Optimal Control of Bunching and Serve on Move WLANs in Intelligent Transportation Systems

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by

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I dedicate this thesis to

My Parents
(Amma Nanna)

Declaration

I declare that this written submission represents my ideas in my own words and where others ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Thesis Approval

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Optimal Control of Bunching and Serve on Move WLANs in Intelligent Transportation Systems

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CERTIFICATE OF COURSE WORK

This is to certify that Mr. M.Venkateswararao.K (Roll No. 11I190012) was admitted to the candidacy of the M.Sc-Ph.D Degree on July 17, 2011 and confirmed his PhD registration on July 12, 2013. He has successfully completed all the courses required for Ph.D Degree. The details of the course work done are given below.

S. No.	Course Code	Course Name	Credits
1	IE 802	Topics in Industrial Engineering and Operations Research	6
2	IE 708	Markov Decision Processes	6
3	IE 606	Analysis & Control of Queues	6
4	MA 408	Measure Theory	8
5	HS 699	Communication and Presentation Skills	4

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Abstract

This report contributes towards two problems related to intelligent transportation systems. We first study the 'bus bunching' problem. In a bus transportation system, the time gap between two successive buses is called headway. When the headways are small (high-frequency bus routes), any perturbation (e.g., variability in load and or traffic conditions) makes the system unstable: the headway variance tends to increase along the route. Eventually the buses can end up bunching, i.e., they start travelling together. One also needs to consider the waiting times of the passengers. The main aim of this thesis is to derive Pareto frontier (a notion of optimality for multi-objective problems) of the two performance measures, bunching probability and passenger average waiting times, under static/stationary, 'partially dynamic' and fully dynamic policies.

Under stationary policies we consider that the buses depart from the depot after equal intervals of time and the aim is to design this constant headway (Pareto) optimally. We derive the required performance under fluid arrival and fluid boarding assumption and obtained optimal policies. Further using Monte-Carlo simulations, we demonstrate that the performance with Poisson arrivals is well approximated by the theoretical results. As the traffic variance increases we observe, it is optimal to reduce the bus frequency. More interestingly even with the increase in load (passenger arrival rates), it is optimal to reduce the bus frequency. This is true in the low load regimes; while for higher loads it is optimal to increase the frequency with increase in load.

In practice, in majority of cases stationary headway policies are used. However when the randomness of the traffic (or load) increases, the static policies may not suffice (bunching might become significantly more). We propose 'partially dynamic' policies that depend upon *easily available state* information; just consider that the current headway depends upon the headways of the previous trips and you would see significant improvement in both (bunching probability and passenger average waiting times) the performances. We refer them as partially dynamic

policies, since they do not depend upon more elaborate and more expensive state information, e.g., load waiting at various stops in the previous trip or the round trip delays of previous bus etc. On solving an appropriate dynamic programming equations, we derive closed form expressions for (Pareto) optimal headway policies. The optimal headway is linear in previous-trip headways. Using Monte-Carlo simulations we compare the performance under partially dynamic and static policies; partially dynamic policies perform significantly better in most of the scenarios that we studied.

We then derive dynamic headways which control the bus frequency based on the observed system state. The observation is a delayed information of the time gaps between successive bus arrivals at various stops, corresponding to some previous trips. We solve the relevant dynamic programming equations to obtain near-optimal policies, and the approximation improves as the load factor reduces. The near-optimal policy turns out to be linear in previous trip headways and the earlier bus-inter-arrival times at various stops corresponding to the latest trip whose information is available. Using Monte Carlo based simulations, we demonstrate that the proposed dynamic policies significantly improve (both) the performance measures, in comparison with the previously proposed partial dynamic policies.

The other problem is performance analysis of serve on the move wireless LANs. Recently proposals are made to mount wireless transceivers on periodically moving vehicles (e.g., public transport system). These vehicles were primarily meant to facilitate human transportation in places like large universities. The idea was to design economical networks using the already existing infrastructure, when delays can be tolerated. The salient feature of these networks is that the wireless server does not stop, rather provides the service to the users while on the move and as long as the contact is available. Thus the service of various users waiting on such networks is interlinked. One cannot model this system with existing continuous/discrete polling models, as the later assume the server to stop and serve before resuming with its journey. We obtain the conditions for stability and then the workload analysis. We also discuss optimal scheduling policies.

Key words: Bus bunching, Pareto frontier, bunching probability, waiting times, dynamic programming, fluid arrivals, polling models, stability, workload, Monte-Carlo simulation

Publications

Journals

1. Koppisetti, M. Venkateswararao, and Veeraruna Kavitha. “Optimal dynamic headway policies to control bunching, waiting, using cheap information”. To be communicated to Transportation Research:B.

2. Koppisetti, M. Venkateswararao, and Veeraruna Kavitha. “Dynamic Bus dispatch Policies, optimizing bunching probability and waiting times” (To be communicated).

Conferences

1. Kavitha, Veeraruna, and K. M. Rao. ”Performance analysis of serve on the move wireless LANs.” Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt), 2015 13th International Symposium on. IEEE, 2015.

2. Koppisetti, M. Venkateswararao, Veeraruna Kavitha, and Urtzi Ayesta. ”Bus schedule for optimal bus bunching and waiting times.” Communication Systems & Networks (COMSNETS), 2018 10th International Conference on. IEEE, 2018. (**Recipient of the best paper award**)

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Chapter 1

Introduction

This thesis contributes to the field of intelligent transportation systems. Broadly, the contribution of the thesis is in two parts. The first and major part of the thesis deals with the bus bunching problem in (public) transportation system. We obtained optimal headway policies, i.e., the optimal bus dispatch policies at depot under different (available information) scenarios.

The last part of the thesis is about the performance analysis of a wireless LAN, which is mounted on a public transport system. These systems are commonly referred to as Ferry based Wireless LANs (Bin Tariq *et al.* (2006); Kavitha and Altman (2009); Boon *et al.* (2011)). These are introduced as a economic alternative network which utilizes the existing public transport system, to transfer the data between locations (wireless) covered by the path of the vehicle and when the data to be transferred is delay tolerant. The salient part of this work is to obtain optimal scheduling policies, specific to the situation where the wireless service is provided while on move.

