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Stationary, Finite horizon Bus schedules for optimal bunching, waiting

Abstract

In a bus transportation system the time gap between two successive buses is called headway. When the headways are small (high-frequency bus routes), any perturbation (e.g., in the number of passengers using the facility, traffic conditions, etc.) makes the system unstable, and the headway variance tends to increase along the route. Eventually, buses end up bunching, i.e, they start travelling together. Bus bunching results in an inefficient and unreliable bus service and is one of the critical problems faced by bus agencies. Another important aspect is the expected time that a typical passenger has to wait before the arrival of its bus. The bunching phenomenon might reduce if one increases the headway, however this can result in unacceptable waiting times for the passengers. We precisely study this inherent trade-off and derive a bus schedule optimal for a joint cost which is a convex combination of the two performance measures. We assume that the passengers arrive according to a fluid process, board at a fluid rate and using gate service, to derive the performance. We derive the stationary as well as the transient performance. Further using Monte-Carlo simulations, we demonstrate that the performance of the system with Poisson arrivals can be well approximated by that of the fluid model.

We make the following interesting observations regarding the optimal operating frequency of the buses. If the randomness in the traffic (variance in travel times) increases, it is optimal to reduce the bus frequency. More interestingly even with the increase in load (passenger arrival rates), it is optimal to reduce the bus frequency. This is true in the low load regimes, while for high loads it is optimal to increase the frequency with increase in load.

Static headway time leads to bunching when the randomness of the traffic and passenger arrival rate to the stops varies with the time. Towards this, we try to implement the model with dynamic headway time which control the bus frequency based on the state of the system. We studied the performance measures under dynamic headway. We derived the optimal dispatch policy, i.e., optimal depot headway time using dynamic programming to overcome the bus bunching and reduce the mean waiting time of passengers.

Keywords: Bus bunching, waiting time

1. Introduction

Public transport plays an important role in any system. We consider public transport systems like that of buses, trams, metros, local trains etc, for brevity we refer them as bus transport systems. In fact in the later cases, bunching (buses travelling together) is a major issue. Bus agencies desire to provide the best service to the passengers due to heavy competition from other transport services and would strive hard to reduce/eliminate the bunching possibilities.

In daily life, majority of people in a city rely on public transport and benefit from affordable service due to subsidized rates from the government. However, significant percentage of people are shifting towards private transport due to unreliable, inefficient service provided by the public systems. This is leading to undesirable issues like traffic congestion, pollution etc. To keep cities green, government and city planners need to enhance the efficiency and reliability of the public transport.

Typically, most popular bus routes have high frequency of buses and buses keep circulating around the route. It is well-known that such transit routes without any intervention or control are unstable ([5]). Any perturbation, typically, in the number of passengers arriving (demand) at the bus stops or in traffic conditions, can cause bus bunching. That is, two (or sometimes more) buses arrive at a bus stop simultaneously and start travelling together. Such a perturbation and hence bus bunching is inevitable and needs to be controlled using some intervention strategy.

Headway can be defined as the time gap between two consecutive buses. This headway is predetermined at the depot (where the buses start their journey) and hence is known. However the headways at the other bus-stops, encountered on the route, have random fluctuations as discussed. When a bus is delayed for a long period at a particular stop (due to larger demand and or larger transit times from previous stops), it has to cater to the increased number of passengers at the next stop. Hence it gets further delayed. Whereas, the following bus observes less passengers than anticipated, as the time gap between the buses is reduced. Hence it further speeds up due to lesser dwell times (the boarding and de-boarding time). Eventually, the headway becomes zero at a certain point along the route and both the buses travel together. This phenomenon is called 'Bus bunching'. Thus, the headway varies as the buses circulate along their path. As a result, passengers waiting at various bus stops experience large variance in their waiting times leading to an unreliable transport service. Further, this results in an inefficient usage of resources (as bunched buses run empty) even for bus operators. Thus efficient control of bunching of buses is important from the perspective of both the public using the service, as well as the operator.

Bus bunching is a critical issue faced by bus agencies and this problem has been thoroughly investigated over past few decades. However, it is still an active area of research as it is challenging to provide a generic, practical, and effective solution. Existing control strategies are based on ideas like skipping the forthcoming bus stops (e.g., [1, 4, 7]), limited boarding (e.g., [2, 3]), holding buses at specific locations (e.g., [1, 8, 2]) etc. Holding control applied at intermediate bus-stops and or skipping of stops may not be comfortable from the perspective of the passengers travelling in the bus. Hence our focus is on the holding control strategy only at depot. Further in papers like [2, 3, 8] etc., authors discuss and control the eventual variance of the error between the ideal schedule and the schedule considering random fluctuations, when the number of stops and buses converges to infinity. They assume no bunching. However in many scenarios it is not possible to completely avoid bunching. Further when the randomness is very high, it is almost impossible to adhere to the ideal schedules. In such scenarios, it is rather important to reduce the probability of bunching, and we precisely consider this probability. For such highly random scenarios, it is also important to consider the passenger waiting times. Further in all scenarios the number of stops and the number of trips is finite, hence such a modelling more realistic. In [1], authors consider finite number of buses looping continuously in the circular path and covering finite number of stops. However they work with expected value of the squared difference between the actual headway and the supposed headway, and not with the probability of bunching. Further the average passenger waiting times are defined as the expected value of the product of the headway and the number of passengers arrived during the headway. This definition does not consider the influence of passenger arrivals spread over the entire (bus inter-arrival) interval. We consider (customer) average of the waiting times of the passengers, the time gaps between their arrival epoch to the stop and the arrival epoch of the bus that they board. We derive theoretical expression for fluid arrivals, we also demonstrate through simulations that the derived expressions well approximate that corresponding to Poisson arrivals.

The main goal of this work is to derive optimal headway between the buses at depot, that minimizes a convex combination of two costs: a) the average passenger waiting times; and b) the probability of bunching. Using our results one can obtain a bus frequency optimal for stationary performance as well as the one that optimizes the performance for finite number of trips. Both the optimizers are the same once the number of trips is greater than the number of stops.

2. System Model

We consider buses moving on a single route and this route has M number of stops represented by $Q_1, Q_2, \ldots, Q_M$. Each stop has infinite waiting capacity. Any bus starts at the depot (Q_0) and travels along a predefined cyclic path, while boarding/de-boarding passengers at the encountered bus stops. Passengers arrive independently of others in each stop Q_i according to a fluid process with rate λ. We also consider Poisson arrivals and show that the fluid model can well approximate the system with Poisson arrivals, using simulations. The passengers board the bus at 'fluid' rate. That is, the time taken to board x number of passengers equals bx, where b is the boarding time per passenger at any stop. Let S_k^i be the time taken by k-th bus to travel from $(i-1)$- th stop Q_{i-1} to i-th stop Q_i. These random travel times, $\{S_k^i\}_k$ (for each stop Q_i), are independent and identically distributed (IID). The buses start at the depot after headway times (interval between two successive bus departures) and traverse through the M stops before concluding their trip.

The total number of trips equal T, where we assume $T > M$. The main purpose of this paper is to obtain the sequence of 'depot-headways' (one for each trip) optimally. We further make the following assumptions to study the problem:
R.1) Surplus number of buses: the next bus can start at any specified headway in the depot, without having to wait for the return of the previous bus.
R.2) Parallel boarding and de-boarding: and the time taken for de-boarding is smaller with probability one.
R.3) Gated service: Only the passengers that arrived before the arrival of the bus can board the bus. Passengers arrived during the boarding process, will wait for the next bus.
R.4) If buses are bunched at any stop then the second bus will wait till the previous bus departs, before it starts boarding its passengers. Thus overtaking of buses does not happen.
R.5) There is no constraint on the capacity of the buses.

In most of the cities the buses have two doors, boarding and de-boarding happens in parallel and hence one can neglect the de-boarding times. Boarding times are negligible compared to bus travel times. Thus we will have negligible number of arrivals during a boarding time, and hence gated service is a reasonable assumption. Assumption **R**.*1 simplifies the model sufficiently and is instrumental in deriving closed-form expressions for performance measures of this kind of a complicated system.* Without this assumption one would have to take care of the looping effects. Assumption of availability of few extra buses can easily ensure **R**.1 is satisfied. Even availability of one/two extra buses can ensure such a condition is satisfied with sufficiently large probability, and is often a well practised method to care of some eventualities. The systems usually operate with very small bunching probabilities, and so the event in assumption **R**.4 is a rare event. Further this is a common practice in many transportation systems like Trams, metros, local train etc and to a large extent even in bus transportation systems. The remaining assumptions are mentioned as and when required.

Bunching

Because of variability in demand and traffic conditions, some buses are delayed. They arrive later than their scheduled time at some stop. Delay of the bus results in more number of passengers to be boarded, and hence longer dwell time at the stop. Thus it gets delayed further for the next stops. This would also imply smaller dwell time for the following bus at the same stop, as it has to board lesser number of passengers. This can continue for the following bus-stops, the bus-stop headway times (time gap between two consequent buses at the same stop) become smaller and can eventually become zero. This phenomenon is called *Bus bunching.* We define bunching probability as the probability of occurrence of this event. *Bunching probability for the k-th trip at i-th stop, b_k^i, is defined as the probability that the dwell time of $(k-1)$-th bus at stop Q_i is greater than the inter arrival time between $(k-1)$ and k-th buses to the same stop.*

Waiting times

Passengers wait for the bus at every bus stop. When a bus is delayed their waiting time increases. If the delay results in bunching, the waiting times can be longer. Further the waiting times also depend upon the depot-headway time. The larger this headway is, the longer are the waiting times. We define the *(passenger) waiting times as the time difference between their arrival instance and the arrival instance of the bus in which they board.* Let $W_{n,k}^i$ be the waiting time of the n-th passenger that arrived to bus stop Q_i in the k-th trip and let X_k^i be the number of passengers that arrived to stop Q_i which board the k-th bus. Thus the number of passengers that availed service in the first T trips at stop Q_i equals $\sum_{k\le T} X_k^i$, and hence the customer average of these waiting times specific to stop Q_i over T trips equals:

$$\frac{\sum_{k\le T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k\le T} X_k^i}. \tag{1}$$

When the depot-headway increases bunching happens less often. However the passengers have to wait longer. We precisely study this trade-off. We derive the required performance measures, and, obtain headway times that optimize a convex combination of the two performance measures with an aim to obtain the 'Pareto frontier'.

3. System Dynamics and Trip-wise Performance

Dwell time is the amount of time spent by a bus in the stop. By **R**.2, it equals total boarding time of passengers waiting at the bus stop. Recall X_k^i is the number of passengers waiting at stop Q_i, at the arrival instance of k-th bus. Because of gated service and fluid boarding at rate b the dwell time equals:

$$V_k^i = X_k^i b. \tag{2}$$

By assumption **R**.4, the buses serve one after the other. That is, even in the event of bunching, the trailing bus starts boarding its customers only after the preceding one departs. In such events the dwell times would be bigger than $X_k^i b$. However transport systems typically operate with small bunching chances (typically less than 10%) and hence *we neglect the influence of these extra terms in the dwell times.*

3.1. Bus inter-arrival times

Let $\mathcal{N}^i(I)$ be the number of passengers that arrived in an interval of length I at Q_i for any $i \leq M$. For fluid arrivals this equals $\mathcal{N}^i(I) = \lambda I$, while, for Poisson arrivals $\mathcal{N}^i(I)$ is Poisson distributed with parameter λI.

We now describe the inter-arrival times of the buses at various stops and for various trips. We begin with stop Q_1 and trips other than the first trip. The k-th bus departs the depot after $(k-1)$-th bus, with a time gap equal to the (depot) headway time h_k, and it reaches stop Q_1 after further travelling for S_k^1 amount of time. Thus the inter arrival time between $(k-1)$-th bus and k-th bus at Q_1 equals,

$$I_k^1 = h_k + S_k^1 - S_{k-1}^1.$$

By Gated service, the number of passengers served by k-th bus at Q_1 exactly equals the number that arrived during this inter-arrival time, i.e., $X_k^1 = \mathcal{N}^1(I_k^1)$. Also, note that the passengers that arrived during the dwell time ($V_k^i = X_k^i b$) of k-th bus would be served by $(k+1)$-th bus.

It is clear that the k-th bus takes an amount of time given by

$$\sum_{1\leq j\leq i} S_k^j + \sum_{1<j<i} V_k^j,$$

to reach stop Q_i after its departure at depot, and the headway (time gap) between k-th and $(k-1)$-th buses at the depot equals h_k. Thus the inter-arrival time between k-th and $(k-1)$-th buses at stop Q_i,

$$I_k^i = h_k + \sum_{1\leq j\leq i} S_k^j + \sum_{1<j<i} V_k^j - \left(\sum_{1\leq j\leq i} S_{k-1}^j + \sum_{1<j<i} V_{k-1}^j \right). \tag{3}$$

Hence the number of passengers waiting at stop Q_i at the arrival instance of k-bus equals, $X_k^i = \mathcal{N}^i(I_k^i)$.

