STABILITY OF PRODUCTION-INVENTORY CONTROL SYSTEMS CONSIDERING INVENTORY SHORTAGES

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Extended Abstract

The modelling and analysis of the production-inventory control systems of manufacturing firms, is critical in understanding the dynamic behaviour of the firm in terms of fluctuations in inventory levels, production releases, stock-outs; and ultimately affecting the profit of the firm. The dynamic behaviour is said to arise from the interaction between the various system components over time (Sterman 2000). Mathematical programming techniques such as linear/non-linear/stochastic programming cannot adequately capture dynamic systems, characterized by delays, feedbacks and nonlinearities. A natural choice to examine the production and inventory dynamics is the application of system dynamics and control theoretic techniques. This often involves capturing of the production–inventory system using feedback-based structures (Forrester 1961, Towill 1982, Sterman 2000) and analysis of the system through the application of control theoretic tools such as block diagrams, and functional transformations (Disney *et al*. 2004).

In this paper, stock-flow diagrams are employed to model the production-inventory system, which is then analyzed using z-transformation techniques. Regions of stability for the cases of infinite inventory coverage and limited inventory coverage are separately identified. The behaviour of the production-inventory system characterized by periods of excess inventory and periods of inventory shortages are analyzed and the results presented.

The underlying model of the production ordering and inventory control system in this work is adapted from the generic stock management structure (Sterman 2000) and the Automated Pipeline Inventory and Order based Production Control System (APIOBPCS) family of models (Towill 1982, Disney *et al*. 2004). In this research, the above ordering rule is adapted to represent the production release ordering rules. However, the model developed in this research improves over the APIOBPCS model in the explicit inclusion of the order fulfilment non-linearity, i.e., the shipment is defined as a function of demand and inventory-on-hand to ensure that the shipment does not exceed the inventory on hand.

The production & inventory management model is as shown in Figure 1, using stock-flow diagram. The sequence of operations in a given time period is as follows: Sales, Forecasting, Calculation of desired WIP and desired inventory, Determining the current period's production release, Determining the production rate, Computing the current WIP and the current inventory levels. The underlying equations governing each function are as shown in Table 1 (Venkateswaran and Son, 2005). In Equation (1), the forecasted demand (*FD*) of the products is modelled as first order exponential smoothing of the customer sales rate (*SALES*), with a smoothing constant ρ. The shipment rate (*SHIP*) of the physical goods is determined as a function of the current demand and the inventory in stock, to ensure that the shipment does not exceed the inventory on hand (Equation (2)).

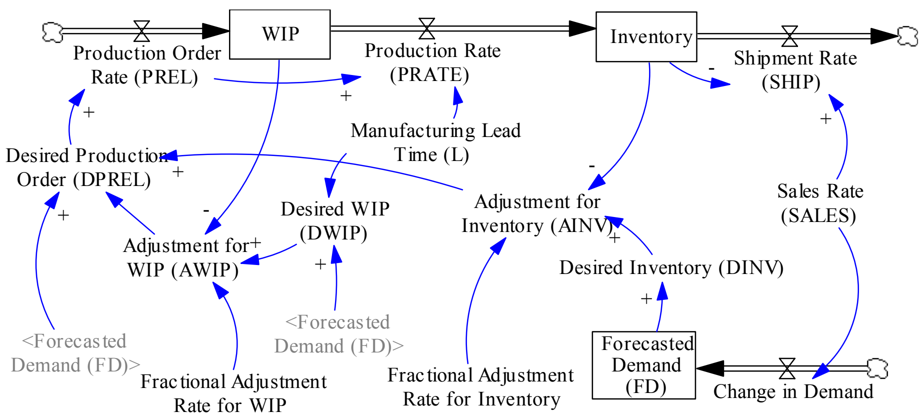

Figure 1: Stock-flow diagram of production and inventory management

Table 1: Equations governing the model

$FD_t = FD_{t-1} + \delta \cdot \rho \cdot (SALES_{t-1} - FD_{t-1})$	(1)
$SHIP_t = MIN[SALES_t, INV_t]$	(2)
$PREL_t = FD_{t-1} + \alpha \cdot (FD_{t-1} \cdot L - WIP_{t-1}) + \beta \cdot (FD_{t-1} - INV_{t-1})$	(3)
$INV_t = INV_{t-1} + (PRATE_t - SHIP_t)$	(4)
$WIP_t = WIP_{t-1} + (PREL_t - PRATE_t)$	(5)
$PRATE = PREL_{t-L}$	(6)