1.1 Bus bunching

We consider (public) transport systems like that of buses, trams, metros, local trains etc, for brevity we refer them as bus transport systems. Majority of people in a city rely on such a transport for their daily activities. However, these systems tend to become inefficient due to various factors like variable demands and or variable traffic conditions etc. Some of the major issues faced by these systems are significant delays in arrival times at various stops, bunching of buses (two or more buses start travelling together) which in turn can lead to huge variations in bus occupancies (some buses are heavily loaded followed by almost empty buses), traffic congestion, pollution etc. There is a huge requirement to design the bus schedules/routes in such a way so as to minimize the above mentioned issues to the best possible extent.

Typically, most popular bus routes have high frequency of buses and the buses keep circulating around the same route repeatedly. The buses are scheduled to depart from the depot (where the buses start their journey), one after the other according to a pre-designed scheule. It is well-known that such transit routes without any intervention or control are unstable (Newell and Potts (1964)). Any perturbation, typically, in the number of passengers arriving (demand) at the bus stops and or in the traffic conditions, can lead to bunching of two or more buses. Sometimes two (or more subsequent) buses get together somewhere along the route (or at some stop) and would start travelling together. The perturbations (variabilities in traffic conditions, demands etc) are inevitable, but the bus schedule should be robust, in the sense one should design the (bus) schedules which minimize the above inefficiency to the best extent possible.

Headway is defined as the time gap between two consecutive buses. This headway is predetermined at the depot and hence is known. However the headways at the other bus-stops, encountered en-route, have random fluctuations as discussed. When a bus is delayed for a long period at a particular stop (due to larger demand and or larger transit times from previous stops), it has to cater to the increased number of passengers at the next stop. Hence it gets further delayed. Whereas, the following bus observes less passengers than anticipated, as the time gap between the buses is reduced. Hence it further speeds up due to lesser dwell times (the boarding and de-boarding time). Eventually, the headway becomes zero at a certain point along the route and the buses start travelling together. This phenomenon is called 'Bus bunching'. As a result of these variable headways (and probable bunching), passengers waiting at various bus stops experience large variance in their waiting times (and in the bus occupancies) leading to an unreliable transport service. Further, this results in an inefficient usage of resources (as some bunched buses run almost empty) even for bus operators. Thus efficient control of bunching of buses is important from the perspective of both the public using the service, as well as the operator.

Bus bunching is a critical issue faced by bus agencies and this problem has been thoroughly investigated over past few decades. However, it is still an active area of research as it is still challenging to provide a generic, practical, and effective solution. Existing control strategies are based on ideas like skipping the forthcoming bus stops (e.g., Suh *et al.* (2002); Fu *et al.* (2003); Sun and Hickman (2005); Cortés *et al.* (2010)), limited boarding (e.g., Delgado *et al.*

(2009, 2012)), holding buses at specific locations (e.g., Yu and Yang (2009); Delgado *et al.* (2009); Cortés *et al.* (2010); Xuan *et al.* (2011); Sánchez-Martínez *et al.* (2016)) etc. Holding control applied at intermediate bus-stops and or skipping of stops may not be comfortable from the perspective of the passengers travelling in the bus. Hence our focus is on the holding control strategy only at depot. In papers like Delgado *et al.* (2009, 2012); Xuan *et al.* (2011) etc., authors discuss and control the eventual variance of the error between the ideal schedule and the schedule considering random fluctuations, when the number of stops and buses tends to infinity. They assume no bunching. However in many scenarios it is not possible to completely avoid bunching. Further when the randomness is very high, it is almost impossible to adhere to the ideal schedules. In such scenarios, *it is rather important to reduce the probability of bunching, and we precisely consider this probability.* For such highly random scenarios, it is also important to consider the *passenger waiting times.* Further in all scenarios the number of stops and the number of trips is finite, hence such a modelling is more realistic.

In Cortés *et al.* (2010), authors consider finite number of buses looping continuously in a circular path and covering finite number of stops. However they work with expected value of the squared difference between the actual headway and the supposed headway, and not with the probability of bunching. Further the average passenger waiting times are defined as the expected value of the product of the headway and the number of passengers arrived during the headway. This definition does not consider the influence of passenger arrivals spread over the entire (bus inter-arrival) interval. We consider (customer) average of the waiting times of the passengers; we define waiting time of a passenger as *the exact time gap between the arrival epoch of that passenger (to the stop) and the arrival epoch of the bus that the passenger boards.* We derive theoretical expression for fluid arrivals, we also demonstrate through simulations that the derived expressions well approximate that corresponding to Poisson arrivals.

The aim of this thesis is to derive a policy of optimal headways between departure epochs of successive buses at depot, that minimizes a weighted combination of the two costs: a) the average passenger waiting times; and b) the probability of bunching. We derived optimal bus frequency (a stationary policy), optimal partially dynamic (control depends upon the readily available information, the previous-trip headways) policies and optimal fully dynamic (control depends upon headways used for few previous trips and the bus-inter-arrival times at various stops corresponding to a previous trip whose information is available) policies. Note that the last case corresponds to that of a problem with delayed information. We thus obtained the Pareto frontier[1] of bunching probability and the expected waiting times, under three type of policies.

Partially dynamic policies using cheap information

We obtained the partially dynamic policies under independent and as well as correlated travel times. We showed that these optimal policies depend linearly upon the headways of the previous trips for both correlated as well as independent travel times. We demonstrated huge improvement with these policies (in comparison with stationary policies): a) with correlated travel times and for low traffic variability conditions, the improvement is between 13% to 23%; while b) for the rest of the conditions the improvement is significantly high (greater than 30% and upto 80%). We refer these policies as partially dynamic policies, since they do not depend upon more elaborate and more expensive state information, e.g., load waiting at various stops in the previous trip or the round trip delays of previous bus etc.

Fully Dynamic policies

One can do much better if one has access to a more informative system state that is influenced by the random fluctuations governing the system. For example, if one can observe the number of passengers waiting in various stops (at bus arrival instances) or an equivalent information in the previous trips, a better headway policy can be designed using this knowledge. In this thesis we consider that the bus-inter-arrival times between various stops (of previous trips) are observable, based on which the headway times of the future trips are decided.

The natural tool to design such policies is the theory of Markov decision processes. However in this system, one will *have access only to delayed information*: the headway decision for the current bus has to be made immediately after the previous bus departs the depot, the information related to the previous bus trajectory (no delay) is obviously not available for this decision epoch. Further one may not even have the information about some of the trips, previous to the trip that just started. We assume that d-delay information is available and derive the optimal policies.