There are *random fluctuations in travel times based on traffic conditions. We model these fluctuations by Gaussian random variables.* To be precise we assume the travel time by k-th bus between i and $(i-1)$-th bus stop to be $S_k^i = s^i + N_k^i$, where $\{N_k^i\}_{i,k}$ are IID Gaussian random variables with mean zero and variance ϵ^2 and $\{s^i\}_i$ are constants. For fluid arrivals[1] and Gaussian travel times, the inter arrival times from equation (3) are (with $\rho := \lambda b$):

$$I_k^i \;=\; h_k + \sum_{1\leq j\leq i} N_k^j + \sum_{1\leq j<i} V_k^j - \sum_{1\leq j\leq i} N_{k-1}^j - \sum_{1<j<i} V_{k-1}^j \;=\; h_k + \sum_{1\leq j\leq i} N_k^j + \rho \sum_{1\leq j<i} \left(I_k^j - I_{k-1}^j \right) - \sum_{1\leq j\leq i} N_{k-1}^j. \tag{4}$$

In the above, by notation, we set $N_k^i = 0$ for all $k \leq 0$.

The above model considers independent travel times (between any fixed pair of stops) across various trips. In section 7 we consider analysis with correlated travel times.

[1]This is approximately true even for Poisson arrivals, as the inter bus-arrival times (at the same stop) are usually large, and then the number of passengers that arrived during this inter bus-arrival time can be approximated by λ times the inter-arrival time by elementary renewal theorem.

First trip details:

These details are different from the other trips and we describe the same now. Assume the system starts at time 0. The passenger arrivals start at time 0 as well as the first bus leaves the depot at time 0. Inter-arrival times $\{I_k^i\}$ were defined to facilitate computing the number of passengers boarding at respective stops and in respective trips. For the first trip, *the equivalent of inter-arrival times is the arrival instance of the first bus at respective stops. This is appropriate because the first bus has to serve all the waiting passengers, that have arrived since the system started.* In other words the first bus serves, at any stop, those passengers that have arrived between time 0 and the arrival instance of the first bus at that stop. Thus we define $\{I_1^i\}$ (for simplicity the same notation used) as actually the arrival time w.r.t 0 at different stops, as below:

$$I_1^i \quad = \quad h_1 + \sum_{1\le j\le i}\left(N_1^j + s^{(j)}\right) + \rho \sum_{1\le j<i} I_1^j, \tag{5}$$

where by definition of the stating process *we set $h_1 = 0$ and $N_k^i = 0$ for all $k \le 0$ and for any $i \le M$*. We would like to stress again that these are not not inter-arrival times, but are the arrival instances corresponding to the first trip.

Thus *even with independent travel times (and passenger arrivals), the bus inter-arrival times and hence the dwell times are correlated for all trips.* One needs to study these correlations to obtain the performance, and we begin with the following:

Lemma 1. *Assume $N_k^i = 0$ when $k \le 0$ and $h_1 = 0$. The inter-arrival times (or arrival times for first trip) can be expressed in terms of the relevant Gaussian walking components $\{N_k^i\}$ and the fixed inter-stop distances $\{s^j\}$ as below (for any trip k and for any stop i):*

$$I_k^i \quad = \quad \sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i\ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} + \sum_{r=0}^{i-1}\left[(1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{k-l}^{i-r}\right], \text{ where} \tag{6}$$

$$\gamma_l^i \quad \triangleq \quad (-1)^l \binom{i-1}{l} \rho^l (1+\rho)^{i-1-l}, \tag{7}$$

$$\mu_l^r \quad \triangleq \quad (-1)^l \sum_{u=0}^{r} \binom{r}{u+l-1}\binom{u+l}{l} \rho^{u+l-1}, \text{ with } \binom{n}{r} := 0 \text{ when } n < r. \tag{8}$$

Thus the mean and variance of inter arrival time for any stop i ($1 \le i \le M$) and for any trip k are given by,

$$E[I_k^i] \;=\; \sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i\ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \text{ and } \left(\varrho_k^i\right)^2 := E\left(I_k^i - E[I_k^i]\right)^2 \;=\; \epsilon^2 \sum_{r=0}^{i-1}\left[(1+\rho)^{2r} + \sum_{l=1}^{\min\{r+1,k-1\}} \left(\mu_l^r\right)^2\right]. \tag{9}$$

Proof is in Appendix A. ■

3.2. Passenger waiting times

As already mentioned, waiting times are the times for which a typical passenger waits, before its bus arrives. We first discuss the trip-wise passenger waiting times. Towards this we gather together the waiting times of the passengers that arrived during one (bus) inter-arrival time. Let the sum of the waiting times of the passengers that arrived during this trip and their customer average respectively be (notations as used in (1)):

$$\bar{W}_k^i \triangleq \sum_{n=1}^{X_k^i} W_{n,k}^i \quad \text{and} \quad \bar{w}_k^i \triangleq \frac{\sum_{n=1}^{X_k^i} W_{n,k}^i}{X_k^i}.$$

Fluid arrivals: The customers are assumed to arrive at regular intervals (of length $1/\lambda$) with large λ. The waiting time of the first passenger during bus inter-arrival period (I_k^i) is approximately[2] I_k^i, that of the second passenger is

[2]The residual passenger inter-arrival times at bus-arrival epochs get negligible as $\lambda \to \infty$.

approximately $I_k^i - 1/\lambda$ and so on. Further, if the time duration I_k^i is large in comparison with $1/\lambda$ then $X_k^i \approx \lambda I_k^i$. Thus as $\lambda \to \infty$, the following (observe it is a Riemann sum) converges:

$$\frac{\bar{W}_k^i}{\lambda} = \frac{1}{\lambda}\sum_{n=0}^{\lambda I_k^i}\left(I_k^i - \frac{n}{\lambda}\right) \to \int_0^{I_k^i}(I_k^i - x)dx = \frac{(I_k^i)^2}{2}. \tag{10}$$

Thus for large λ,

$$X_k^i = \mathcal{N}^i(I_k^i) \approx \lambda I_k^i, \quad \bar{W}_k^i \approx \lambda\frac{(I_k^i)^2}{2} \quad \text{and} \quad \bar{w}_k^i \approx \frac{I_k^i}{2}, \tag{11}$$

and we *use this approximation throughout as the fluid approximation.*

Poisson arrivals: For Poisson arrivals, due to memoryless property, the conditional expectation as given by Lemma 8 of Appendix,

$$E\left[\bar{W}_k^i \middle| I_k^i\right] = \lambda\frac{\left(I_k^i\right)^2}{2} \text{ and } E\left[\bar{w}_k^i \middle| I_k^i\right] = \frac{I_k^i}{2}. \tag{12}$$

In all (for Poisson as well as Fluid arrivals), the expected values of the trip-wise quantities corresponding to waiting times of passengers of stop Q_i in the k-trip equal

$$E\left[\bar{W}_k^i\right] = \lambda\frac{E\left(I_k^i\right)^2}{2} \text{ and } E[\bar{w}_k^i] = \frac{E[I_k^i]}{2}. \tag{13}$$

The quantity $E[\bar{w}_k^i]$ is the expected value of the customer average of the passenger waiting times corresponding to trip k and stop i. Thus these performance measures are the same for fluid as well as Poisson arrivals, however the analysis of the inter-arrival times $\{I_k^i\}$ is available only for fluid arrivals.

Reaching stationarity: From Lemma 1, we observe that the variance of I_k^i and its expected value depend upon the trip number. The system is in transient behaviour for the first M trips and reaches a kind of (variance) stationarity after M trips: *for the first M trips the variance changes with trip, while the variance is the same for the rest.* When one considers constant/deterministic headway times (as required for stationary analysis) then the process becomes completely stationary after the first M trips. To be precise from (13) and Lemma 1, *the trip-wise waiting time performance measures are the same for all trips $k > M$, under constant headways.*

3.3. Bunching probability

Bunching is said to occur at a stop when two buses meet, i.e., when the headway time (time gap between buses) of consequent buses becomes zero at that stop. Bunching probability, b_k^i, of k-th bus at i-th stop is the probability that the dwell time V_{k-1}^i (equation (2)) of $(k-1)$-th bus is greater than the inter arrival time (equation (4)) between $(k-1)$ and k-th buses. Thus for fluid arrivals (by (11)):

$$b_k^i = P(\mathcal{N}^i(I_{k-1}^i)b > I_k^i) \;=\; P(I_k^i - \rho I_{k-1}^i < 0). \tag{14}$$

By first conditioning on (I_k^i, I_{k-1}^i) and simplifying, we get the same expression for bunching probability with Poisson arrivals[3]. The above expression is true because of assumption **R**.4. Thus we require the (marginal) distribution of $(I_k^i - \rho I_{k-1}^i)$ for computing the bunching probability at any stop Q_i and for any trip k:

Lemma 2. *Recall the constant from equations (7) and assume $N_k^i = 0$ when $k \le 0$, $h_1 = 0$. Then*

$$I_k^i - \rho I_{k-1}^i \;=\; \sum_{l=0}^{\min\{i,k-1\}}\tilde{\gamma}_l^i h_{k-l} + \varpi_k^i + \sum_{r=0}^{i-1}\left((1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-l}^{i-r}\right) - \rho\sum_{r=0}^{i-1}\left((1+\rho)^r N_{k-1}^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-1-l}^{i-r}\right) \tag{15}$$

[3]However, as already mentioned, further analysis is applicable only for fluid arrivals. Thus to extend the analysis to Poisson arrivals, one needs to first study the inter-arrival times $\{I_k^i\}$ given by (4) with Poisson arrivals, for which the dwell times do not satisfy $V_k^i = \rho I_k^i$.

where,

$$\tilde{\gamma}^i_l \;\triangleq\; \gamma^i_l - \rho\gamma^i_{l-1}\mathbb{1}_{l>0} \text{ and } \varpi^i_k = \begin{cases} \mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma^{i-j+1}_{k-1}s^{(j)} - \mathbb{1}_{i\geq k-1}\rho\sum_{j=1}^{i-k+2}\gamma^{i-j+1}_{k-2}s^{(j)} & \text{if } k \leq M \\ \rho\sum_{j=1}^{i-M+1}\gamma^{i-j+1}_{M-1}s^{(j)} & \text{if } k = M+1 \\ 0 & \text{if } k > M+1. \end{cases} \tag{16}$$

The mean and variance for $k \geq 2$ respectively given by,

$$E[I^i_k] - \rho E[I^i_{k-1}] \;=\; \sum_{l=0}^{\min\{i-1,k-1\}} \tilde{\gamma}^i_l h_{k-l} + \varpi^i_k \quad \text{and} \quad \left(\sigma^i_k\right)^2 = \epsilon^2 \sum_{r=0}^{i-1}\left\{(1+\rho)^{2r} + \sum_{l=1}^{\min\{r+2,k-1\}}\left(\tilde{\mu}^r_l\right)^2\right\},$$

where,

$$\tilde{\mu}^r_l \triangleq \mu^r_l - \rho\mu^r_{l-1}\mathbb{1}_{l>0}.$$

Proof: is direct from Lemma 1. ■

From equation (14) and Lemma 2, the bunching probability for any $2 \leq k \leq T$ and $1 \leq i \leq M$ is equals to,

$$b^i_k = 1 - \Phi_k\left(E[I^i_k] - \rho E[I^i_{k-1}]\right), \tag{17}$$

where Φ_k is the cdf of a normal random variable with mean 0 and variance $\left(\sigma^i_k\right)^2$:

$$\Phi_k(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\left(\sigma^i_k\right)^2}} \exp\left(\frac{-t^2}{2\left(\sigma^i_k\right)^2}\right) dt.$$

Once again the variances $\{\sigma^i_k\}_k$ are the same for trips $k > M+1$ (we need one more trip because of dependency on I^i_{k-1}) and hence observe that $\Phi_k = \Phi_M$ for all trips greater than, $k > M+1$. Once again under constant/stationary deterministic headways, the bunching probability is the same for all trips with $k > M+1$. We refer the analysis with constant headways as stationary analysis and the same is considered in section 4, while the analysis with general head-way policies (headways depend on trips) are referred to as transient analysis and the corresponding study is available from section 5 onwards.

