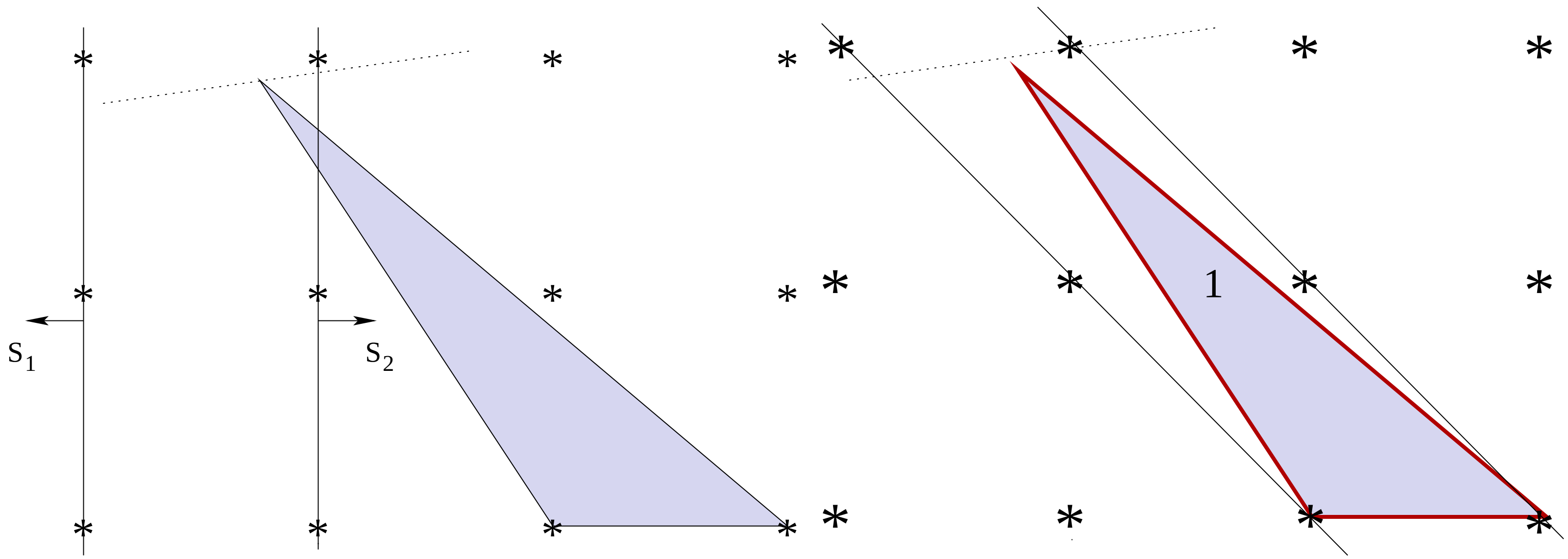
Lower bound after "application" of a disjunction can be at least K if and only if

$$\begin{aligned} pA - s_L c - \pi &= 0 \\ qA - s_R c + \pi &= 0 \\ pb - s_L K - \pi_0 &\geq \delta \\ qb - s_R K + \pi_0 &\geq -1 + \delta \\ p, q, s_L, s_R &\geq 0 \\ (\pi, \pi_0) \in \mathbb{Z}^d \times \{0\}^{n-d} &\times \mathbb{Z}^1, \end{aligned} \tag{2}$$

is feasible for some $\delta > 0$.

Solve (2) over different values of K to get the best one.