We obtained closed form expressions for an ε-optimal policy, that is near optimal under

[1] In any multi-objective problem like ours, a pair of performance measures (e.g., achieved through a headway-policy) is said to be efficient/Pareto optimal, if there exists no other policy which can strictly improve one of the performance measures, without degrading the other. A Pareto frontier is the set of all efficient pairs of performance measures.

small load factors. Interestingly the policy is linear in the previous trip headways and the bus-inter-arrival times at various stops of the previous trip (information about which is available). We showed numerically that this dynamic policy has significant improvement in comparison with respect to the partially dynamic optimal policies in all the example scenarios that we studied. This improvement is significant even for considerable load factors (upto 0.5). We used Monte-Carlo based simulations to estimate the two performance measures. The observation process (getting access to the required state process) might be complicated and expensive, but the complexity of the proposed policy is negligible. Thus cost to be paid is only for procuring the information.

In summary, both the versions of the dynamic policies are linear in the (available) state/information and hence are simple to implement (we also have simple expressions for coefficients). Both of them improve significantly in comparison with the commonly used schedules of constant headways between successive buses. One can do much better with fully dynamic policies, if one has access to more information (exact bus-inter-arrival times at various stops).

1.2 Ferry based wireless LANs: Serve on Move

In applications like Ferry Based Wireless LANs (FWLAN), service is offered by a moving server in distributed fashion, whenever it encounters a waiting customer on/near its path. The Ferries are originally meant for carrying human beings (or goods) in a cyclic path repeatedly (e.g., public transportation systems) and the idea was to construct economic communication networks exploiting its repeated journey. Wireless LANs are fitted into such ferries which serve the wireless users in the background. One of the important techniques to analyze such systems is via polling models. However FWLAN offers the service while on the move, where as in existing polling models (Bin Tariq *et al.* (2006); Kavitha and Combes (2013) and references therein) the server stops to attend the encountered customer. We consider the analysis of the former. Thus our system models the FWLANs realistically.

Of all the polling models, studied so far in literature, the one nearest to the serve on the move (SoM) FWLAN would be the time limited polling system, in which the queue visit times are limited. The time limited polling systems are studied under the assumptions of exponential visit times (e.g., Leung (1994); De Souza e Silva *et al.* (1995); Frigui and Alfa (1998); Eliazar and Yechiali (1998); Al Hanbali *et al.* (2012) etc.,), where as the visit times of the SoM FWLAN hardly exhibit any memoryless property. In fact they result because of traversing more or less the same length of interval in all the cycles and hence the random variations in the visit times can be better modeled using either constant times, or using bounded random variables with moderate values of variance. We consider an autonomous version of time limited system (Al Hanbali *et al.* (2008); de Haan *et al.* (2009) etc.,), wherein the server visit times are independent of the workload conditions in the queue visited. And this is exactly how a visit by the server to a queue occurs in the FWLAN.

The performance of SoM FWLAN predominantly depends upon the service intervals of various queues and their intersections. We showed that the system is stable when the combined load into a set of the intersecting queues is less than the load drained out per cycle, while the server is traversing over the combined service interval. We would also require stability condition of any subset of the intersecting queues. Further, we obtain approximate workload analysis of general time limited polling systems, and demonstrate the accuracy of the approximation using simulations. Finally, we obtain workload analysis of SoM FWLAN for some scheduling policies and also discuss optimality.

1.3 Organisation of the thesis

This thesis is organised as follows. Chapter 2 presents the system description and preliminary analysis. Sections 2.1-2.2 describe the model and related performance measures, bunching probability and passenger waiting times are discussed. Chapter 3 presents the stationary policies of bus bunching problem. Stationary analysis is considered in section 3.1. Important observations are made by using Monte Carlo based simulations in section 3.1.4.

Chapter 4 presents partially dynamic optimal dispatch policies. Time varying policies that do not depend upon system state are considered. We view the previous (trip) decisions as state and obtain policies which we refer as partially dynamic policies. Performance measures and overall cost are discussed in section 4.1. The optimal depot head way policy is obtained by solving appropriate dynamic programming equations using backward induction method. Monte-Carlo based study of the derived policies and their comparison is in section 4.3. The analysis corresponding to correlated travel times is in section 4.4.

Chapter 5 presents the fully dynamic policies related to bus bunching problem. We derived a near-optimal dynamic policy for small load factors by solving the corresponding finite horizon dynamic programming equations, using backward induction algorithm. These depend linearly upon the previous trip headways and the bus-inter arrival times at various stops corresponding to the latest trip whose information is available.

Chapter 6 presents the performance analysis of a serve on the move FWLAN. We derived the stability conditions when the service intervals (visibility interval of the user) are intersecting as well as for the case when they are not intersecting. We discussed the workload analysis of general time limited polling systems and obtained the workload under different optimal policies.

Conclusions and future research directions were presented in Chapter 7. Proofs and some relevant theorems and lemmas are in the Appendices A and B.

Chapter 2

Bus bunching: System description and preliminary analysis

In this chapter, we describe the system and derive some important preliminary performance measures: passenger waiting times and bunching probability.

2.1 System Model

We consider buses moving on a single route and this route has M number of stops represented by $Q_1, Q_2, \ldots, Q_M$. Each stop has infinite waiting capacity. Any bus starts at the depot (Q_0) and travels along a predefined cyclic path, while boarding/de-boarding passengers at the encountered bus stops. Passengers arrive independently of others in each stop Q_i according to a fluid process with rate λ. We also consider Poisson arrivals and show that the fluid model can well approximate the system with Poisson arrivals, using simulations. The passengers board the bus at 'fluid' rate. That is, the time taken to board x number of passengers equals bx, where b is the boarding time per passenger at any stop. Let S_k^i be the time taken by k-th bus to travel from $(i-1)$- th stop Q_{i-1} to i-th stop Q_i. These random travel times, $\{S_k^i\}_k$ (for each stop Q_i), are independent and identically distributed (IID). The buses start at the depot after given headway times (interval between two successive bus departures) and traverse through the M stops before concluding their trip. The total number of trips equal T, where we consider $T > M$. The main purpose of this Chapter is to obtain the sequence of 'depot-headways' (one for each trip) optimally. We further make the following assumptions to study the problem:

R.1) Surplus number of buses: the next bus can start at any specified headway in the depot, without having to wait for the return of the previous bus.

R.2) Parallel boarding and de-boarding: the time taken for de-boarding is smaller with probability one.