4. Stationary Analysis (Trip/Customer averages)

When T, the total number of trips, is sufficiently large in comparison with M, the number of stops, the stationary analysis would be sufficient. For this case, it is sufficient to consider constant/stationary headway policies (i.e., $h_k = h$ for some $h < \infty$ and for all trips k). At the end of section 5 we will show (for the case with large number of trips T) that the optimal among stationary policies provides near optimal performance, even when optimized among non-stationary headway policies (see Figure 6 in section 6). Further, the stationary policies have independent importance: practically it is more convenient to implement stationary policies. In all, we consider analysis using stationary policies in this section.

Under such stationary policies, from (6) and (9) of Lemma 1 the inter-arrival times and their moments for trips

($k > M+1$), simplify as below:

$$
\begin{aligned}
I_k^i &= h + \sum_{r=0}^{i-1}\left[(1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-l}^{i-r}\right], \text{ with expected value,} \qquad (18)\\
E[I_k^i] &= h \text{ and variance, } \left(\varrho_k^i\right)^2 = \varrho_i^2 \text{ with } \varrho_i^2 := \epsilon^2 \sum_{r=0}^{i-1}\left[(1+\rho)^{2r} + \sum_{l=1}^{r+1}\left(\mu_l^r\right)^2\right] \text{and further,}\\
I_k^i - \rho I_{k-1}^i &= h(1-\rho) + \sum_{r=0}^{i-1}\left((1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-l}^{i-r}\right) - \rho\sum_{r=0}^{i-1}\left((1+\rho)^r N_{k-1}^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-1-l}^{i-r}\right)\\
E[I_k^i - \rho I_{k-1}^i] &= h(1-\rho) \text{ and } \left(\sigma_k^i\right)^2 = \left(\sigma^i\right)^2 \text{ with } \left(\sigma^i\right)^2 := \epsilon^2(1+\rho^2)\sum_{r=0}^{i-1}\left\{(1+\rho)^{2r} + \sum_{l=1}^{r+1}\left(\mu_l^r\right)^2\right\}.
\end{aligned}
$$

It is important to observe here that all the above moments do not depend upon the trip number for trips $k > M+1$. In other words, $\{I_k^i\}_{k>M+1}$ and $\{I_k^i - \rho I_{k-1}^i\}_{k>M+1}$ form stationary Gaussian sequence for any given stop Q_i. In fact $\{I_k^i\}_{k>M}$ itself is stationary, however it is convenient to consider common set of indices $\{k \geq M+1\}$.

Define the following Gaussian vectors, corresponding to trip k and trips k-l respectively as below,

$$\mathbb{N}_k := [N_k^1, N_k^2, \cdots, N_k^M] \text{ and } \mathbb{N}_l^k := [\mathbb{N}_k, \mathbb{N}_{k-1}, \cdots, \mathbb{N}_l]$$

The bus inter-arrival times $\{I_k^i\}_{k>M+1}$, for any given stop Q_i, are stationary but are not independent as seen from equation (18). Nevertheless, from the same equations, inter-arrival times of a trip, $\mathbb{I}_k := [I_k^1, I_k^2, \cdots, I_k^M]$, depend only upon $\mathbb{N}_{k-M}^k$ and so the sampled inter-arrival times

$$\{I_{j+kl}^i\}_{k\geq 1} = I_{l+j}^i, I_{2l+j}^i \cdots (\text{ for any stop } Q_i),$$

with $l > M+1$ and for any $0 \leq j \leq l-1$ form an IID Gaussian sequence.

Thus the *system is in transience only for the first $M+1$ trips, under stationary headway policies.* To be more precise, any expected performance measure related to a single trip is the same for all trips other than the first M trips. Passenger waiting times related to a trip can be one such example. On the other hand for performance measures like bunching probability, which depend upon two consecutive trips, the stationarity is reached after $M+1$ trips. Hence again the bunching probability of k-th trip equals stationary bunching probability, for all $k > M+1$.

One can also derive the time (trip) average of any performance measure, using the above 'block' IID characteristics. By Law of large numbers, for any (integrable) performance f that depends (for example) upon one trip (almost surely (a.s.)):

$$
\begin{aligned}
\bar{f} &:= \lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K} f(\mathbb{I}_k) \qquad (19)\\
&= \lim_{K\to\infty}\frac{1}{K(M+1)}\sum_{j=1}^{M+1}\sum_{k=1}^{K} f(\mathbb{I}_{j+k(M+1)}) \overset{a.s.}{=} E[f(\mathbb{I}_{M+1})].
\end{aligned}
$$

In the above the expectation is with respect to the Gaussian random variables of equation (18). One can derive trip average of the performance measures that depend upon finite number of consecutive trips (e.g., bunching probability) in a similar way.

We find the stationary optimal depot headway time and the non stationary case is discussed in the next section.

4.1. Customer average of Waiting times

This performance is important from the perspective of passengers and hence it is more appropriate to consider the 'passenger' average of the waiting times defined in (1) using the estimates of (13). We also consider the average defined using $\{\bar{w}_k^i\}$, given by equation (13), because of two reasons: a) it will be seen that it is relatively easier to handle the second averages in optimization problems; b) the two sets of optimization problems are approximately equivalent, as shown in the following (for Poisson as well as Fluid arrivals).

Lemma 3. *For any stop Q_i, under any stationary policy $\{h\}$:*

$$\bar{W}^i = \lim_{T\to\infty} \frac{\sum_{k=1}^T \bar{W}_k^i}{\sum_{k=1}^T X_k^i} = \lim_{T\to\infty} \frac{T}{\sum_{k=1}^T X_k^i} \frac{\sum_{k=1}^T \bar{W}_k^i}{T} \overset{a.s.}{=} \frac{E[(I_k^i)^2]}{2E[I_k^i]} = \frac{\left(\varrho_k^i\right)^2 + \left(E[I_k^i]\right)^2}{2E[I_k^i]} = \frac{\left(\varrho_k^i\right)^2}{2E[I_k^i]} + \frac{E[I_k^i]}{2} \text{ and thus} \quad (20)$$

$$\bar{W}^i = \frac{(\varrho_i)^2}{2h} + \frac{h}{2} \text{ and similarly,} \quad (21)$$

$$\bar{w}^i = \lim_{T\to\infty} \frac{1}{T} \sum_{k\le T} \bar{w}_k^i = \lim_{T\to\infty} \frac{1}{T} \sum_{k\le T} \frac{\bar{W}_k^i}{X_k^i} = \frac{E[I_k^i]}{2} = \frac{h}{2} \text{ a.s.} \quad (22)$$

Proof: is in Appendix A. ■

Under suitable conditions, for example when the depot headway time is large and/or the small traffic variability (ϵ), one can neglect the first term in the equation (21). Hence, $\bar{W}^i \approx \bar{w}^i = h/2$.

We would like to give equal importance to passengers of all stops. Hence we consider the following for optimization purposes:

$$\bar{w} = \sum_{i=1}^{M} \bar{w}^i = \sum_{i=1}^{M} \frac{E[I_k^i]}{2}. \quad (23)$$

4.2. *Bunching probability under stationarity*

From (14) we would require analysis of terms $\{I_k^i - \rho I_{k-1}^i\}$ and towards this we first consider the following lemma, which can directly be derived using Lemma 1:

As already discussed, various trips can be correlated, however the bunching probabilities in different trips remains the same. This is because the bunching probabilities depend only upon $I_k^i - \rho I_{k-1}^i$ and because these are identically distributed for all $k > M + 1$. For all such trips the bunching probability of a stop in a trip is the same and equals,

$$b_k^i = b^i \text{ with } b^i := 1 - \Phi_M\left(h(1-\rho)\right). \quad (24)$$

As mentioned already, this also represents *the bunching probability of a trip under stationarity.* Note that the bunching probabilities of initial trips can be different, and these can be computed in a similar way if required.

4.3. *Total cost and optimization*

Our aim is to minimize a joint cost that considers both the factors (23) and (17). Towards this we consider a weighted average of the two costs with α and $\{\beta^i\}_i$ representing the weights for various components as below:

$$r = \sum_{i=1}^{M} \left\{ \frac{h}{2} + \alpha\beta^i \left(1 - \Phi_M\left(h(1-\rho)\right)\right) \right\}. \quad (25)$$

Here α is the trade-off parameter between bunching probability and waiting times, while $\{\beta^i\}_i$ determine the trade-off for bunching probabilities of various stops[4]. Let h^* be the minimizer for total cost (25). The total cost is a differentiable function and hence h^* is the zero of the following derivative (obtained using Leibniz rule):

$$\frac{dr(h)}{dh} = \frac{M}{2} - \sum_{i=1}^{M} \left(\frac{\alpha\beta^i}{\sqrt{2\pi}} \exp\left(\frac{-h^2(1-\rho)^2}{2\left(\sigma^i\right)^2} \right) \frac{1-\rho}{\sigma^i} \right). \quad (26)$$

Also, one can easily verify that $d^2r(h)/dh^2 > 0$ for all h and $\alpha > M\sqrt{2\pi}\sigma^M/2(1-\rho)$, hence that h^* *is the unique minimizer.*

[4]Note that the bunching probability is low at initial stops and increases with stop number (see Lemma 2) and hence the need for different $\{\beta^i\}_i$.

We consider the special case with $\beta^M = 1$ and $\beta^m = 0$ for all $m \neq M$ to obtain a good approximation for the above optimizer which is equals to,

$$h_s^* \approx \sqrt{\frac{-\log(C_1)}{C_2}},\ C_1 = \frac{M\sqrt{2\pi}\sigma^M}{2\alpha(1-\rho)},\ C_2 = \frac{(1-\rho)^2}{2\,(\sigma^M)^2}. \tag{27}$$

We use sub-script h_s^* to indicate that this is the 'stationary optimal policy'.
Remarks: It is immediately clear that the optimal bus frequency (inverse of h_s^*) decreases as the number of stops increase. This is in fact true even for the general case, as seen from (26). Similarly, the optimal frequency decreases with increase in traffic variability factor ϵ (see (27)). We compute the optimizers for remaining cases using numerical computations in the next section to derive some more interesting inferences.

4.4. Simulations

In this section, we verify the derived performance measures of the proposed model through Monte-Carlo simulations. We emulate the buses travelling on a single route with 8 bus stops, boarding a random number of passengers using gated service and avoiding parallel boarding (into two or more buses simultaneously) of passengers at any stop. The travelling times between stops are perturbed by normally distributed noise with zero mean and variance ϵ^2. Passenger arrivals are either due to fluid arrivals or due to Poisson arrivals, and we consider fluid boarding. We conduct the simulations with $M = 8, \lambda = 200,\ \rho = 0.3, s = 50$ and $\alpha = 150$ in Figure 1.

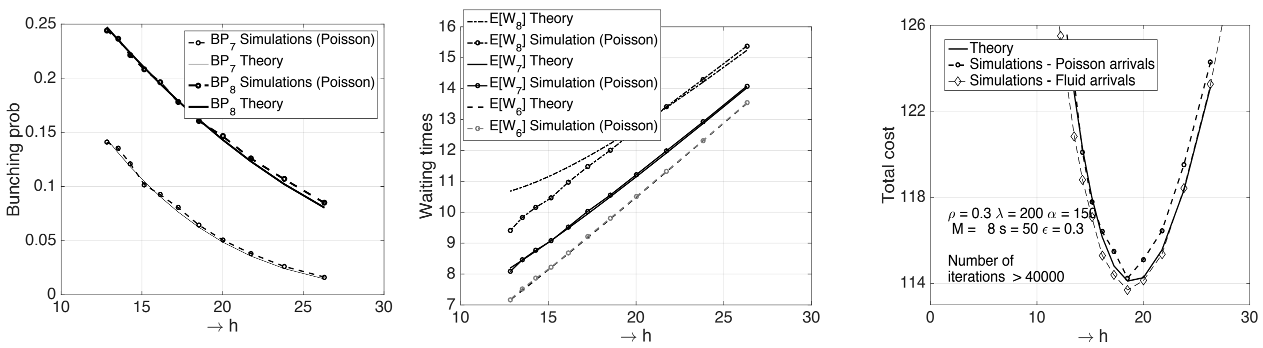

Figure 1. Bunching probability, Waiting time and Total cost vs headway

In Figure 1, we compared the theoretical quantities with the ones estimated through simulations. We plot bunching probability, average passenger waiting times and total cost respectively as a function of headway. We find a good match between theory (curves without markers) and simulations considering Poisson arrivals (curves with circular markers). We also conducted simulations using fluid arrivals. The simulation results with fluid arrivals better match the theoretical counterparts. We included the simulation based results for fluid arrivals (curves with diamond markers) in the third sub-figure, i.e., for total cost. We notice a good match between (both) the simulated quantities and the theoretical expressions in majority of the cases. These observations affirm the theory derived.