Examples

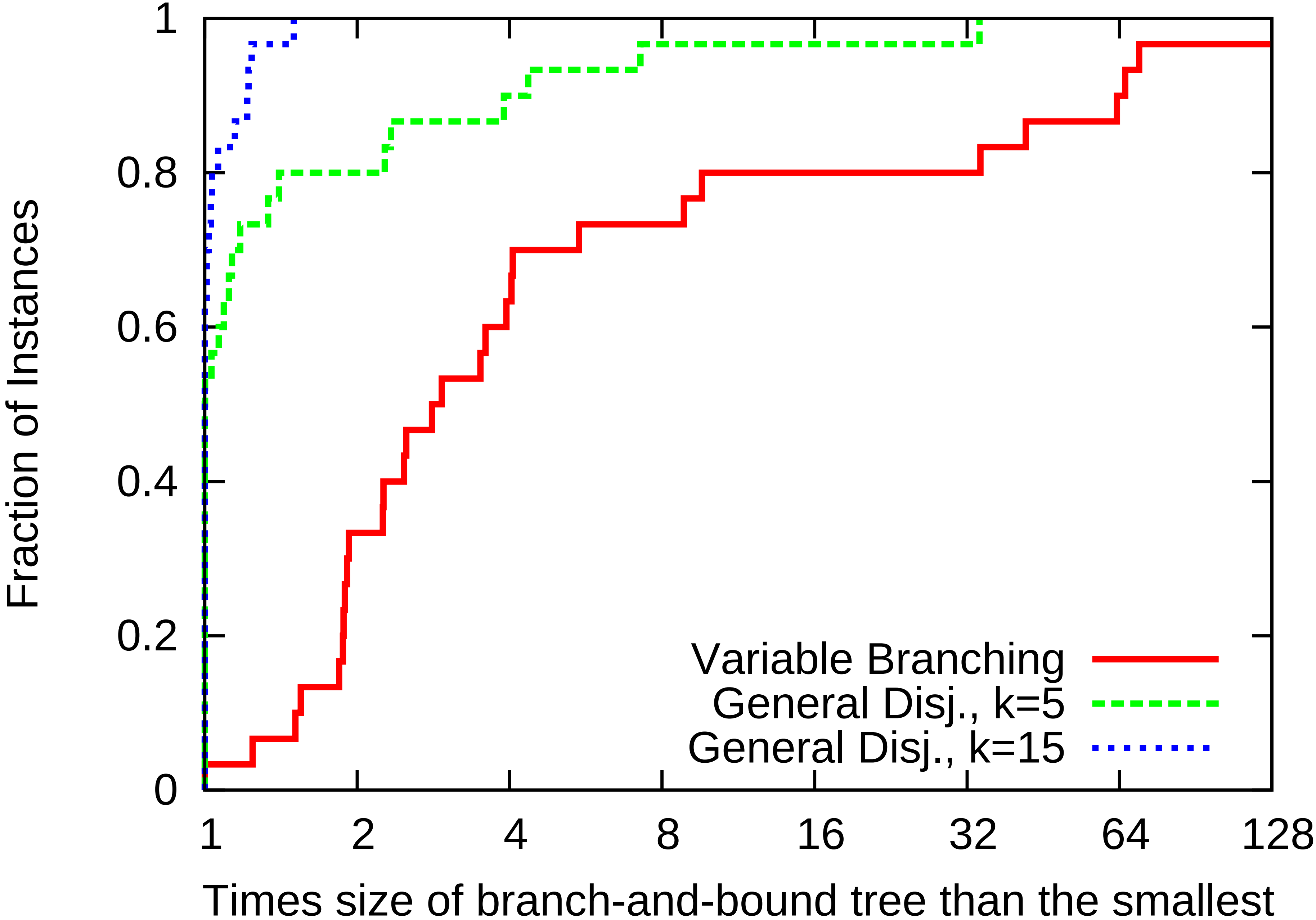


(No. of disjunctions: 4) (No. of disjunctions: 1)

Tree size for selected instances:

Instance	Variable	Gen. Disj.	Improvement
10teams	115	12	9.58
bell3a	16387	259	63.27
flugpl	394	6	65.67
gt2	340	10	34
mod008	2840	68	41.76
vpm1	263111	20	13155.55

Comparison over 30 instances (MIPLIB and Mittleman)