R.3) Gated service: Only the passengers that arrived before the arrival of the bus can board the bus. Passengers arrived during the boarding process, will wait for the next bus.

R.4) If buses are bunched at any stop then the second bus will wait till the previous bus departs, before it starts boarding its passengers. Thus overtaking of buses does not happen.

R.5) There is no constraint on the capacity of the buses.

In most of the cities the buses have two doors, boarding and de-boarding happens in parallel, and hence, one can neglect the de-boarding times. Boarding times are negligible compared to bus travel times. Thus we will have negligible number of arrivals during a boarding time, and hence gated service is a reasonable assumption. Assumption **R**.*1 simplifies the model sufficiently and is instrumental in deriving closed-form expressions for performance measures of this kind of a complicated system.* Without this assumption one would have to take care of the looping effects. Assumption of availability of few extra buses can easily ensure **R**.1 is satisfied. Even availability of one/two extra buses can ensure such a condition is satisfied with sufficiently large probability, and is often a well practised method to care of some eventualities. The systems usually operate with small bunching probabilities, and so the event in assumption **R**.4 is a rare event. Further this is a common practice in many transportation systems like Trams, metros, local train etc and to a large extent even in bus transportation systems. Assumption **R**.5 can be restrictive, but is a commonly made assumption in literature (Hickman (2001); Fu *et al.* (2003); Sun and Hickman (2005); Xuan *et al.* (2011); He (2015)) and one can consider relaxation of this for the future work.

Bunching

Because of variability in demand and traffic conditions, some buses are delayed. They arrive later than their scheduled time at some stop. Delay of the bus results in more number of passengers to be boarded, and hence longer dwell time at the stop. Thus it gets delayed further for the next stops. This would also imply smaller dwell time for the following bus at the same stop, as it has to board lesser number of passengers. This can continue for the following bus-stops, the bus-stop headway times (time gap between two consequent buses at the same stop) become smaller and can eventually become zero. This phenomenon is called *Bus bunching*. We define bunching probability as the probability of occurrence of this event. *Bunching probability for the k-th trip at i-th stop, b_k^i, is defined as the probability that the dwell time of $(k-1)$-th bus at stop Q_i is greater than the inter arrival time between $(k-1)$ and k-th buses to the same stop.*

Waiting times

Passengers wait for the bus at every bus stop. When a bus is delayed their waiting time increases. If the delay results in bunching, the waiting times can be longer. Further the waiting times also depend upon the depot-headway time. The larger this headway is, the longer are the waiting times. We define the *(passenger) waiting times as the time difference between their arrival instance (at their stop) and the arrival instance of the bus in which they board.* Let $W_{n,k}^i$ be the waiting time of the n-th passenger that arrived to bus stop Q_i in the k-th trip and let X_k^i be the number of passengers that arrived to stop Q_i which board the k-th bus. Thus the number of passengers that availed service in the first T trips at stop Q_i equals $\sum_{k\leq T} X_k^i$, and hence the customer average of these waiting times specific to stop Q_i over T trips equals:

$$\frac{\sum_{k\leq T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k\leq T} X_k^i}. \tag{2.1}$$

When the depot-headway increases bunching happens less often. However the passengers have to wait longer. We precisely study this trade-off. We derive the required performance measures, and, obtain headway times that optimize a weighted combination of the two performance measures with an aim to obtain the 'Pareto frontier'.

2.2 System Dynamics and Trip-wise Performance

We define $\phi = \{h_k\}_{1\leq k\leq T}$ as the headway policy, which is a sequence of headway gaps between consequent trips at depot; h_k is the time gap between $(k-1)$-th and k-th bus departures at depot. We first derive the trip-wise performance for a given (deterministic) headway policy[1].

Dwell time is the amount of time spent by a bus in the stop. By **R**.2, it equals total boarding time of passengers waiting at the bus stop. Recall X_k^i is the number of passengers waiting at stop Q_i, at the arrival instance of k-th bus. Because of gated service and fluid boarding at rate b the dwell time equals:

[1] Performance under fully dynamic (true-state dependent) headway policies is considered in Chapter 5.

$$V_k^i = X_k^i b. \tag{2.2}$$

By assumption **R**.4, the buses serve one after the other. That is, in the event of bunching, the trailing bus starts boarding its customers only after the preceding one departs. In such events the dwell times would be bigger than $X_k^i b$. However transport systems typically operate with small bunching chances (typically less than 10%) and hence *we neglect the influence of these extra terms in the dwell times.* We will in fact show via simulation that in-spite of this inaccurate modelling the theoretical performance well matches the one estimated using Monte Carlo simulations in (sub) sections 3.1.4 from Chapter 3 and 4.3 from Chapter 4.

2.2.1 Bus inter-arrival times

Let $\mathcal{N}^i(I)$ be the number of passengers that arrived in an interval of length I at Q_i for any $i \leq M$. For fluid arrivals this equals $\mathcal{N}^i(I) = \lambda I$, while, for Poisson arrivals $\mathcal{N}^i(I)$ is Poisson distributed with parameter λI.

We now describe the inter-arrival times of the buses at various stops and for various trips. We begin with stop Q_1 and trips other than the first trip. The k-th bus departs the depot after $(k-1)$-th bus, with a time gap equal to the (depot) headway time h_k (see Figure 2.1), and it reaches stop Q_1 after further travelling for S_k^1 amount of time. Thus the inter arrival time between $(k-1)$-th bus and k-th bus at Q_1 equals,

$$I_k^1 = h_k + S_k^1 - S_{k-1}^1.$$

By Gated service, the number of passengers served by k-th bus at Q_1 exactly equals the number that arrived during this inter-arrival time, i.e., $X_k^1 = \mathcal{N}^1(I_k^1)$. Also, note that the passengers that arrived during the dwell time ($V_k^i = X_k^i b$) of k-th bus would be served by $(k+1)$-th bus.