From the sub-figures of Figure 1, we observe that the bunching probability improves (decreases) with depot-headway (h), while, the passenger waiting times degrade (increase). This is the *inherent trade-off that needs to be considered to design an efficient system*. We plot optimal depot-headways for some examples in Figures 2-3, which are estimated using numerical simulations (dashed curves with circular markers). We also plot the approximate h_s^* given by (27) in the same figures (solid curves). We notice that the approximate optimizer well matches the ones estimated using numerical simulations, for small load factors (ρ) and or small traffic variability (ϵ). When variability increases the first factor in waiting times (23) becomes significant and then the theoretical h_s^* (27) is no more a good approximation (for load factors bigger than 0.5 in Figure 2 and traffic variance $\epsilon > 5$ in Figure 3).

As the load factor increases, or equivalently when the customer arrival rates increase, one would anticipate an increased bus-frequency to be optimal. On the contrary we notice that the optimal headway increases (initially) with increase in load-factor in Figure 2. This is because the passenger waiting times for any given headway h, approximately equal $h/2$ when the variability components $\left(\{\varrho_k^i\}_i\right)$ due to traffic and or load conditions) are negligible (see equation 23). Thus with increase in load factor, the bunching probabilities increase sharply, while waiting times are less influenced and hence an increase in optimal depot-headway. However as seen in the same figure, when load increases beyond 0.5, the variability components in waiting times also become significant and now we notice that the optimal depot-headways are smaller. *To summarize, the optimal frequency of the buses decreases initially with increase in load and for higher range of load factors it increases with load.*

On the other hand, with increase in traffic variability factor ϵ, we notice that the optimal depot-headway always increases (see Figure 3). *Thus it is optimal to decrease the frequency of buses, when traffic variability increases.* These results are true as long as the buses have sufficient capacity (to board all the customers).

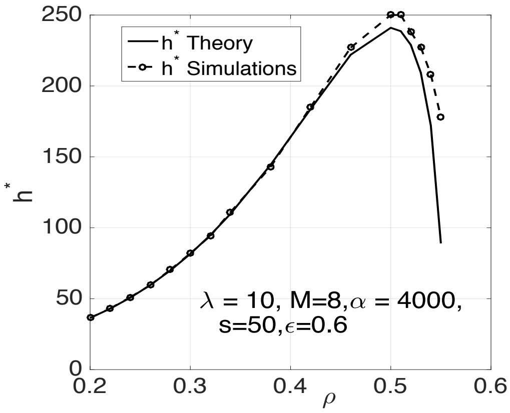

Figure 2. Optimal h versus load

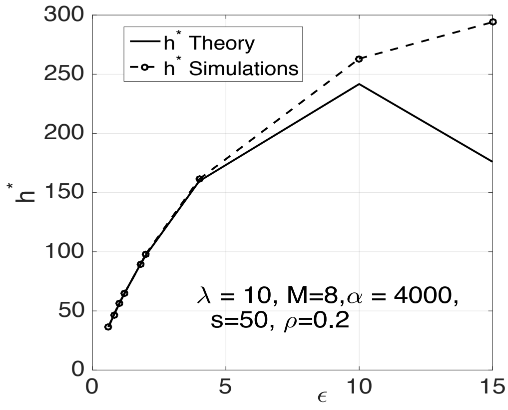

Figure 3. h^* versus traffic variability

5. Finite Horizon (Transient) Problem

As opposed to the case study in the previous section, we now consider the case when the number of trips T is not very large in comparison with M, the number of stops. As discussed previously, even with stationary policies, system is in transience behaviour for the first $M+1$ trips and stationary performance is reached only after $M+1$ trips. Further it may not be sufficient to consider stationary policies, when T is comparable with M. Thus we consider headway policies that vary with time, however these do not depend upon (any) system state[5]. In all, we have time varying policies $\phi := \{h_k\}_{1\le k\le T}$ with h_k representing the depot headway between trip $(k-1)$ and k, and, we seek the optimal among such policies that minimizes a total cost, similar to the one defined in the previous section.

We require optimal policies among non-stationary (but state independent) policies, thus we pose it as an finite horizon (T) sequential control problem. We begin with derivation of the performance measures under these non-stationary (and finite horizon) policies. That is, we define the running cost corresponding to each trip and eventually the cost corresponding to all the trips.

[5]Generalization to state/history dependent policies is out scope of this work, and our aim is to consider such policies in future.

5.1. Passenger waiting times

The customer average (and the corresponding expected value) of the waiting times of the passengers belonging to Q_i and corresponding to the T trips equals:

$$\frac{\sum_{k=1}^{T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k=1}^{T} X_k^i} \text{ and } E^{\phi}\left[\frac{\sum_{k=1}^{T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k=1}^{T} X_k^i}\right],$$

where E^{ϕ} is the expectation with respect to probability measure P^{ϕ} that is resultant when policy ϕ is used. In view of the discussions following Lemma 3, we consider the following modified (approximate) average which also makes it mathematically tractable:

$$\frac{E^{\phi}\left[\sum_{k=1}^{T}\left(\frac{\sum_{n=1}^{X_k^i} W_{n,k}^i}{X_k^i}\right)\right]}{T} = \frac{1}{T}\sum_{k=1}^{T} E^{\phi}\left[\bar{w}_k^i\right].$$

Thus the waiting time-component of the running cost corresponding to trip k (corresponding to all stops) equals, $\sum_{i=1}^{M} E\left[\bar{w}_k^i\right]$. From Lemma 1 and equation (13), this component equals,

$$\sum_{i=1}^{M} E^{\phi}\left[\bar{w}_k^i\right] = \sum_{i=1}^{M}\frac{1}{2}E^{\phi}\left[I_k^i\right] = \sum_{i=1}^{M}\frac{1}{2}\left(\sum_{l=0}^{\min\{i-1,k-1\}}\gamma_l^i h_{k-l} + \mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right). \tag{28}$$

5.2. Bunching probability

The other component of running cost, under any given policy $\phi := \{h_k\}_{1\leq k\leq T}$, is related to bunching. The average bunching cost across all the T trips at stop Q_i can be defined as the expected value of the fraction of trips that resulted in bunching at stop Q_i, i.e., as the following:

$$E^{\phi}\left[\frac{\sum_{k=1}^{T}\mathbb{1}_{V_{k-1}^i > I_k^i}}{T}\right] = \frac{\sum_{k=1}^{T} P^{\phi}\left(I_k^i - \rho I_{k-1}^i < 0\right)}{T}.$$

Thus the bunching component of the running cost corresponding to trip k (and stop Q_i) equals the corresponding bunching probability, $P\left(I_k^i - \rho I_{k-1}^i < 0\right)$ (see (14)). From Lemma 2 and equation (17), this component equals,

$$P^{\phi}\left(I_k^i - \rho I_{k-1}^i < 0\right) \quad = \quad 1 - \Phi_k\left(\sum_{l=0}^{\min\{i,k-1\}}\tilde{\gamma}_l^i h_{k-l} + \varpi_k^i\right). \tag{29}$$

As discussed in the previous section, the chances of bunching at initial stops is low, while that at later stops is significant. Thus the running cost component corresponding to bunching, for trip k is given by:

$$1 - \Phi_k\left(\sum_{l=0}^{\min\{M,k-1\}}\tilde{\gamma}_l^M h_{k-l} + \varpi_k^M\right). \tag{30}$$

5.3. Pareto frontier and overall running cost

We thus have multiple objective functions, which are time averages of the bunching probability of last stop,

$$P_B(\phi) := \frac{1}{T}\sum_{k=1}^{T} P^{\phi}\left(I_k^M - \rho I_{k-1}^M < 0\right) \text{ and passenger average waiting times at all stops } E_W(\phi) := \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M} E^{\phi}\left[\bar{w}_k^i\right]$$

We are naturally interested in the Pareto frontier, the space of all 'efficient' points. Pareto frontier is the efficient sub-region of any achievable region which consists of dominating performance vectors. In our context we say

$(P_B(\phi^*), E^*_W(\phi^*))$, corresponding to policy ϕ^*, is a dominating pair of performance measures, if there exists no other policy ϕ which achieves a strict performance pair, i.e., such that

$$P_B(\phi) < P_B(\phi^*), \text{ as well as } E_W(\phi) < E_W(\phi^*).$$

One of the common technique to obtain Pareto frontier is to solve a convex combination of the two costs. Thus we consider the optimization of the following convex combination, parametrized by α (from equations (28)-(30)):

$$\begin{aligned} P_B(\phi) + \alpha E_W(\phi) &= \frac{1}{T}\sum_{k=1}^{T} r_k \text{ with} \\ r_k &:= \sum_{i=1}^{M}\left\{\frac{1}{2}\left(\sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right) + \alpha\left(1-\Phi_k\left(\sum_{l=0}^{\min\{M,k-1\}}\tilde{\gamma}_l^M h_{k-l} + \varpi_k^M\right)\right)\right\}, \end{aligned}$$

where r_k represents the overall running cost of trip k. This running cost depends upon the headway policy $\phi = \{h_k\}_{k\leq T}$ and can be re-written as,

$$\begin{aligned} r_k(\phi;\alpha) &= \sum_{i=1}^{M}\frac{1}{2}\left(\sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right) + \alpha\left(1-\Phi_k\left(\sum_{l=0}^{\min\{M,k-1\}}\tilde{\gamma}_l^M h_{k-l} + \varpi_k^M\right)\right) \\ &= \sum_{l=0}^{\min\{k,M\}-1}\bar{\gamma}_l h_{k-l} + \vartheta_k^M + \alpha\left(1-\Phi_k\left(\sum_{l=0}^{\min\{k-1,M\}}\tilde{\gamma}_l^M h_{k-l} + \varpi_k^M\right)\right) \text{ with,} \qquad (31) \\ \bar{\gamma}_l &:= \frac{1}{2}\sum_{i=l+1}^{M}\gamma_l^i \quad \text{and} \qquad \vartheta_k^M := \sum_{i=1}^{M}\frac{1}{2}\left(\mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right). \qquad (32) \end{aligned}$$

Our objective is to equivalently optimize the summation of running costs (as T is a fixed number) corresponding to all trips in consideration (recall h_1 is set to 0, as the first bus starts immediately by convention):

$$\min_{\phi=\{h_2,\cdots,h_T\}} \sum_{k=1}^{T} r_k(\phi;\alpha). \qquad (33)$$

This is like the well known sequential control problem and the most convenient solution concept for this is the Dynamic Programming equations. We follow the same approach to solve this problem.

5.4. Dynamic Programming equations, solved by backward induction

We are interested to find the optimal policy i.e., the optimal headway times between the buses at the depot for all trips considered, which optimizes (33). As already mentioned, the optimal policy can be obtained by solving Dynamic Programming (DP) using backward induction. Towards this we discuss the necessary ingredients, like running cost, transition probabilities etc.