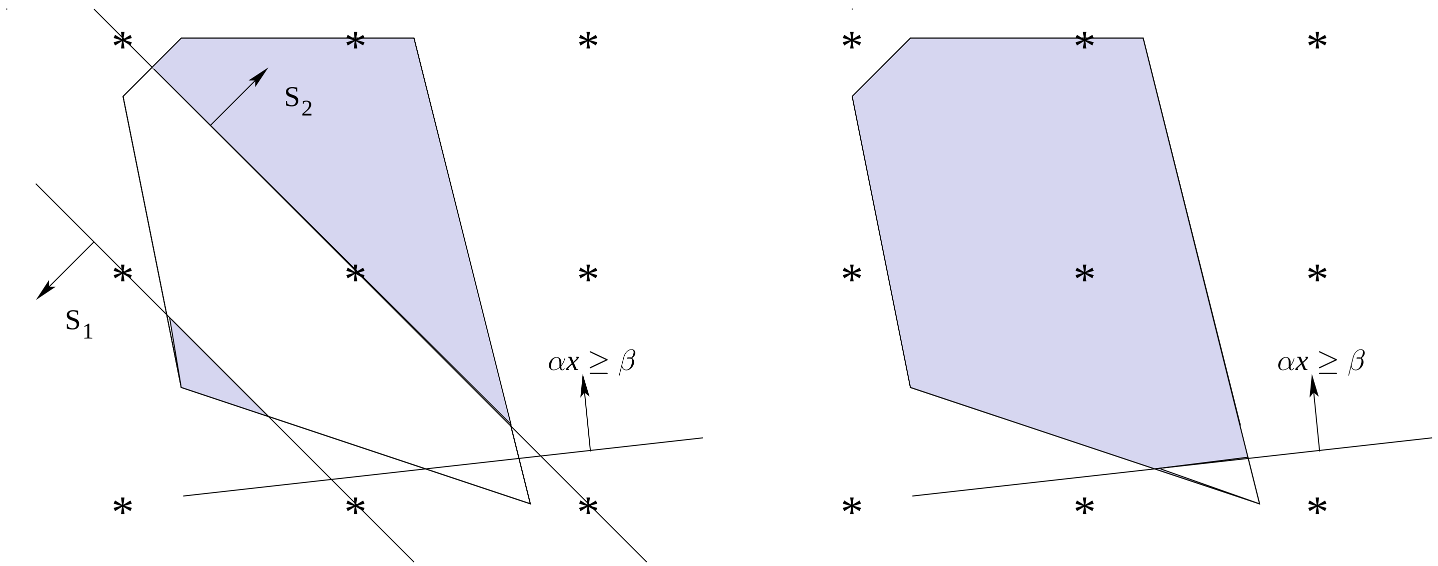
Computational Complexity

The problem of finding the "best" disjunction is:

- $\mathcal{NP}-$complete in general (Reduction from Number Partitioning Problem)
- $\mathcal{NP}-$complete when $\pi \in \{0, 1\}^n$
- $\mathcal{NP}-$complete when $\pi \in \{-1, 0, 1\}^n$
- $\mathcal{NP}-$complete when $(\pi, \pi_0) \in \{0, 1\}^{n+1}$
- $\mathcal{NP}-$complete when $x \in \{0, 1\}^n$ and either of above four conditions hold (Reduction from 1-IN-3SAT)

. . . lead to similar results for some problems of generating *split inequalities*