It is clear that the k-th bus takes an amount of time given by

$$\sum_{1 \leq j \leq i} S_k^j + \sum_{1 < j < i} V_k^j,$$

to reach stop Q_i after its departure at depot, and the headway (time gap) between k-th and $(k-1)$-th buses at the depot equals h_k. Thus the inter-arrival time between k-th and $(k-1)$-th buses at stop Q_i,

$$I_k^i = h_k + \sum_{1 \leq j \leq i} S_k^j + \sum_{1 < j < i} V_k^j - \left(\sum_{1 \leq j \leq i} S_{k-1}^j + \sum_{1 < j < i} V_{k-1}^j \right). \tag{2.3}$$

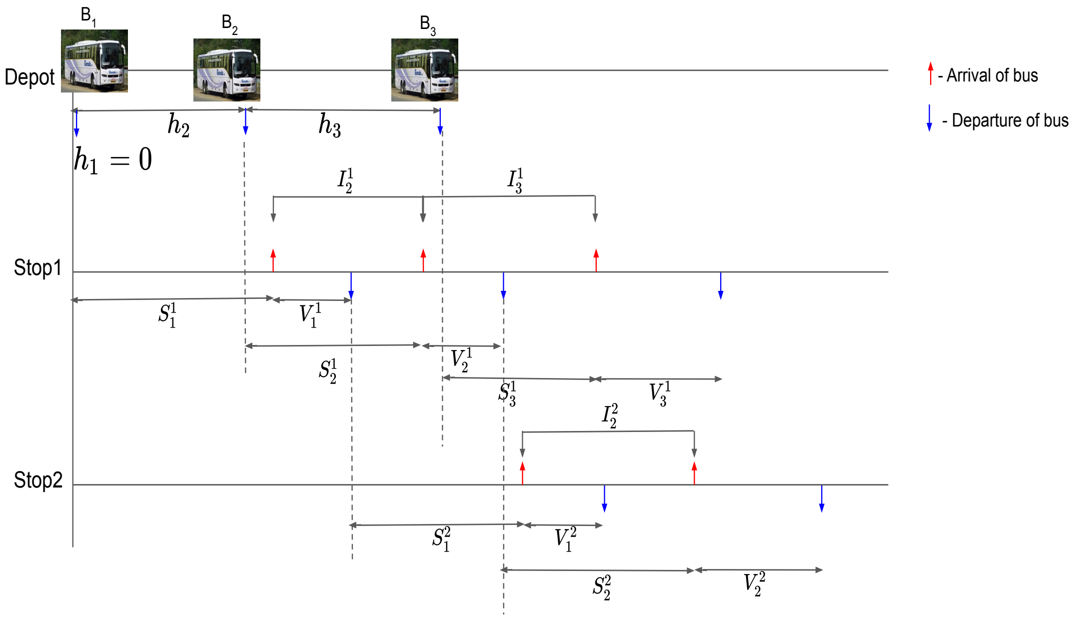

Figure 2.1: System model

Hence the number of passengers waiting at stop Q_i at the arrival instance of k-th bus equals, $X_k^i = \mathcal{N}^i(I_k^i)$.

There are *random fluctuations in travel times based on traffic conditions. We model these fluctuations by Gaussian random variables.* To be precise we assume the travel time by k-th bus between i and $(i-1)$-th bus stop to be $S_k^i = s^i + N_k^i$, where $\{N_k^i\}_{i,k}$ are IID Gaussian random variables with mean zero and variance β^2 and $\{s^i\}_i$ are constants. For fluid arrivals[2] and Gaussian travel times, the inter arrival times from equation (2.3) are (with $\rho := \lambda b$):

$$
\begin{aligned}
I_k^i &= h_k + \sum_{1 \le j \le i} N_k^j + \sum_{1 \le j < i} V_k^j - \sum_{1 \le j \le i} N_{k-1}^j - \sum_{1 < j < i} V_{k-1}^j, \\
&= h_k + \sum_{1 \le j \le i} N_k^j + \rho \sum_{1 \le j < i} \left(I_k^j - I_{k-1}^j \right) - \sum_{1 \le j \le i} N_{k-1}^j. \qquad (2.4)
\end{aligned}
$$

In the above, by notation, we set $N_k^i = 0$ for all $k \le 0$.

The above model considers independent travel times across various trips. In section 4.4 we consider analysis with correlated travel times.

[2] This is approximately true even for Poisson arrivals, as the inter bus-arrival times (at the same stop) are usually large, and then the number of passengers that arrived during this inter bus-arrival time can be approximated by λ times the inter-arrival time by elementary renewal theorem.

First trip details:

These details are different from the other trips and we describe the same now. Assume the system starts at time 0. The passenger arrivals start at time 0 as well as the first bus leaves the depot at time 0. Inter-arrival times $\{I_k^i\}$ were defined to facilitate computing the number of passengers boarding at respective stops and in respective trips. For the first trip, *the equivalent of inter-arrival times is the arrival instance of the first bus at respective stops. This is appropriate because the first bus has to serve all the waiting passengers, that have arrived since the system started.* In other words the first bus serves, at any stop, those passengers that have arrived between time 0 and the arrival instance of the first bus at that stop. Thus we define $\{I_1^i\}$ (for simplicity the same notation is used) as actually the bus arrival time w.r.t 0 at different stops, as below:

$$I_1^i \;=\; h_1 + \sum_{1 \le j \le i} \left(N_1^j + s^{(j)} \right) + \rho \sum_{1 \le j < i} I_1^j, \tag{2.5}$$

where by definition of the starting process *we set $h_1 = 0$ and $N_k^i = 0$ for all $k \le 0$ and for any $i \le M$.* We would like to stress again that these are not inter-arrival times, but are the arrival instances corresponding to the first trip.

From the above discussions, *even with independent travel times (and passenger arrivals), the bus inter-arrival times and hence the dwell times are correlated for all trips.* One needs to study these correlations to obtain the performance, and we begin with the following:

Lemma 2.1. *Assume $N_k^i = 0$ when $k \le 0$ and $h_1 = 0$. The inter-arrival times (or the arrival times for the first trip) can be expressed in terms of the relevant Gaussian walking components $\{N_k^i\}$ and the fixed inter-stop distances $\{s^j\}$ as below (for any trip k and for any stop i):*

$$I_k^i \;=\; \sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i \ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} + \sum_{r=0}^{i-1} \left[(1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{k-l}^{i-r} \right], \tag{2.6}$$

where

$$\gamma_l^i \;\triangleq\; (-1)^l \binom{i-1}{l} \rho^l (1+\rho)^{i-1-l}, \tag{2.7}$$

$$\mu_l^r \;\triangleq\; (-1)^l \sum_{u=0}^{r} \binom{r}{u+l-1} \binom{u+l}{l} \rho^{u+l-1}, \text{ with } \tbinom{n}{r} := 0 \text{ when } n < r. \tag{2.8}$$