From (31), the running cost of any trip depends (at maximum) only upon the headways of the last M trips, i.e., for trip k it depends at maximum upon $\mathbf{h}_k := [h_{k-M}, \cdots h_k]$ for $k > M$ and $\mathbf{h}_k := [0, \cdots, 0, h_1, h_2, \cdots, h_k]$ when $k \leq M$ (required number of zeros are inserted to make them same length vectors). Note that $\mathbf{h}_1 = [0, \cdots, 0, h_1] = [0, \cdots, 0, 0]$ (as h_1 set to 0 without loss of generality), $\mathbf{h_2} = [0, \cdots, 0, h_2]$ and $\mathbf{h_3} = [0, \cdots, 0, h_2, h_3]$ and so on. The dynamic programming equations, for any $k \leq T$ are given by ([6]):

$$v_k(\mathbf{h}_{k-1}) = \min_{h_k}\Big(r_k(\mathbf{h}_k) + v_{k+1}(\mathbf{h}_k)\Big) \text{ and } v_{T+1}(\mathbf{h}_{T+1}) = 0. \qquad (34)$$

In the above $v_k(\mathbf{h}_{k-1})$ represents the optimal cost from trip k till the last trip (T), when the previous trip depot-headways are given by $\mathbf{h}_{k-1}$. The running/trip-wise costs are given by(31) and hence these equations can be rewritten as (for any $k \leq M+1$):

$$v_k(\mathbf{h}_{k-1}) = \min_{h_k}\left\{\sum_{l=0}^{k-1} h_{k-l}\bar{\gamma}_l + \vartheta_k^M + \alpha\left(1-\Phi_k\left(\sum_{l=0}^{k-1}\tilde{\gamma}_l^M h_{k-l} + \varpi_k^M\right)\right) + v_{k+1}(\mathbf{h}_k)\right\}. \qquad (35)$$

We solved these optimality equations for a special case (mentioned in the hypothesis) and derived the optimal policy:

Theorem 4. *Assume $T > M + 1$. Define the following constants backward recursively for all $0 \leq k < T - 1$: first set $\delta^{T+1} = 0$, $\eta_l^{T+1} = 0$ for all $0 \leq l \leq M$ and then set*

$$\begin{aligned} \eta_l^{T-k} &= \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1} \frac{\tilde{\gamma}_{l+1}^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_l, \qquad a_{T-k} = \sigma_{T-k}^M \sqrt{-2 \log \left(\frac{\eta_0^{T-k} \sqrt{2\pi} \sigma_{T-k}^M}{\tilde{\gamma}_0^M \alpha} \right)}, \\ \delta^{T-k} &= \alpha \left(1 - \Phi_k \left(a_{T-k}\right)\right) + \vartheta_k^M + \frac{\eta_0^{T-k}(a_{T-k} - \varpi_{T-k}^M)}{\tilde{\gamma}_0^M} + \delta^{T-k+1}. \end{aligned} \tag{36}$$

If for any trip k, the following condition is satisfied

$$\alpha > e^{\left(\sum_{l=1}^M h_{T-k-l} \tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j} \sqrt{2\pi} \sigma_{T-j}^M}{\tilde{\gamma}_0^M} \text{ for all } 0 \leq j < k \text{ and } \eta_0^{T-k} > 0, \tag{37}$$

then the optimal policy and the value function are respectively given by:

$$h_{T-k}^*(\mathbf{h}_{T-k-1}) = \frac{1}{\tilde{\gamma}_0^M} \left[-\sum_{l=1}^M h_{T-k-l} \tilde{\gamma}_l^M - \varpi_{T-k}^M + a_{T-k} \right], \text{ and} \tag{38}$$

$$v_{T-k}^*(\mathbf{h}_{T-k-1}) = \sum_{l=1}^M h_{T-k-l} \left(\frac{\tilde{\gamma}_0^M \eta_l^{T-k} - \eta_0^{T-k} \tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \right) + \delta^{T-k}. \tag{39}$$

Proof: is in Appendix A. ■

From (38), the optimal policy (once the hypothesis is satisfied) is linear in previous trip depot-headways and depends only on values of previous M trips. The coefficients of this linear dependence $\{\tilde{\gamma}_l^M / \tilde{\gamma}_0^M\}_l$ are the same for all trips. The affine component ϖ_{T-k}^M disappears for later trips (i.e., for trips $k > M + 1$). Interestingly the only component that depends upon the trip number are the constants $\{a_{T-k}\}_k$, which is the main element of non-stationarity in the optimal policy.

Low load factors

We will now prove that the hypothesis of the above theorem are satisfied for low load factors, under some simple assumptions. Typical operating conditions work under such low load factors. We obtained the optimal depot headway policy (see (38)) under the condition (37). We have the following lemma to prove that the condition (37) is satisfied under low load factors.

Lemma 5. *Assume $\alpha > \epsilon M \sqrt{2\pi}/2$. Then there exists an upper bound, $1 \geq \bar{\rho}_\eta > 0$, such that condition (37) is satisfied (for all k) for all load factors $\rho \leq \bar{\rho}_\eta$. For all such ρ the optimal policy is completely defined by Theorem 4, i.e., by equations (38)-(39) when we start with $h_1^* = 0$.*

Proof: is in Appendix A. ■

5.5. Algorithms

The Lemma 5 guarantees the applicability of Theorem 4, for low load factors which are more relevant for practical scenarios. And then, one can obtain the optimal policy using the algorithm defined in Algorithm 1, constructed using Theorem 4.

Using DP equations directly :

For general conditions, one needs to solve the DP equations (35) using well known numerical techniques (see e.g., [6]). We will need to consider discrete choices for the headways.

Algorithm 1: Defined using Theorem 4

1. Input parameters: M (number of stops), T (number of trips), ρ (load factor), α (trade-off factor) , ϵ (traffic variability) and $\{s^i\}$ (inter-stop walking times)
2. Compute coefficients $\gamma_k^i, \tilde{\gamma}_k^i, \bar{\gamma}_k^i, \varpi_k^i$ for all k, i, using equations (7), (16) and (32).
3. Compute iterative coefficients (backward recursion)
   - Initial values: Set $\eta_l^{T+1} = 0$ for each l
   - for $k = 0$ to T-1
     - Compute η_l^{T-k} for each l, using (36) of Theorem 4
     - Compute a_{T-k} using η_0^{T-k} and the other coefficients
4. Computation of Optimal policy (forward manner)
   - Set initial trip bus headways to zero, i.e., $h_1 = 0$. By notation set even $h_l = 0$ for all $l \leq 0$.
   - for $k = 2$ to T-1
     - Compute trip k headway, h_k from (38), using the previous trip headways $\mathbf{h}_{k-1}$ and coefficients calculated above

5.6. *Asymptotic ($T \to \infty$) analysis and Simplified algorithm*

From Algorithm 1, it is a complicated procedure to compute the trip-wise-constants, $\{\eta_0^t\}_t$ and $\{a_t\}_t$. One can have a much simplified (approximate) algorithm if there is a possibility to (well) approximate these by appropriate constants. We attempt this using large trips ($T \to \infty$) approximation: compute the required coefficients under this limit. We verify the accuracy of this approximation, later using numerical examples.

Recall from Theorem 4,

$$\eta^{T-k} = \bar{\gamma}_l + \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \text{ for all } l \geq 1. \tag{40}$$

Summing over $l = 0 \cdots M$ we get (from (32))

$$\sum_{l=0}^{M}\eta_l^{T-k} = \sum_{l=0}^{M}\bar{\gamma}_l + \sum_{l=0}^{M}\eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{\sum_{l=0}^{M}\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} = \frac{M}{2} + \sum_{l=0}^{M}\eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{1-\rho}{(1+\rho)^{M-1}}, \tag{41}$$

and the above is true because, by a simple exchange of the indices of the two summations involved one can prove the following:

$$\begin{aligned}\sum_{l=0}^{i-1}\gamma_l^i &= 1, \sum_{l=0}^{M}\bar{\gamma}_l = \sum_{l=0}^{M}\sum_{i=l+1}^{M}\gamma_l^i = \sum_{i=1}^{M}\sum_{l=0}^{i-1}\gamma_l^i = \sum_{i=1}^{M}1/2 = M/2 \text{ and}\\ \sum_{l=0}^{M}\tilde{\gamma}_l^M &= \sum_{l=0}^{M}\left(\gamma_l^M - \rho\gamma_{l-1}^M\mathbb{1}_{l>0}\right) = \sum_{l=0}^{M-1}(1-\rho)\gamma_l^M = 1-\rho.\end{aligned}$$

If there exists a limit $\eta_l^{T-k} \to \eta_l^*$ (for each $0 \leq l \leq M$) as T converges to ∞, then $\sum_{l=0}^{M}\eta_{l+1}^{T-k+1} - \sum_{l=0}^{M}\eta_{l+1}^{T-k}$ converges to 0 and then the limit η_0^* is given by the following (see equation (41)):

$$\eta_0^* = \frac{M(1+\rho)^{M-1}}{2(1-\rho)}, \text{ and subsequently,} \tag{42}$$

$$a_* = \sigma_M^M\sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma_M^M}{2(1-\rho)\alpha}\right)}. \tag{43}$$

Substituting this limit in (40), by using induction, one can easily show that the limits η_l^T for each l exists. Thus we have the limit for η_0^T, which is given by the above equation. One can propose an approximate algorithm using this limit, which is much more simplified than Algorithm 1. The approximate algorithm is provided in 2.

Algorithm 2: Using approximate iterative co-efficients

1. Input parameters: $M, T, \rho, \alpha, \sigma_M^M$
2. Approximate iterative coefficients $\eta_0^{T-j} \approx \eta_0^*$ and $a^{T-j} \approx a_*$ for all j, using η_0^*, a_* defined in (42)-(43)
3. Computation of Optimal policy (forward manner)
   - Set initial trip bus headways to zero, i.e., $h_1 = 0$. By notation set even $h_l = 0$ for all $l \leq 0$.
   - for $k = 2$ to T-1
     - Compute trip k headway, h_k from (38), using the previous trip headways $\mathbf{h}_{k-1}$ and coefficients calculated above

5.6.1. Comparison with Stationary analysis

In section 4 we considered optimal stationary policy, wherein the same depot-headway is applicable for all trips. This policy is optimal when the number of trips is considerable (infinity), the system reaches stationarity, and hence the system performance is close to the stationary performance. That is, the performance with huge number of trips does not depend upon initial trips (i.e., on transient behaviour). In section 5, we considered the case with a finite/small number of trips. In this case, the performance of all trips (including initial transient trips) is significant. As anticipated we have an optimal policy that is non-stationary, the headway depends upon the trip number (equation (38) of Theorem 4).

We now investigate if the transient optimal policy of section 5 approaches the stationary policy of section 4. To be more precise, we study the headways related to the last few trips (i.e., h_{T-k}^*, for k small) given by Theorem 4 as $T \to \infty$. Our aim is to verify if these headways approximately coincide with the stationary headway h^* (27) obtained in section 4.

In the previous subsection, we computed the limits of the iterative coefficients defining the optimal policy (38), and the idea is to use these asymptotic constants for the above mentioned verification. If ever the algorithm reaches stationary regime, i.e., say $h_{T-k} \approx h_s^*$ (for all small enough k), then this stationary h_s^* should satisfy (approximately) the following from Theorem 4 (after replacing iterative coefficients with their approximations given in (42)-(43)):

$$h_s^* \;=\; \frac{1}{\tilde{\gamma}_0^M}\left(-\sum_{l=1}^{M} h_s^* \tilde{\gamma}_l^M + a_*\right) \text{, in other words, } h_s^* \;=\; \frac{a_*}{1-\rho} \;=\; \frac{\sigma_M^M}{1-\rho}\sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma_M^M}{2(1-\rho)\alpha}\right)}. \tag{44}$$

From the above, one can observe that the optimal policy obtained from finite horizon analysis under stationarity is the same as the policy derived from the stationary analysis (27).

6. Numerical Analysis

We now consider few numerical examples to study and compare the policies derived in previous sections. We study various aspects related to the optimal policies like, the structure of the optimal policy, the performance comparison with different policies etc. In Figure 1 we observed that the theoretical performance (derived after simplifying assumptions) well approximates the Monte-Carlo estimates for practical values of load factors. The approximation is good even for Poisson (passenger) arrivals. Thus we anticipate the optimal policies (derived using those theoretical expressions) to improve the performance even for real systems. Towards this, in all the experiments described in this section, we have compared the performance improvement obtained using Monte-Carlo simulation based estimates. These simulations are carried using a similar procedure as explained in subsection 4.4.

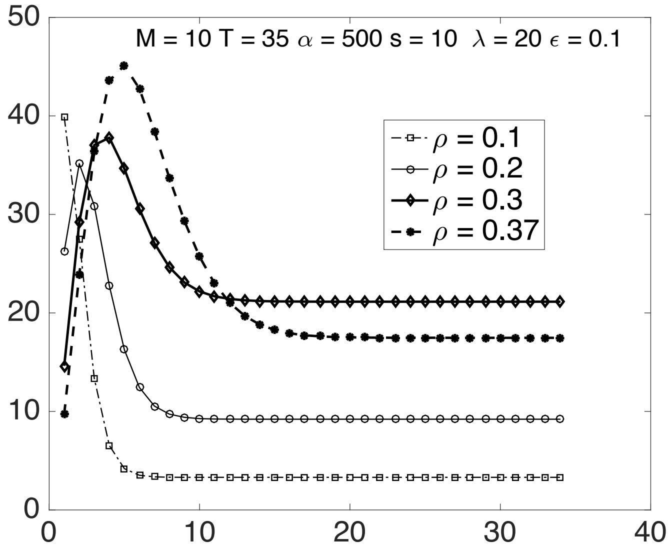

Figure 4. Expected waiting time verses bunching probability

6.1. Structure of optimal policy

In Figure 4, we plot the optimal policy for small load factors, which is computed using Algorithm 1 (and given by Theorem 4). Basically we plot the optimal depot headway h_k^* (time gap between starting instances of k and $k-1$ buses at depot), as a function of the trip number k. As observed from the figure, the structure of the optimal policy is different for different load factors. For very small loads ($\rho = 0.1$ in Figure 4) the depot headway is decreasing and finally converges to the corresponding stationary optimal headway (h_s^*). And for larger load factors ($\rho \geq 0.2$), the depot headway is increasing initially and then starts decreasing to optimal h_s^*. The transient behaviour is different for different regimes of load factors, however in all cases the headway settles down (as the trip number increases) to the corresponding stationary optimal value h_s^* given by (27). Recall that this stationary optimal value is also the limit of non-stationary policy as derived in (44), thus the observation of the Figure 4 reinforce the discussion of subsection 5.6.1.