It is noted that all unfulfilled demand is assumed to be lost. The production ordering aspect (Equation 3) determines the production release quantities (*PREL*) using the ordering rule, based upon the forecasted demand, the difference between the desired level of WIP and the current WIP level, and difference between the desired level of inventory and the current inventory level (*INV*). Based on the Little's Law, the desired WIP in the system is set to yield the desired throughput, given the lead time (*L*) (see Equation 3). The desired throughput is set equal to the forecasted demand. To provide adequate coverage of inventory, the manufacturer seeks to maintain a desired level of inventory set equal to the forecasted demand (see Equation 3). The fractional adjustment rate for WIP (α) describes how much of the discrepancy between the desired and current levels of WIP is to be added to the production release order. The fractional adjustment rate for inventory (β) describes how much of the discrepancy between the desired and current levels of inventory is to be added to the production release order. Also, the inventory level (*INV*) accumulates the difference in the production rate (*PRATE*) and the shipment rate (Equation 4). In Equation 5, *WIP* accumulates the difference in the production release rate and the production rate. The production process is modelled as a typically fixed pipeline delay in Equation 6.

Production and inventory control systems can be readily viewed as a system sampled at regular discrete intervals, since the ordering rules are evaluated only at discrete points in time, such as every day or every week. The underlying equations are discrete, and hence *z*-transform can be readily applied to obtain generalized transfer functions (Disney and Towill 2002, Venkateswaran and Son 2006) of the production release order (and later the stability conditions). The transfer functions are obtained in terms of the following system parameters: (1) fractional adjustment of WIP, (2) fractional adjustment of inventory, (3) exponential smoothing constant for forecast, and (4) production lead time.

Now, a necessary step before the *z*-transform analysis is the linearization of the non-linear functions present in the system model. Linearization is important as the exact solution using *z*-transform analysis can be obtained only for linear system. The non-linear functions are found to arise in the inventory and production control systems due to 'saturation effect'. The saturation effect results in sharp discontinuities in the output response to a varying input. In the model, the shipment rate is a typical non-linear function characterized by the saturation effect. In this paper, such non-linear functions are linearized using local linearization technique, in which the non-linear function is separated into piecewise linear functions. In the model presented above, the shipment rate (Equation 2) is given as a function of the sales and the inventory available. Assuming a sharply discontinuous function, it is seen that when the sales is less than the inventory available, the shipment rate equals the sales rate; and when the sales rate is more than the inventory available, the shipment rate equals the inventory available, as shown below:

$$SHIP_t = MIN\{SALES_t, INV_t\} \quad \Rightarrow \quad SHIP_t = \begin{cases} SALES_t & ,\text{if } SALES_t \le INV_t \quad (7a) \\ INV_t & ,\text{if } SALES_t > INV_t \quad (7b) \end{cases}$$

These two distinct regimes of inventory operations are separately analyzed using *z*-transform techniques. In the first operational regime, it is assumed that there is always sufficient inventory coverage to meet the desired shipments. In the second operational regime, it assumed that there is not sufficient inventory coverage to meet the desired shipments. It is noted that the dynamic behaviour of the system often results in transition between the operational regimes, which are not captured in such separate analysis. However, useful insights can be drawn from such segregated analysis. The *z*-transforms of the production ordering and inventory control system (Equations 1, 3–6) were obtained. *Z*-transforms were also obtained for the piece-wise linearized form of Equation (2), (7a) and (7b).

For the case of sufficient inventory coverage (when *SALES* <= *INV*), the transfer function for *PREL/SALES* has been obtained by solving the *z*-transforms of equations (1), (3-6) and (7a) simultaneously. For the case of insufficient inventory coverage (when *SALES* > *INV*), the transfer function for *PREL/SALES* has been obtained by solving the *z*-transforms of equations (1), (3-6) and (7b) simultaneously.