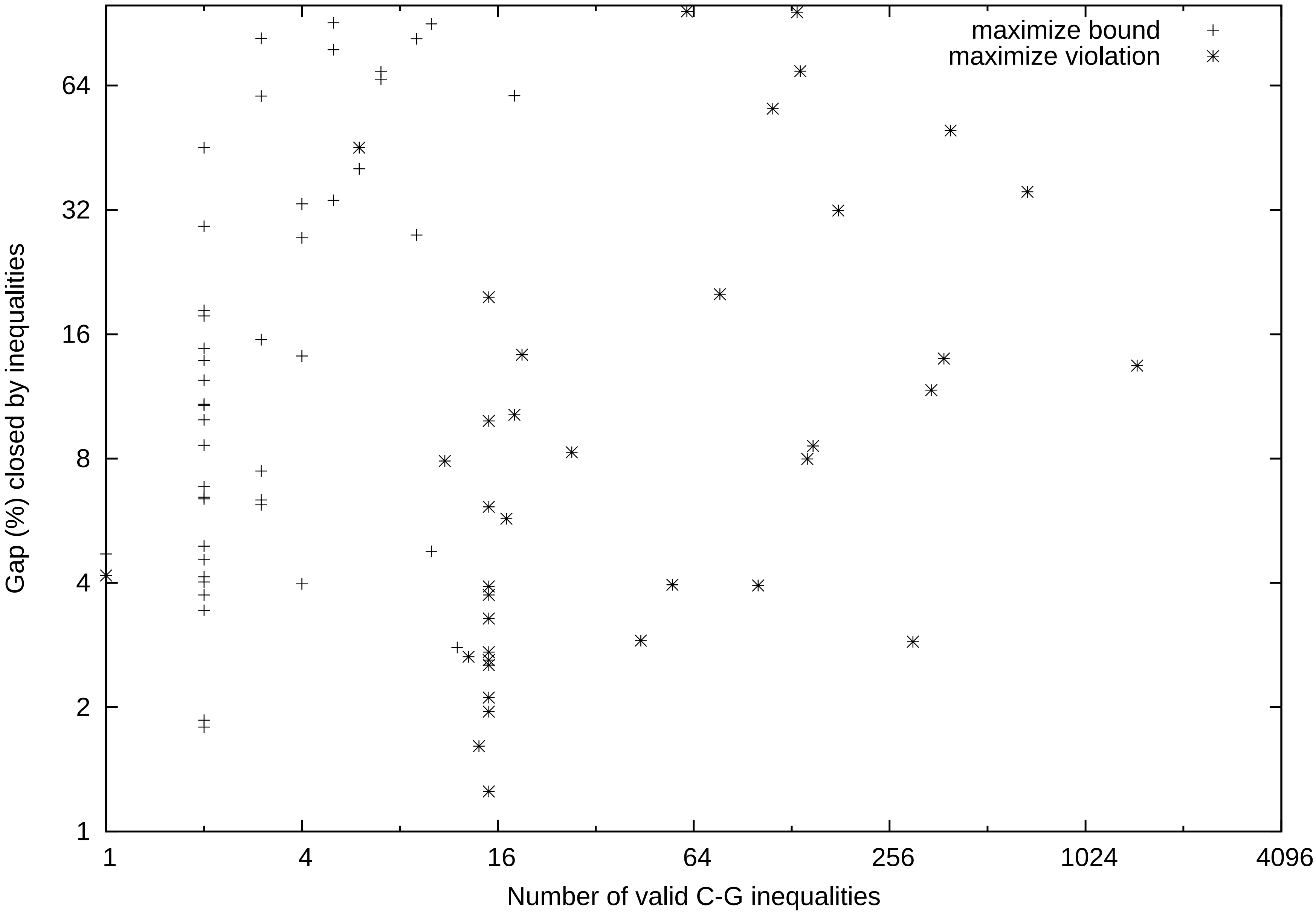
Disjunctions for the Cutting-Plane Algorithm