The policy might settle down eventually to h_s^*, however as noticed we have significant transience prior to this and this leads to substantial improvement. We will observe in subsequent examples a significant improvement in both the performance measures (bunching probability as well as expected waiting times), when one uses optimal non-stationary policies in place of optimal stationary policies.

6.2. Comparison of the Algorithms

In this subsection, we compare the performance measures obtained under various algorithms; Algorithm 1, Algorithm 2.

ρ	Exact DP		Approximate DP		Error(%)	
	BP	WT	BP	WT	BP	WT
0.05	0.00066	1.459	0.00068	1.459	3.03	0
0.1	0.00124	2.230	0.00123	2.235	0.81	0.22
0.2	0.00479	6.038	0.00483	6.078	0.84	0.66
0.3	0.01942	16.399	0.01953	16.494	0.57	0.58
0.35	0.03910	25.730	0.03981	25.783	1.82	0.21
0.4	0.07976	37.930	0.08262	37.668	3.59	0.69

Table 1. Solutions and Performance of Exact and Approximate DP equations

ρ	Exact DP		Approximate DP		Error(%)	
	BP	WT	BP	WT	BP	WT
0.05	0.00141	2.377	0.00142	2.377	0.71	0
0.1	0.00265	3.790	0.00262	3.794	1.13	0.11
0.2	0.01042	10.596	0.01045	10.631	0.29	0.33
0.3	0.04352	27.542	0.04427	27.538	1.72	0.01
0.35	0.09274	40.423	0.09445	40.078	1.84	0.85
0.4	0.20972	50.542	0.21544	48.417	2.73	4.2

Table 2. Solutions and Performance of Exact and Approximate DP equations

In Tables 1 and 2 we compare the performance measures corresponding to Algorithm 1 and Algorithm 2 through numerical simulations for different load factors (ρ), which is tabulated in the first column. The performance measures obtained using exact dynamic programming equations are tabulated in the next two columns, while that with the approximate algorithm are in the fourth and fifth columns. The percentage errors are calculated and are tabulated in the last two columns. For both the tables, we set $M = 10, T = 35, \lambda = 20, \alpha = 2000$. We consider different traffic variabilities in the two tables; we set $\epsilon = 0.1$ in Table 1 and $\epsilon = 0.2$ in Table 2. We notice from both the Tables that the performance measures with approximate algorithm well matches that obtained by solving exact DP equations. By Lemma 5 and related discussions, we proved that the approximation is good for low load factors. However we notice from the tables that the approximation is good even for load factors upto $\rho = 0.4$. In fact the approximation error is within 10%, where the percentage error is calculated as below:

$$\text{percentage error} = \frac{|\text{approximate value} - \text{exact value}|}{\text{exact value}} \times 100.$$

Thus the approximate algorithm is a good substitute for sufficiently large range of load factors. Further these simulations also indicate that the DP equations given by Theorem 4 are valid for a large range of load factors.

6.3. Insufficiency of stationary policies

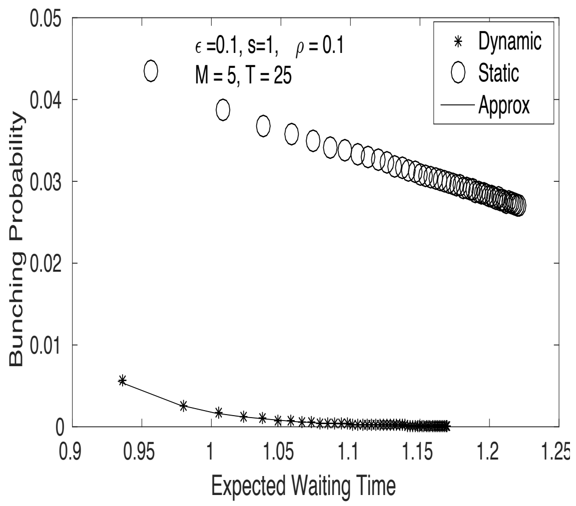

Figure 5. Expected waiting time verses bunching probability

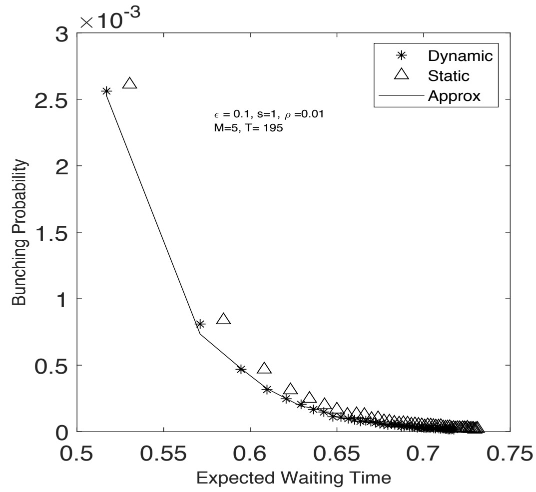

Figure 6. Expected waiting time verses bunching probability

We now compare the two dynamic policies with the optimal stationary policy h_s^*. Towards this, we plot the Pareto frontier of performance measures in Figures 5 and 6. The Pareto frontier is obtained by the Monte-Carlo estimates of both the performance measures, under the optimal policies of (33) for different α. The figure also shows the performance under various optimal stationary policies (27), corresponding to the same set of trade-off factors α. We conduct the experiment with $M = 5, T = 30, s = 1, \epsilon = 0.1$ and $\rho = 0.1$ in Figure 5 and $\rho = 0.01$ in Figure 6. We first observe that the approximate algorithm (Algorithm 2) once again well matches with with the Algorithm 1 in all examples.

We can also observe that the performance corresponding to both the algorithms is significantly better than the corresponding stationary optimal performance. However the stationary policies would still be important if one requires a simpler policy.

7. Correlated Travel times

Previously we considered independent travel times, i.e., we had the following model:

$$S^j_{k+1} = s^j + N^j_{k+1}$$

where s^j was fixed across various trips. In certain scenarios, especially when the frequency of buses is high, it is more appropriate to consider correlated travel times as below:

$$S^j_{k+1} = S^j_k + N^j_{k+1}.$$

One can easily extend the analysis to this case and we mention the required changes. Firstly we have the following result regarding the inter-arrival times:

Lemma 6. *For correlated travel times, we have:*

$$\begin{aligned}
I^i_k &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma^i_l h_{k-l} + \sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W^j_{k-l} \\
I^i_k - \rho I^i_{k-1} &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma^i_l h_{k-l} + \sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W^j_{k-l} - \rho \sum_{l=0}^{\min\{k-2,i-1\}} \gamma^i_l h_{k-1-l} + \sum_{l=0}^{\min\{k-2,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W^j_{k-1-l} \\
E[I^i_k] &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma^i_l h_{k-l} \\
(\sigma^i_{c,k})^2 &= variance\left(I^i_k - \rho I^i_{k-1}\right) = \epsilon^2 \left(\sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \left(\tilde{\gamma}_l^{i-j+1}\right)^2 + \frac{1-\rho^{2i}}{1-\rho^2} \right).
\end{aligned}$$

■

As with independent travel times, the inter-arrival times and the difference $\{I^i_k - \rho I^i_{k-1}\}_k$ become stationary with constant headways (i.e., when $h_k \equiv h$ for some $h > 0$), once $k > M$. Further the dependency of various terms on the history is almost as in Lemmas 1-2 for independent travel times, except for the variances. One can notice that $(\sigma^i_k)^2$ ($variance\left(I^i_k - \rho I^i_{k-1}\right)$ corresponding to independent travel times given in Lemma 2) is different from $(\sigma^i_{c,k})^2$ given in the above lemma, but the rest of the coefficients are exactly the same.

One can again consider the analysis of first few trips and derive optimal control when one starts with the first trip. In this sub-section we present a slightly different alternative which is of independent importance. We consider control of some T-trips after some t_0 initial uncontrolled trips with $t_0 > M$. There could be more turbulences after some initial period and control may become important after t_0 trips. For this case the optimal control policy can be computed as in the previous case and we have the following:

Theorem 7. *Assume $t_0 > M$ and $T \geq 1$. Define the following constants using the variance $\left(I^i_k - \rho I^i_{k-1}\right)$ corresponding to correlated travel times (with remaining coefficients as in Table 3) for all $T - t_0 - 1 \geq k \geq 1$*

$$a_{c,T-k} = \sigma^M_{c,T-k}\sqrt{-2\log\left(\frac{\eta_0^{T-k}\sqrt{2\pi}\sigma^M_{c,T-k}}{\tilde{\gamma}^M_0\alpha}\right)}, \tag{45}$$

$$\delta_c^{T-k} = \alpha\left(1 - \Phi_k\left(a_{c,T-k}\right)\right) + \vartheta^M_k + \frac{\eta_0^{T-k}(a_{c,T-k} - \varpi^M_{T-k})}{\tilde{\gamma}^M_0} + \delta_c^{T-k+1}.$$

If for any trip $T - k$ (with $T - t_0 - 1 \geq k \geq 1$), the following condition is satisfied

$$\alpha > e^{\left(\sum_{l=1}^M h_{T-k-l}\tilde{\gamma}^M_l\right)^2}\frac{\eta_0^{T-j}\sqrt{2\pi}\sigma^M_{c,T-j}}{\tilde{\gamma}^M_0} \text{ for all } 0 \leq j < k \text{ and } \eta_0^{T-k} > 0, \tag{46}$$

then the optimal policy and the value function are respectively given by:

$$h^*_{T-k}(\mathbf{h}_{T-k-1}) = \frac{1}{\tilde{\gamma}^M_0}\left[-\sum_{l=1}^M h_{T-k-l}\tilde{\gamma}^M_l - \varpi^M_{T-k} + a_{c,T-k}\right], \text{ and} \tag{47}$$

$$v^*_{T-k}(\mathbf{h}_{T-k-1}) = \sum_{l=1}^M h_{T-k-l}\left(\frac{\tilde{\gamma}^M_0\eta_l^{T-k} - \eta_0^{T-k}\tilde{\gamma}^M_l}{\tilde{\gamma}^M_0}\right) + \delta_c^{T-k}. \tag{48}$$

Proof follows as in the case of Theorem 4. ■

One can again prove the validation of the hypothesis of this theorem for low-load factors exactly as in Lemma 5. We finally conclude that the optimal policy for the case with correlated travel times is similar to the ones in Algorithms 1-2, except for the variance term and its influence on the coefficients $\{a_{c,T-k}\}_k$. It is trivial to realize that Algorithms 1-2 can also work with intermediate trips control (i.e., for the case $t_0 > M$), for which the terms like $\{\varpi_{T-k}\}_k$ are excluded in (38).

8. Conclusions

We modelled the bus bunching problem with Gaussian bus travel times and fluid arrivals. We studied the related performance measures. We discussed stationary as well as transient (suitable for finite trip problems) performance measures. Using numerical simulations, we showed that the performance of the system with Poisson arrivals can be well approximated with the derived theoretical expressions, when the arrival rates are large. We obtained the optimal depot headway time, i.e., the optimal bus frequency as a function of parameters like load conditions (passenger arrival rates), number of bus stops, traffic variability conditions (variance of the travel times) etc. We made the following observations using the theoretical as well as numerical study: a) When bus frequency decreases, bunching probability decreases and passenger waiting times increase; b) Optimal bus frequency decreases with increase in traffic variability and load conditions; and c) If the load is significantly larger, then the optimal bus frequency actually increases with load.