Thus the mean and variance of inter arrival time for any stop i ($1 \le i \le M$) and for any trip k

are given by,

$$
\begin{aligned}
E[I_k^i] &= \sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i\ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \text{ and} \\
\left(\varrho_k^i\right)^2 &:= E\left(I_k^i - E[I_k^i]\right)^2 = \beta^2 \sum_{r=0}^{i-1} \left[(1+\rho)^{2r} + \sum_{l=1}^{\min\{r+1,k-1\}} \left(\mu_l^r\right)^2 \right].
\end{aligned} \tag{2.9}
$$

Proof is in Appendix A. ■

2.2.2 Passenger waiting times

As already mentioned, waiting times are the times for which a typical passenger waits, before its bus arrives. We first discuss the trip-wise passenger waiting times. Towards this we gather together the waiting times of the passengers that arrived during one (bus) inter-arrival time. Let the sum of the waiting times of the passengers that arrived during this trip and their customer average respectively be (notations as used in (2.1)):

$$
\bar{W}_k^i \triangleq \sum_{n=1}^{X_k^i} W_{n,k}^i \quad \text{and} \quad \bar{w}_k^i \triangleq \frac{\sum_{n=1}^{X_k^i} W_{n,k}^i}{X_k^i}.
$$

Fluid arrivals: The customers are assumed to arrive at regular intervals (of length $1/\lambda$) with large λ. The waiting time of the first passenger during bus inter-arrival period (I_k^i) is approximately[3] I_k^i, that of the second passenger is approximately $I_k^i - 1/\lambda$ and so on. Further, if the time duration I_k^i is large in comparison with $1/\lambda$ then $X_k^i \approx \lambda I_k^i$. Thus as $\lambda \to \infty$, the following (observe it is a Riemann sum) converges:

$$
\frac{\bar{W}_k^i}{\lambda} = \frac{1}{\lambda} \sum_{n=0}^{\lambda I_k^i} \left(I_k^i - \frac{n}{\lambda} \right) \to \int_0^{I_k^i} (I_k^i - x)dx = \frac{(I_k^i)^2}{2}. \tag{2.10}
$$

Thus for large λ,

$$
X_k^i = \mathcal{N}^i(I_k^i) \approx \lambda I_k^i, \quad \bar{W}_k^i \approx \lambda \frac{(I_k^i)^2}{2} \quad \text{and} \quad \bar{w}_k^i \approx \frac{I_k^i}{2}, \tag{2.11}
$$

and we *use this approximation throughout as the fluid approximation.*

Poisson arrivals: For Poisson arrivals, due to memoryless property, the conditional expectation as given by Lemma 8.1 of Appendix A,

$$
E\left[\bar{W}_k^i \middle| I_k^i\right] = \lambda \frac{\left(I_k^i\right)^2}{2} \text{ and } E\left[\bar{w}_k^i \middle| I_k^i\right] = \frac{I_k^i}{2}. \tag{2.12}
$$

[3] The residual passenger inter-arrival times at bus-arrival epochs get negligible as $\lambda \to \infty$.

In all (for Poisson as well as Fluid arrivals), the expected values of the trip-wise quantities corresponding to waiting times of passengers of stop Q_i in the k-trip equal

$$E\left[\bar{W}_k^i\right] = \lambda \frac{E\left(I_k^i\right)^2}{2} \text{ and } E[\bar{w}_k^i] = \frac{E[I_k^i]}{2}. \tag{2.13}$$

The quantity $E[\bar{w}_k^i]$ is the expected value of the customer average of the passenger waiting times corresponding to trip k and stop i. Thus these performance measures are the same for fluid as well as Poisson arrivals, however the analysis of the inter-arrival times $\{I_k^i\}$ is available only for fluid arrivals.

Reaching stationarity: From Lemma 2.1, we observe that the variance of I_k^i and its expected value depend upon the trip number. The system is in transient behaviour for the first M trips and reaches a kind of (variance) stationarity after M trips: *for the first M trips the variance changes with trip, while the variance is the same for the rest.* When one considers constant/deterministic headway times (as required for stationary analysis) then the process becomes completely stationary after the first M trips. To be precise from (2.13) and Lemma 2.1, *the trip-wise waiting time performance measures are the same for all trips $k > M$, under constant headways.*

2.2.3 Bunching probability

Bunching is said to occur at a stop when two buses meet, i.e., when the headway time (time gap between buses) of consequent buses becomes zero at that stop. Bunching probability, b_k^i, of k-th bus at i-th stop is the probability that the dwell time V_{k-1}^i (equation (2.2)) of $(k-1)$-th bus is greater than the inter arrival time (equation (2.4)) between $(k-1)$ and k-th buses. Thus for fluid arrivals (by (2.11)):

$$b_k^i = P(\mathcal{N}^i(I_{k-1}^i)b > I_k^i) \; = \; P(I_k^i - \rho I_{k-1}^i < 0). \tag{2.14}$$

By first conditioning on (I_k^i, I_{k-1}^i) and simplifying, we get the same expression for bunching probability with Poisson arrivals[4]. The above expression is true because of assumption **R**.4. Thus we require the (marginal) distribution of $(I_k^i - \rho I_{k-1}^i)$ for computing the bunching probability at any stop Q_i and for any trip k:

[4] However, as already mentioned, further analysis is applicable only for fluid arrivals. Thus to extend the analysis to Poisson arrivals, one needs to first study the inter-arrival times $\{I_k^i\}$ given by (2.4) with Poisson arrivals, for which the dwell times do not satisfy $V_k^i = \rho I_k^i$.

Lemma 2.2. *Recall the constants from equations (2.7) and assume $N_k^i = 0$ when $k \leq 0$, $h_1 = 0$. Then*

$$
\begin{aligned}
I_k^i - \rho I_{k-1}^i &= \sum_{l=0}^{\min\{i,k-1\}} \tilde{\gamma}_l^i h_{k-l} + \varpi_k^i + \sum_{r=0}^{i-1} \left((1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{k-l}^{i-r} \right) \\
&\quad - \rho \sum_{r=0}^{i-1} \left((1+\rho)^r N_{k-1}^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{k-1-l}^{i-r} \right), \qquad (2.15)
\end{aligned}
$$

where, $\tilde{\gamma}_l^i \triangleq \gamma_l^i - \rho \gamma_{l-1}^i \mathbb{1}_{l>0}$ *and*

$$
\varpi_k^i = \begin{cases} \mathbb{1}_{i \geq k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} - \mathbb{1}_{i \geq k-1} \rho \sum_{j=1}^{i-k+2} \gamma_{k-2}^{i-j+1} s^{(j)} & \text{if } k \leq M \\ \rho \sum_{j=1}^{i-M+1} \gamma_{M-1}^{i-j+1} s^{(j)} & \text{if } k = M+1 \\ 0 & \text{if } k > M+1. \end{cases} \qquad (2.16)
$$