Now, it is important to understand how the production ordering and inventory control system responds to any change in its input (i.e. sales rate), especially under a fluctuating market. Does the response result in increasing amplitude oscillations and chaos in general, or does the response appear controllable and damped? Thus it becomes essential to know under what conditions the system is stable or unstable. Hence, the general conditions for the system stability from the *PREL* transfer functions in terms of the various design parameters, were derived. It is noted that the system given by its closed form transfer function is said to be stable if all the roots (poles) of the transfer function's denominator polynomial lie within the unit circle in the complex plan. In this research, Jury's Test (Jury 1964) is employed to determine the location of the roots. Though this method enables a solution, it still involves tedious calculations, which are hence performed by the authors by using Mathematica®. The stability conditions for the two different operational regimes considered are obtained in terms of the control parameters: fractional adjustment rates of WIP (α) and inventory (β).

The conditions for stability for a fixed pipeline delay of L = 3, and ρ = 1 under the first operational regime (when *SALES* <= *INV*) and second operations regime (when *SALES* > *INV*) have been derived as shown in Equations (8) and (9):

$$\alpha > \frac{-2-4\beta+3\beta^2 \pm (-2+\beta)\sqrt{4+\beta^2}}{2(-3+2\beta)} \text{ and } \alpha < \frac{2+\beta}{2} \quad (8)$$

$$\beta > -1+\alpha \text{ and } \beta < 1 \quad (9)$$

Figures 2a and 2b illustrates the stable (grey area) region and unstable regions on the parameter plane for each case. The system guarantees to be stable when the values of α and β are restricted to the stable region, and critically stable when α and β lie on the boundary lines.

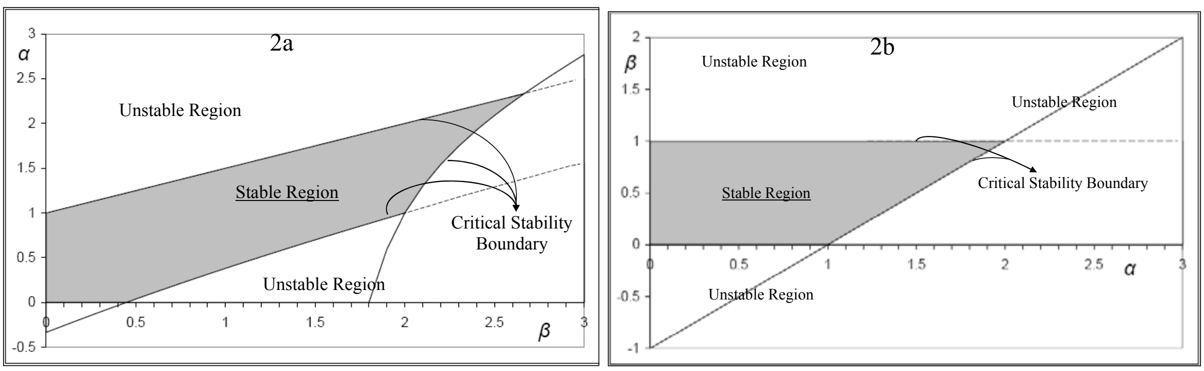

Figure 2: Stable and unstable regions for (a) sufficient inventory coverage, (b) insufficient inventory coverage

Now, it is important to present the dynamic behaviour of the original non-linear system in response to time-varying system input. Stability regions have been obtained for two distinct instances of system operations: (1) under sufficient inventory coverage and (2) under insufficient inventory coverage. However, the dynamic behaviour of the actual system often results in transition between one operational regime to the other, which is not captured in such separate analysis. It is of interest to validate the applicability of such separate analysis when the system switches from one operational regime to another i.e. *SHIP*= Min(*INV*, *SALES*). The system response (*PREL*) to a step input demand pattern for various settings of α and β si studied, the results of which are as shown in Table 2.

From Table 2, it is seen that the actual system clearly demarcates itself into stable and unstable region, without the presence of a critically stable region. Some of the most interesting results are obtained when (1) α=1 and β=1; (2) α=1 and β=1.182; (3) α=0 and β=0.445; (4) α=0.381 and β=1. The results for each are as shown in Figures 3a to 3d, where, *PREL_1* (thin solid line) corresponds production release quantity under sufficient inventory case, and *PREL* (thick solid line) corresponds production release in the actual system.