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For all $0 \le l \le M, 0 \le r \le M, 1 \le i \le M, 0 \le k \le T-1$

$$
\begin{aligned}
\gamma_l^i &\triangleq (-1)^l \binom{i-1}{l} \rho^l (1+\rho)^{i-1-l}, \quad \mu_l^r \triangleq (-1)^l \sum_{u=0}^{r} \binom{r}{u+l-1} \binom{u+l}{l} \rho^{u+l-1}, \text{ with } \binom{n}{r} := 0 \text{ when } n < r. \\
\bar{\gamma}_l &:= \frac{1}{2} \sum_{i=l+1}^{M} \gamma_l^i \quad \text{and} \qquad \vartheta_k^M := \sum_{i=1}^{M} \frac{1}{2} \left(\mathbb{1}_{i \ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \right) \\
\tilde{\gamma}_l^i &\triangleq \gamma_l^i - \rho \gamma_{l-1}^i \mathbb{1}_{l>0} \text{ and } \varpi_k^i = \begin{cases} \mathbb{1}_{i \ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} - \mathbb{1}_{i \ge k-1} \rho \sum_{j=1}^{i-k+2} \gamma_{k-2}^{i-j+1} s^{(j)} & \text{if } k \le M \\ \rho \sum_{j=1}^{i-M+1} \gamma_{M-1}^{i-j+1} s^{(j)} & \text{if } k = M+1 \\ 0 & \text{if } k > M+1. \end{cases} \\
\eta_l^{T-k} &= \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1} \frac{\tilde{\gamma}_{l+1}^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_l, \quad a_{T-k} = \sigma_{T-k}^M \sqrt{-2 \log \left(\frac{\eta_0^{T-k} \sqrt{2\pi} \sigma_{T-k}^M}{\tilde{\gamma}_0^M \alpha} \right)}, \\
\delta^{T-k} &= \alpha \left(1 - \Phi_k \left(a_{T-k}\right)\right) + \vartheta_k^M + \frac{\eta_0^{T-k} (a_{T-k} - \varpi_{T-k}^M)}{\tilde{\gamma}_0^M} + \delta^{T-k+1}.
\end{aligned}
$$

Table 3. Notations and coefficients summarized

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Appendix A: Proofs

Proof of Lemma 1: The proof is based on mathematical induction. We begin with first trip, i.e., $k = 1$ and prove equation (6) by induction for all stops, i.e., for all $1 \le i \le M$. First note that (6) simplifies to the following when $k = 1$:

$$
I_1^i = \gamma_0^i h_1 + \sum_{j=1}^{i} \gamma_0^{i-j+1} s^{(j)} + \sum_{r=0}^{i-1} \left[(1+\rho)^r N_1^{i-r} \right]. \tag{49}
$$

We will prove the equation (49) again by mathematical induction. We begin with $i = 1$ then,

$$
I_1^1 = h_1 + s^{(1)} + N_1^1 \text{ where } \gamma_0^1 = 1.
$$

Hence (49) is true for $i = 1$. Assuming the result is true for $i = n$, i.e.,

$$
I_1^n = \gamma_0^n h_1 + \sum_{j=1}^{n} \gamma_0^{n-j+1} s^{(j)} + \sum_{r=0}^{n-1} \left[(1+\rho)^r N_1^{n-r} \right].
$$

Now we prove it for $i = n + 1$. From equation (4) ,

$$\begin{aligned}
I_1^{n+1} &= I_1^n + N_1^{n+1} + s^{(n+1)} + \rho I_1^n \\
&= (1+\rho)I_1^n + N_1^{n+1} + s^{(n+1)} \\
&= (1+\rho)\gamma_0^n h_1 + (1+\rho)\sum_{j=1}^{n}\gamma_0^{n-j+1}s^{(j)} + \sum_{r=0}^{n-1}\left[(1+\rho)^{r+1}N_1^{n-r}\right] + N_1^{n+1} + s^{(n+1)} \\
&= \gamma_0^{n+1}h_1 + \sum_{j=1}^{n+1}\gamma_0^{n+1-j+1}s^{(j)} + \sum_{r=0}^{n}\left[(1+\rho)^r N_1^{n+1-r}\right] \text{ because } \binom{m}{0} = 1 \ \ \forall m.
\end{aligned}$$

Hence equation (49) is verified for $i = n + 1$. Hence lemma is true for $k = 1$ and $1 \le i \le M$. Assuming the equation (6) is true for $k = m$ and for any $1 \le i \le M$, i.e.,

$$I_m^i = \sum_{l=0}^{\min\{i-1,m-1\}}\gamma_l^i h_{m-l} + \mathbb{1}_{i\ge m}\sum_{j=1}^{i-m+1}\gamma_{m-1}^{i-j+1}s^{(j)} + \sum_{r=0}^{i-1}\left[(1+\rho)^r N_m^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{m-l}^{i-r}\right]. \tag{50}$$

Now we prove it for $k = m + 1$ and any $1 \le i \le M$. We will prove this again by sub induction. The sub induction is true for $i = 1$ and $k = m + 1$, because: using (4) again we have:

$$I_{m+1}^1 = h_{m+1} + N_{m+1}^1 - N_m^1 \text{ where } \gamma_0^1 = 1.$$

Now, assuming the result is true for $i = n$ and $k = m + 1$, i.e.,

$$I_{m+1}^n = \sum_{l=0}^{\min\{n-1,m\}}h_{m+1-l}\gamma_l^n + \mathbb{1}_{i\ge m+1}\sum_{j=1}^{n-m}(-1)^m s^{(j)}\gamma_m^{n-j+1} + \sum_{r=0}^{n-1}\left[(1+\rho)^r N_{m+1}^{n-r} + \sum_{l=1}^{r+1}\mu_l^r N_{m+1-l}^{n-r}\right].$$

From above, equation (50) (with $i = n$) and equations (4)-(7) we have:

$$\begin{aligned}
I_{m+1}^{n+1} &= I_{m+1}^n + N_{m+1}^{n+1} - N_m^{n+1} + \rho I_{m+1}^n - \rho I_m^n \quad \text{[from equation (4)]} \\
&= (1+\rho)I_{m+1}^n + N_{m+1}^{n+1} - N_m^{n+1} - \rho I_m^n \\
&= (1+\rho)\left\{\sum_{l=0}^{\min\{n-1,m\}}\gamma_l^n h_{m+1-l} + \mathbb{1}_{i\ge m+1}\sum_{j=1}^{n-m}\gamma_m^{n-j+1}s^{(j)} + \sum_{r=0}^{n-1}\left[(1+\rho)^r N_{m+1}^{n-r} + \sum_{l=1}^{r+1}\mu_l^r N_{m+1-l}^{n-r}\right]\right\} + N_{m+1}^{n+1} \\
&\quad -N_m^{n+1} - \rho\left(\sum_{l=0}^{\min\{n-1,m-1\}}\gamma_l^n h_{m-l} + \mathbb{1}_{i\ge m}\sum_{j=1}^{n-m+1}\gamma_{m-1}^{n-j+1}s^{(j)} + \sum_{r=0}^{n-1}\left[(1+\rho)^r N_m^{n-r} + \sum_{l=1}^{r+1}\mu_l^r N_{m-l}^{n-r}\right]\right) \\
&= \underbrace{\sum_{l=0}^{\min\{n-1,m\}}(1+\rho)\gamma_l^n h_{m+1-l} - \sum_{l=0}^{\min\{n-1,m-1\}}\rho\gamma_l^n h_{m-l}} + \underbrace{\mathbb{1}_{i\ge m+1}\sum_{j=1}^{n-m}(1+\rho)\gamma_m^{n-j+1}s^{(j)} - \mathbb{1}_{i\ge m}\sum_{j=1}^{n-m+1}\rho\gamma_{m-1}^{n-j+1}s^{(j)}} \\
&\quad \underbrace{+N_{m+1}^{n+1} + \sum_{r=0}^{n-1}(1+\rho)^{r+1}N_{m+1}^{n-r}} \underbrace{-N_m^{n+1} + (1+\rho)\sum_{r=0}^{n-1}\mu_1^r N_m^{n-r} - \rho\sum_{r=0}^{n-1}(1+\rho)^r N_m^{n-r}} \\
&\quad +(1+\rho)\sum_{r=0}^{n-1}\sum_{l=2}^{r+1}\mu_l^r N_{m+1-l}^{n-r} - \rho\sum_{r=0}^{n-1}\sum_{l=1}^{r+1}\mu_l^r N_{m-l}^{n-r} \\
&= \sum_{l=0}^{\min\{n,m\}}\gamma_l^{n+1}h_{m+1-l} + \mathbb{1}_{i\ge m+1}\sum_{j=1}^{n-m+1}\gamma_m^{n+1-j+1}s^{(j)} + \sum_{r=0}^{n}(1+\rho)^r N_{m+1}^{n+1-r} + \sum_{r=0}^{n}\mu_1^r N_m^{n+1-r} + \sum_{r=0}^{n}\sum_{l=2}^{r+1}\mu_l^r N_{m+1-l}^{n+1-r} \\
&= \sum_{l=0}^{\min\{n,m\}}\gamma_l^{n+1}h_{m+1-l} + \mathbb{1}_{i\ge m+1}\sum_{j=1}^{n-m+1}\gamma_m^{n+1-j+1}s^{(j)} + \sum_{r=0}^{n}\left[(1+\rho)^r N_{m+1}^{n+1-r} + \sum_{l=1}^{r+1}\mu_l^r N_{m+1-l}^{n+1-r}\right].
\end{aligned}$$

For the above, we need the following, which can easily be verified from equations (7) and (8):

$$(1+\rho)\gamma_l^n 1_{l<n} - 1_{l>0}\rho\gamma_{l-1}^n = \gamma_l^{n+1} \text{ with } \gamma_l^n = 0 \text{ if } l > n.$$

Hence the above equation is true for $i = n+1$. Hence lemma is verified.

■

Lemma 8. *For Poisson arrivals,*

$$E\left[\bar{W}_k^i\right] = \lambda \frac{E\left[(I_k^i)^2\right]}{2} \text{ and } E\left[\bar{w}_k^i\right] = E\left[\frac{\bar{W}_k^i}{X_k^i}\right] = \frac{E[I_k^i]}{2} \text{ a.s.}$$

Proof: Note that:

$$E\left[\bar{W}_k^i\right] = E\left[\left[E[W_k^i \middle| I_k^i]\right]\right].$$

Now consider,

$$E\left[\bar{W}_k^i \middle| I_k^i\right] = E\left[\sum_{n=1}^{X_k^i} W_n^i \middle| I_k^i\right]$$

Passengers can arrive to the stop at any time epoch during the bus inter arrival times. The number of passenger arrivals during the bus inter arrival time I_k^i is X_k^i. First condition on I_k^i. Further conditioning on the number arrived X_k^i during I_k^i, the unordered arrivals are uniformly distributed in the interval I_k^i. i,e., the conditional waiting time of any passenger

$$W_n^i \sim I_k^i - Unif(0, I_k^i) \sim Unif(0, I_k^i).$$

The above is true for Poisson arrivals. Hence,

$$\begin{aligned}
E\left[\bar{W}_k^i \middle| I_k^i, X_k^i\right] &= X_k^i \frac{I_k^i}{2} \quad \text{a.s. and so} \\
E\left[\bar{W}_k^i \middle| I_k^i\right] &= E\left[E\left[\bar{W}_k^i \middle| I_k^i, X_k^i\right] \middle| I_k^i\right] = \lambda \frac{(I_k^i)^2}{2} \text{ and} \\
E\left[\bar{w}_k^i \middle| I_k^i\right] &= E\left[E\left[\frac{\bar{W}_k^i}{X_k^i} \middle| I_k^i, X_k^i\right] \middle| I_k^i\right] = \frac{I_k^i}{2} \text{ a.s.}
\end{aligned}$$

Finally:

$$E\left[\bar{W}_k^i\right] = \lambda \frac{E\left[(I_k^i)^2\right]}{2} \quad \text{and} \quad E\left[\bar{w}_k^i\right] = \frac{E\left[I_k^i\right]}{2}.$$

■

Theorem 9. *Let g be function of the following type:*

$$g(\mathbf{h}_M) = \sum_{l=0}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1 - \Phi\left(\sum_{l=0}^{M} \tilde{z}_l h_{M-l}\right)\right) + \delta$$

where $\{\bar{z}_l\}$, $\{\tilde{z}_l\}$ and δ are positive constants and where $\mathbf{h}_M = [h_1, h_2, \dots, h_M]$ is an M dimensional vector. Consider the following optimization problem:

$$g^*(\mathbf{h}_{M-1}) := \min_{h_M} g(\mathbf{h}_M) \text{ and } h_M^* := \arg\min_{h_M} g(\mathbf{h}).$$

Then there exists an unique optimizer to this problem. The optimizer is given by:

$$h_M^* = \frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right], \text{ if } \alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2} \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0} \text{ with } a := \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}.$$

If the optimizer is an interior point then

$$g^*(\mathbf{h}_{M-1}) = \sum_{l=1}^{M-1} h_{M-l}\left(\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}\right) + \delta \textit{ with } \delta = \alpha\left[1-\Phi(a)\right] + \frac{\bar{z}_0 a}{\tilde{z}_0}.$$

Proof: Consider the objective function,

$$g^*(\mathbf{h}_{M-1}) = \min_{h_M} \sum_{r=0}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1-\Phi\left(\sum_{l=0}^{M}\tilde{z}_l h_{M-l}\right)\right) + \delta. \tag{51}$$