The mean and variance for $k \geq 2$ respectively given by,

$$
\begin{aligned}
E[I_k^i] - \rho E[I_{k-1}^i] &= \sum_{l=0}^{\min\{i-1,k-1\}} \tilde{\gamma}_l^i h_{k-l} + \varpi_k^i \quad \text{and} \\
\left(\sigma_k^i\right)^2 &= \beta^2 \sum_{r=0}^{i-1} \left\{ (1+\rho)^{2r} + \sum_{l=1}^{\min\{r+2,k-1\}} \left(\tilde{\mu}_l^r\right)^2 \right\}. \qquad (2.17)
\end{aligned}
$$

where (see 6.2),

$$
\tilde{\mu}_l^r \triangleq \mu_l^r - \rho \mu_{l-1}^r \mathbb{1}_{l>0}.
$$

Proof: is direct from Lemma 2.1. ■

From equation (2.14) and Lemma 2.2, the bunching probability for any $2 \leq k \leq T$ and $1 \leq i \leq M$ is equals to,

$$
b_k^i = 1 - \Phi_k^i \left(E[I_k^i] - \rho E[I_{k-1}^i] \right), \qquad (2.18)
$$

where Φ_k^i is the cdf of a normal random variable with mean 0 and variance $\left(\sigma_k^i\right)^2$ given by (2.17):

$$
\Phi_k^i(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi \left(\sigma_k^i\right)^2}} \exp\left(\frac{-t^2}{2\left(\sigma_k^i\right)^2} \right) dt.
$$

Once again the variances $\{\sigma_k^i\}_k$ are the same for trips $k > M+1$ (we need one more trip because of dependency on I_{k-1}^i) and hence observe that $\Phi_k^i = \Phi_M^i$ for all trips greater than, $k > M+1$. Once again under constant/stationary deterministic headways, *the bunching probability is the same for all trips with $k > M+1$.*

In this chapter, we described the system and obtained stationary as well as transient (suitable for finite trip problems) performance measures. We will refer the analysis with constant headways as stationary analysis and the same is considered in Chapter 3, while the analysis with general head-way policies (headways depend on trips) is referred to as transient analysis and the corresponding study is available from Chapter 4 onwards.

Chapter 3

Bus bunching: stationary policies

In this chapter, we primarily derive the optimal stationary headway between successive buses at depot, that minimizes the given weighted combination of the two costs: a) the average passenger waiting times; and b) the probability of bunching. By a stationary policy of headways, we meant that the buses are dispatched at equal intervals of times from the depot and by optimal stationary policy we meant the time gap between successive bus departures that minimizes the given weighted combination of the two costs.

3.1 Stationary Analysis (Trip/Customer averages)

When T, the total number of trips, is sufficiently large in comparison with M, the number of stops, the stationary analysis might be sufficient. For this case, it is sufficient to consider constant/stationary headway policies (i.e., $h_k = h$ for some $h < \infty$ and for all trips k). In Chapter 4 (section 4.2.2) we will show (for the case with large number of trips T) that the optimal among stationary policies provides (sufficiently) near optimal performance, even when optimized among non-stationary headway policies. However this observation is true for very limited cases, for e.g., the case with small load factors and high traffic variability (see Figure 4.3, in section 4.3 from Chapter 4). Further, the stationary policies have independent importance: practically it is more convenient to implement stationary policies which is a common practice in many systems. *In all, we begin with analysis using stationary policies.*

Under such stationary policies, from (2.6) and (2.9) of Lemma 2.1 from Chapter 2 the inter-arrival times and their moments for trips ($k > M + 1$), simplify as below (from Lemmas

Results of this chapter have been published in the proceedings of COMSNETS-2018 (Koppisetti *et al.* (2018)).

2.1-2.2 and equation (2.18) from Chapter 2 with constant headways and note $\sum_{l=0}^{i} \tilde{\gamma}_l^i = 1 - \rho$ etc.):

$$
\begin{aligned}
I_k^i &= h + \sum_{r=0}^{i-1}\left[(1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{k-l}^{i-r}\right], \text{ with expected value,} \quad (3.1)\\
E[I_k^i] &= h \text{ and variance, } \left(\varrho_k^i\right)^2 = \varrho_i^2 \text{ with}\\
\varrho_i^2 &:= \beta^2 \sum_{r=0}^{i-1}\left[(1+\rho)^{2r} + \sum_{l=1}^{r+1}(\mu_l^r)^2\right] \text{ and further,}\\
I_k^i - \rho I_{k-1}^i &= h(1-\rho) + \sum_{r=0}^{i-1}\left((1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-l}^{i-r}\right)\\
&\quad -\rho \sum_{r=0}^{i-1}\left((1+\rho)^r N_{k-1}^{i-r} + \sum_{l=1}^{r+1}\mu_l^r N_{k-1-l}^{i-r}\right)\\
E[I_k^i - \rho I_{k-1}^i] &= h(1-\rho) \text{ and } \left(\sigma_k^i\right)^2 = \left(\sigma^i\right)^2 \text{ with}\\
\left(\sigma^i\right)^2 &:= \beta^2 \sum_{r=0}^{i-1}\left\{(1+\rho)^{2r} + \sum_{l=1}^{r+2}(\tilde{\mu}_l^r)^2\right\}.
\end{aligned}
$$

It is important to observe here that all the above moments do not depend upon the trip number for trips $k > M+1$. In other words, $\{I_k^i\}_{k>M+1}$ and $\{I_k^i - \rho I_{k-1}^i\}_{k>M+1}$ form stationary Gaussian sequences for any stop Q_i. In fact $\{I_k^i\}_{k>M}$ itself is stationary, however it is convenient to consider common set of indices $\{k \geq M+1\}$.

Thus the *system is in transience only for the first* $M+1$ *trips, under stationary headway policies.* To be more precise, any expected performance measure related to a single trip is the same for all trips other than the first M trips. Passenger waiting times related to a trip can be one such example. On the other hand for performance measures like bunching probability, which depend upon two consecutive trips, the stationarity is reached after $M+1$ trips. Hence again the bunching probability of k-th trip equals stationary bunching probability, for all $k > M+1$.