Table 2: Stability responses of the actual system versus the stability responses of the operational regimes

α	1	1	1	0	0.381	0.5	1.3	0	0.25
β	0.445	1	1.182	0.445	1	1.182	0.5	1	1.182
Sufficient Inventory	Stable	Stable	Stable	Critically Stable	Critically Stable	Critically Stable	Unstable	Unstable	Unstable
Insufficient Inventory	Stable	Critically Stable	Unstable	Stable	Critically Stable	Unstable	Stable	Critically Stable	Unstable
Actual System	Stable	Stable	Stable	Stable	Stable	Unstable	Unstable	Unstable	Unstable

These results provide with some of the following key insights:

- In the case of sufficient inventory, firms ordering only based on the end inventory levels ignoring the current WIP or supply line (α=0 and β>0) become unstable even if they fully account for the inventory levels (β=1).
- When operating under insufficient inventory, system is critically stable if it accounts entirely for the inventory and WIP levels (α=1 and β=1).
- Increased weightages can be given to the inventory levels (β>1) for an aggressive ordering policy in the actual system, as long as the chosen α and β satisfy the stability conditions in the sufficient inventory case.
- For the actual system, if the α and β are chosen such that they satisfy the stability conditions of insufficient inventory case, amplitude of variation in ordering can be reduced.

It is hence concluded that the proposed piecewise linearization based stability analysis approach is a valid approximation to the analysis of production and inventory ordering systems with non-linearities.

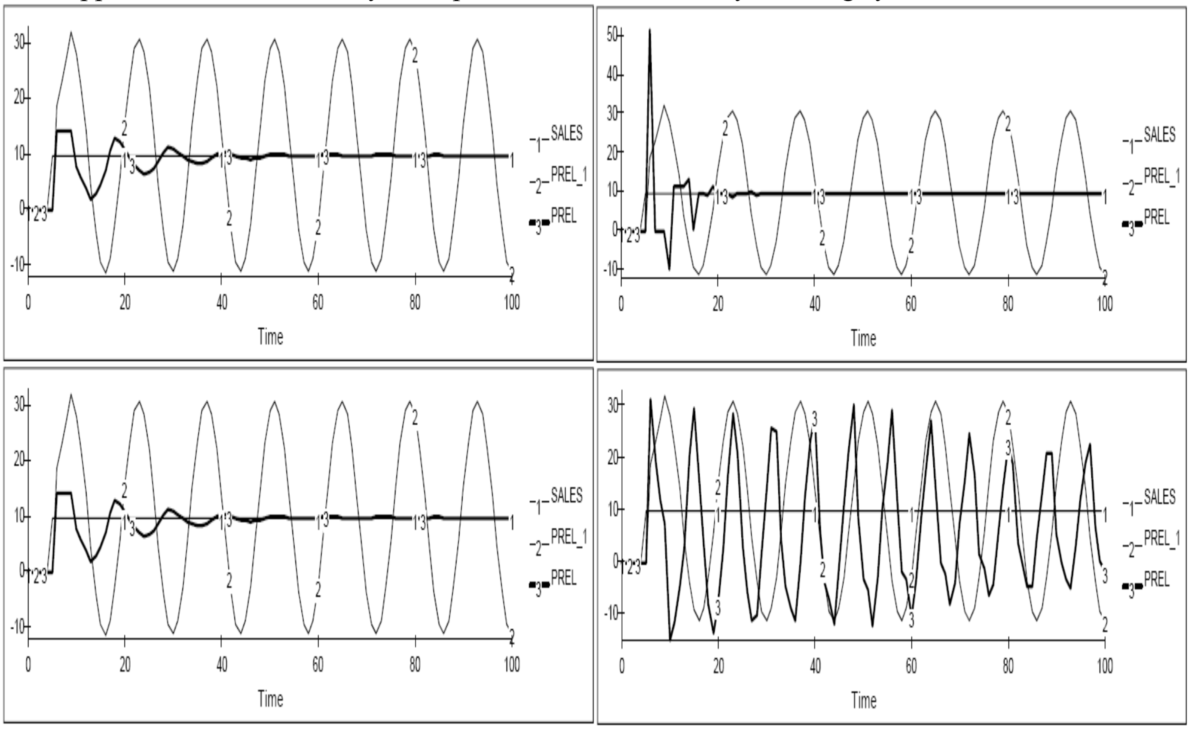

Figure 3: Production release rates for different settings of α and β

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