Let h_M^* be the optimal policy for equation (51), i.e.,

$$\begin{aligned} h_M^* &\in \arg\min(g(\mathbf{h}_M)), \text{ and either} \\ \Rightarrow \frac{d}{dh_M}(g(\mathbf{h}_M))\Big|_{h_M=h_M^*} &= 0 \text{ or } h_M^* \text{ is on the boundary} \end{aligned}$$

The first derivative is given by:

$$\frac{d}{dh_M}(g(\mathbf{h}_M)) = \bar{z}_0 - \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{\left(\sum_{l=0}^{M}\tilde{z}_l h_{M-l}\right)^2}{2}\right)\tilde{z}_0.$$

By hypothesis $\bar{z}_0 > 0$. Under this condition, if $\alpha < \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}$, the first derivative is always positive and hence the optimizer is at lower boundary $\underline{h} = 0$. Otherwise, there exists a zero of the derivative as below:

$$\Rightarrow \sum_{l=0}^{M} h_{M-l}\tilde{z}_l = \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}.$$

This interior point is the minimizer (when unconstrained) as the second derivative is positive, and further we have unique minimizer for any given condition. This interior point is a positive quantity if

$$\frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right] > 0,$$

and it is easy to verify that the above is always true if $\sum_{l=1}^{M} h_{M-l}\tilde{z}_l < 0$ and in case $\sum_{l=1}^{M} h_{M-l}\tilde{z}_l > 0$, the above is true if

$$\alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2}\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}.$$

Thus if the above condition is satisfied, the interior point is always positive (write some more). Thus in all,

$$h_M^*(\mathbf{h}_{M-1}) = \frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right], \text{ if } \alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2}\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0} \text{ with } a := \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}.$$

Substituting the above, the value function equals:

$$\begin{aligned} g^*(\mathbf{h}_{M-1}) &= h_M^*\bar{z}_0 + \sum_{l=1}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1-\Phi\left(h_M^*\tilde{z}_0 + \sum_{l=1}^{M} h_{M-l}\tilde{z}_l\right)\right) + \delta \\ &= \sum_{l=1}^{M} h_{M-l}\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0} + \frac{\bar{z}_0 a}{\tilde{z}_0} + \alpha\left(1-\Phi(a)\right) + \delta \\ &= \sum_{l=1}^{M} h_{M-l}\left(\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}\right) + \delta, \text{ where} \\ \delta &= \alpha\left[1-\Phi(a)\right] + \frac{\bar{z}_0 a}{\tilde{z}_0}. \end{aligned}$$

■

Proof of Theorem 4: The proof is based on mathematical induction. We begin with $i = 0$. Then the corresponding DP equations are given by,

$$
\begin{aligned}
v_T(\mathbf{h}_{T-1}) &= \min_{h_T} r_T(\mathbf{h}_T) \text{ and recall} \\
r_T(\mathbf{h}_T) &= \sum_{l=0}^{M-1} h_{T-l}\eta_l^T + \alpha\left(1 - \Phi_M\left(\sum_{l=0}^{M} \tilde{\gamma}_l^M h_{T-l}\right)\right).
\end{aligned}
$$

The above objective function is like the objective function of Theorem 9 with $\tilde{z}_l = \tilde{\gamma}_l^M, \bar{z}_l = \eta_l^T$ for all l and $\delta = 0$. By this theorem, the optimal policy and value function are respectively,

$$
\begin{aligned}
h_T^*(\mathbf{h}_{T-1}) &= \frac{1}{\tilde{\gamma}_0^M}\left[-\sum_{l=1}^{M} h_{T-l}\tilde{\gamma}_l^M + a_T\right], a_T := \sigma_M^M\sqrt{-2\log\left(\frac{\eta_0^T\sqrt{2\pi}\sigma_M^M}{\tilde{\gamma}_0^M\alpha}\right)} \\
&\text{and} \\
v_T^*(\mathbf{h}_{T-1}) &= \sum_{l=1}^{M} h_{T-l}\left(\frac{\tilde{\gamma}_0^M\eta_l^T - \eta_0^T\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M}\right) + \delta^T, \text{ where } \delta^T = \alpha\left(1 - \Phi_M(a_T)\right) + \frac{\eta_0^T a_T}{\tilde{\gamma}_0^M}.
\end{aligned}
$$

Hence the theorem is verified for $i = 0$.

Assuming result is true for $i = n$ and $n > T - M$, we prove it for $i = n + 1$.

The corresponding DP equation is given by,

$$
v_{T-n-1} = \min\left\{\sum_{l=0}^{M-1} h_{T-n-1-l}\eta_l^{T-n-1} + \vartheta_{T-n-1}^M + \alpha\left(1 - \Phi_{T-n-1}\left(\sum_{l=0}^{M} h_{T-n-1-l}\tilde{\gamma}_l^M + \varpi_{T-n-1}^M\right)\right) + \delta^{T-n}\right\}.
$$

The above objective function is like the objective function of Theorem 9 with $\tilde{z}_l = \tilde{\gamma}_l, \bar{z}_l = \eta_l^{T-n-1}$ for all l and $\delta = \delta^{T-n}$. Note that $\bar{z}_0 = \eta_0^{T-n-1} > 0$ by the given hypothesis. By this theorem, the optimal policy and value function are respectively,

$$
h_{T-n-1}^*(\mathbf{h}_{T-n-2}) = \frac{1}{\tilde{\gamma}_0^M}\left[-\sum_{l=1}^{M} h_{T-n-1-l}\tilde{\gamma}_l^M - \varpi_{T-n-1}^M + a_{T-n-1}\right], \text{ where } a_{T-n-1} = \sigma_{T-n-1}^M\sqrt{-2\log\left(\frac{\eta_0^{T-n-1}\sqrt{2\pi}\sigma_{T-n-1}^M}{\tilde{\gamma}_0^M\alpha}\right)}.
$$

Substituting the above, the value function equals,

$$
\begin{aligned}
v_{T-n-1}^*(\mathbf{h}_{T-n-2}) &= \sum_{l=1}^{M} h_{T-n-1-l}\left(\frac{\tilde{\gamma}_0^M\eta_l^{T-n-1} - \eta_0^{T-n-1}\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M}\right) + \delta^{T-n-1} \text{ where} \\
\delta^{T-n-1} &= \alpha\left(1 - \Phi_{T-n-1}(a_{T-n-1})\right) + \vartheta_{T-n-1}^M + \frac{\eta_0^{T-n-1}\left(a_{T-n-1} - \varpi_{T-n-1}^M\right)}{\tilde{\gamma}_0^M} + \delta^{T-n}.
\end{aligned}
$$

Hence theorem is verified for $i = n + 1$ when $n > T - M$.

■

Proof of Lemma 5: Define the following function,

$$f_\eta : (0, 1] \to \mathbb{R}^{T+1}$$

i.e., $\rho \xrightarrow{f_\eta} \left(\{\eta_0^t\}_{t\geq 0}\right)$. For every $\rho > 0$, $f_\eta(\rho)$ is well defined and continuous (see Table 3). Now we define the continuous extension of f_η at $\rho = 0$, i.e.,

$$f_\eta(0) \triangleq \lim_{\rho\to 0} f_\eta(p).$$

Here $f_\eta(0)$ is not corresponding to any system. It is a continuous function on $[0,1]$ by construction.

Before we proceed further one can easily verify the following (see Table 3):

$$\frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} \to 0 \text{ and for all } i > 0, \bar{\gamma}_i \to 0, \bar{\gamma}_0 \to \frac{M}{2}, \text{ as } \rho \to 0,$$

which would be instrumental in proving this lemma.

If one proves $f_\eta(0) = \lim_{\rho\to 0} f_\eta(\rho) > 0$ (i.e., all components are strictly positive), then by continuity there exists a $\bar{\rho}_\eta > 0$ such that $f_\eta(\rho) > 0$ for all $\rho \le \bar{\rho}_\eta$. This proves the second part of assumption (37). Towards this we begin with $f_\eta(0)$. This part of the proof is based on backward induction. We begin with $i = 0$, i.e., with η_0^T and observe immediately that:

$$\lim_{\rho\to 0} \eta_0^T = \lim_{\rho\to 0} \bar{\gamma}_o = \lim_{\rho\to 0} \frac{(1+\rho)^M - 1}{2\rho} = \frac{M}{2} > 0, \text{ by using L'Hospital's rule.}$$

Further it is clear that

$$\lim_{\rho\to 0} \eta_l^T = 0 \text{ for all } l > 0.$$

In a similar way, for $i = 1$, i.e.,

$$\begin{aligned}
\lim_{\rho\to 0} \eta_0^{T-1} &= \lim_{\rho\to 0} \left(\eta_1^T - \eta_0^T \frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_0 \right) = \lim_{\rho\to 0} \left(\bar{\gamma}_1 - \frac{M}{2} \frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \frac{M}{2} \right) \\
&= \lim_{\rho\to 0} \left(-((1+\rho)^{M-1} - 1) - \frac{M}{2} \frac{-\rho(1+\rho)^{M-2} - \rho(1+\rho)^{M-1}}{(1+\rho)^{M-1}} \right) + \frac{M}{2} \\
&= \frac{M}{2} > 0 \text{ and similarly } \lim_{\rho\to 0} \eta_l^{T-1} = 0 \text{ for all } l > 0
\end{aligned}$$

Now assume the result is true for $i = n-1$ i.e., assume $\lim_{\rho\to 0} \eta_0^{T-n+1} = M/2 > 0$ and $\lim_{\rho\to 0} \eta_l^{T-n+1} = 0$ for all $l > 0$. Then for $i = n$:

$$\begin{aligned}
\lim_{\rho\to 0} \eta_0^{T-n} &= \lim_{\rho\to 0} \left(\eta_1^{T-n+1} - \eta_0^{T-n+1} \frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_0 \right) \\
&= \lim_{\rho\to 0} \eta_1^{T-n+1} + \frac{M}{2} = \frac{M}{2} > 0 \text{ and similarly } \lim_{\rho\to 0} \eta_l^{T-n} = 0 \text{ for all } l > 0.
\end{aligned}$$

Hence, $\eta_0^t > 0$ for all $\rho \le \bar{\rho}_\eta$, t and hence a_t (for all t) are well defined for all such ρ. Now define the following function,

$$f : (0, \bar{\rho}_\eta] \to \mathbb{R}^{2T+1}$$

i.e., $\bar{\rho}_\eta \xrightarrow{\text{f}} \left(\{h_t^*\}_{t\ge 1},\ \{\eta_0^t\}_{t\ge 0} \right)$, where $\{h_t^*\}_t$ are defined recursively in Theorem 4. Observed here that $\{h_t^*\}_{t\ge 1}$ is well defined using (38) (whether or not the condition (37) is true) and hence the function $f(\cdot)$ is well defined and continuous.

It remains to prove (recursively, i.e., using backward induction) that

$$\alpha > \lim_{\rho\to 0} e^{\left(\Sigma_{l=1}^M h_{T-i-l}^* \tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} \text{ for all } 0 \le j < i,$$

as by continuity of f the result follows. And this is immediate since $\tilde{\gamma}_l^M \to 0$ for all $l \ge 1$, and, hence[6]

$$\lim_{\rho\to 0} e^{\left(\Sigma_{l=1}^M h_{T-i-l}^* \tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} = \lim_{\rho\to 0} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} = \epsilon \frac{M\sqrt{2\pi}}{2},$$

[6]Observe that $\{h_t^*\}$ are bounded for all $\rho \le \bar{\rho}_\eta$ (forward) recursively as below (because of finite horizon problem):

$$\sup_{\rho\le\bar{\rho}_\eta} h_2^* \le \left(\max_l \sup_{\rho\le\bar{\rho}_\eta} \tilde{\gamma}_l^M \right) h_1 + \sup_{\rho\le\bar{\rho}_\eta} \sup_t \varpi_t + \sup_{\rho\le\bar{\rho}_\eta} \sup_t a_t < \infty,\ \sup_{\rho\le\bar{\rho}_\eta} h_3^* \le 2 \sup_{\rho\le\bar{\rho}_\eta} \sup_t \varpi_t + 2 \sup_{\rho\le\bar{\rho}_\eta} \sup_t a_t,\ \cdots,\ \sup_{\rho\le\bar{\rho}_\eta} h_t^* \le (t-1) \sup_{\rho\le\bar{\rho}_\eta} \sup_t \varpi_t + (t-1) \sup_{\rho\le\bar{\rho}_\eta} \sup_t a_t.$$

The above are bounded by continuity of the function defining coefficients $\{\varpi_t\}_t$ and $\{a_t\}_t$ with respect to ρ.

by previous result and because α is greater than RHS by given hypothesis. $\blacksquare$