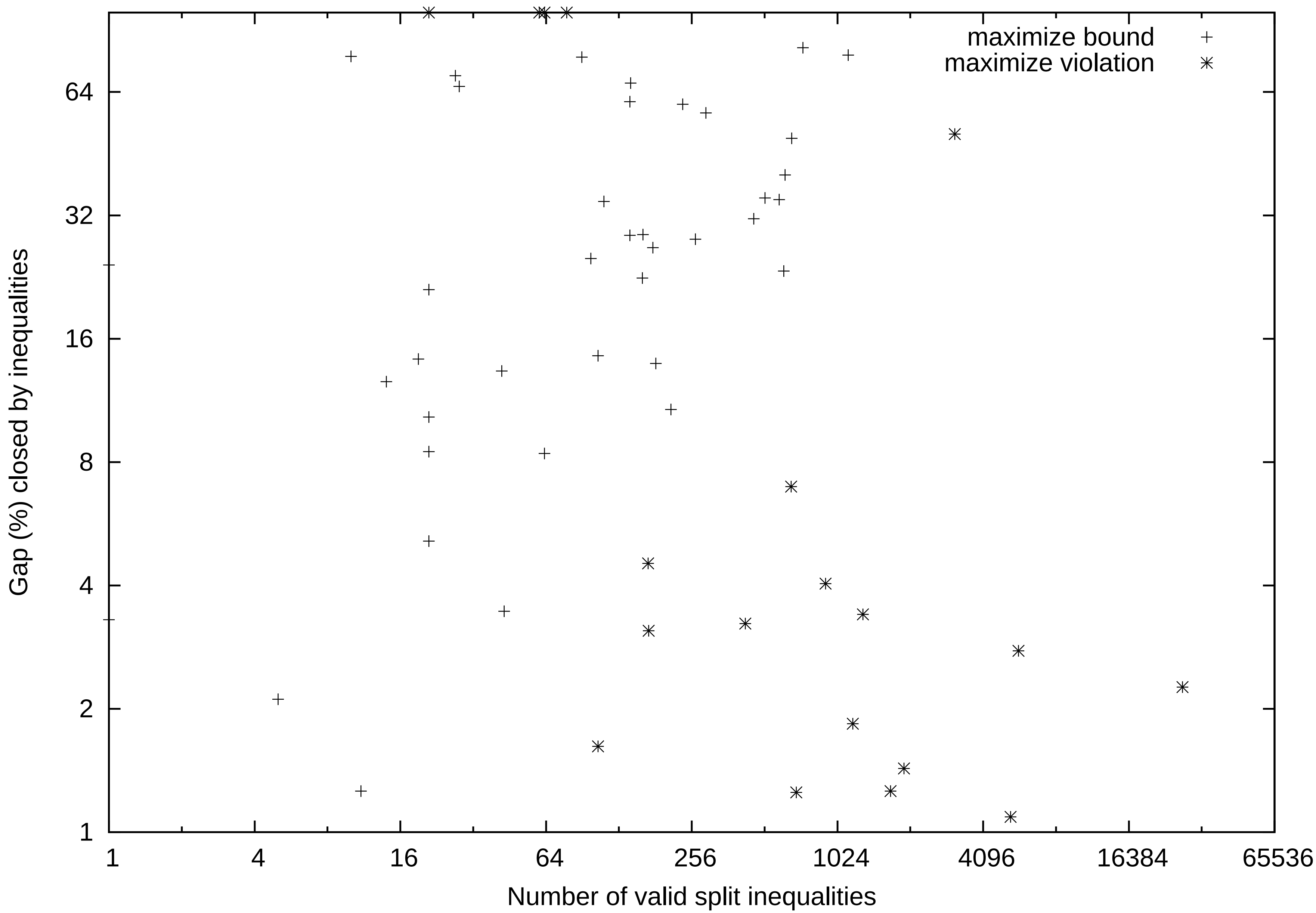
$\alpha x \geq \beta$ is a "split-cut" that separates the LP solution from the feasible region.

- Most valid inequalities are types of *Split Inequalities*: C-G, GMI, Lift and Project, MIR ....
- Given an inequality $\alpha x \geq \beta$, is it an elementary split inequality? $\mathcal{NP}-$Complete.
- Given a polyhedron $\mathcal{P}$ and $K \in \mathbb{R}$, does there exist an elementary split inequality such that the LP bound after adding it is at least K? $\mathcal{NP}-$Complete.

Using Formulations for “best” C-G Inequalities



Using Formulations for “best” Split Inequalities



Conclusions and Future Work

- Selecting the "best" disjunction is $\mathcal{NP}-$hard in the general case and also for several natural restrictions.
- These disjunctions can reduce significantly the size of branch-and-bound tree.
- Inequalities obtained from such disjunctions can also increase the bound much better than maximally violated inequalities.
- However, fast heuristics to discover such disjunctions need to be developed.
- What is best way to use a disjunction: branch or generate valid inequalities?