Define the following Gaussian vectors, corresponding to trip k and trips k to l respectively as below,

$$\mathbb{N}_k := [N_k^1, N_k^2, \cdots, N_k^M] \text{ and } \mathbb{N}_l^k := [\mathbb{N}_k, \mathbb{N}_{k-1}, \cdots, \mathbb{N}_l]\,.$$

The bus inter-arrival times $\{I_k^i\}_{k>M+1}$, for any given stop Q_i, are stationary but are not independent as seen from equation (3.1). Nevertheless, from the same equations, inter-arrival times of a trip, $\mathbb{I}_k := [I_k^1, I_k^2, \cdots, I_k^M]$, depend only upon $\mathbb{N}_{k-M}^k$ and so the sampled inter-arrival times

$$\{I_{j+kl}^i\}_{k\geq 1} = I_{l+j}^i, I_{2l+j}^i \cdots (\text{ for any stop } Q_i),$$

with $l > M+1$ and for any $0 \leq j \leq l-1$ form an IID Gaussian sequence.

One can also derive the time (trip) average of any performance measure, using the above 'block' IID characteristics. By Law of large numbers, for any (integrable) performance f that depends (for example) upon one trip (almost surely (a.s.)):

$$
\begin{aligned}
\bar{f} &:= \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} f(\mathbb{I}_k) \\
&= \lim_{K\to\infty} \frac{1}{K(M+1)} \sum_{j=1}^{M+1} \sum_{k=1}^{K} f(\mathbb{I}_{j+k(M+1)}) \stackrel{a.s.}{=} E[f(\mathbb{I}_{M+1})].
\end{aligned} \tag{3.2}
$$

In the above the expectation is with respect to the Gaussian random variables of equation (3.1). One can derive trip average of the performance measures that depend upon finite number of consecutive trips (e.g., bunching probability) in a similar way.

We find the stationary optimal depot headway time and the non stationary case is discussed in the next section.

3.1.1 Customer average of Waiting times

This performance is important from the perspective of passengers and hence it is more appropriate to consider the 'passenger' average of the waiting times defined in (2.1) from Chapter 2. Observe that equation (2.1) from Chapter 2 refers to the customer average corresponding to all the T trips. We however consider the average defined in the second term of the equation (2.13) from Chapter 2, i.e., consider the averages $\{\bar{w}_k^i\}_i$. These terms are the customer average corresponding to a specific trip k, and we later consider the average of these averages across all T trips. This is done because of the following two reasons: a) it will be seen that it is relatively easier to handle these second averages in optimization problems; b) the two sets of optimization problems are approximately equivalent, as shown in the following (for Poisson as well as Fluid arrivals, the two expected values are approximately the same, as $(\varrho_i)^2/2h$ is typically negligible).

Lemma 3.1. *For any stop Q_i, under any stationary policy $\{h\}$:*

$$\begin{aligned}
\bar{W}^i &= \lim_{T\to\infty} \frac{\sum_{k=1}^T \bar{W}_k^i}{\sum_{k=1}^T X_k^i} = \lim_{T\to\infty} \frac{T}{\sum_{k=1}^T X_k^i} \frac{\sum_{k=1}^T \bar{W}_k^i}{T} \stackrel{a.s.}{=} \frac{E[(I_k^i)^2]}{2E[I_k^i]} = \frac{\left(\varrho_k^i\right)^2 + \left(E[I_k^i]\right)^2}{2E[I_k^i]} \\
&= \frac{\left(\varrho_k^i\right)^2}{2E[I_k^i]} + \frac{E[I_k^i]}{2} \text{ and thus} \qquad (3.3) \\
\bar{W}^i &= \frac{\left(\varrho_i\right)^2}{2h} + \frac{h}{2} \text{ and similarly,} \qquad (3.4) \\
\bar{w}^i &= \lim_{T\to\infty} \frac{1}{T} \sum_{k\le T} \bar{w}_k^i = \lim_{T\to\infty} \frac{1}{T} \sum_{k\le T} \frac{\bar{W}_k^i}{X_k^i} = \frac{E[I_k^i]}{2} = \frac{h}{2} \text{ a.s.} \qquad (3.5)
\end{aligned}$$

Proof: is in Appendix A. ■

Remarks: 1) Under suitable conditions, for example when the depot headway time is large and/or the small traffic variability (β), one can neglect the first term in the equation (3.4). Hence, $\bar{W}^i \approx \bar{w}^i = h/2$;

2) We would like to give equal importance to passengers of all stops. Hence we consider the following for optimization purposes:

$$\bar{w} = \sum_{i=1}^M \bar{w}^i = \sum_{i=1}^M \frac{E[I_k^i]}{2}. \qquad (3.6)$$

3.1.2 Bunching probability under stationarity

As already discussed, various trips can be correlated, however the bunching probabilities in different trips remains the same. This is because the bunching probabilities depend only upon $I_k^i - \rho I_{k-1}^i$ and because these are identically distributed for all $k > M + 1$, under stationarity. For all such trips the bunching probability of a stop in a trip is the same and equals (see (3.1)),

$$b_k^i = b^i \text{ with } b^i := 1 - \Phi_M^i \left(h(1-\rho)\right). \qquad (3.7)$$

As mentioned already, this also represents *the bunching probability of a trip under stationarity.* Note that the bunching probabilities of initial trips can be different, and this is considered in section 4.1 of Chapter 4.

3.1.3 Total cost and optimization

Our aim is to minimize a joint cost that considers both the components given by (3.6) and (3.7). Towards this we consider a weighted average of the two costs with α representing the trade-off parameter:

$$r=\sum_{i=1}^{M}\frac{h}{2}+\alpha\left(1-\Phi_M^M\left(h(1-\rho)\right)\right)=\frac{Mh}{2}+\alpha\ \left(1-\Phi_M^M\left(h(1-\rho)\right)\right). \tag{3.8}$$

Bunching probability is maximum at last stop, other stops have much lesser probability[1] and hence we consider only bunching probability of the last stop. *To simplify the notation we refer Φ_M^M by Φ_M.* The performance at optimizers for different α gives Pareto frontier as will be discussed in sub section 4.1.3 of Chapter 4. Let h_s^* be the minimizer for total cost (3.8). The total cost is a differentiable function and hence h_s^* is either the zero of the following derivative (obtained using Leibniz rule) or is on the boundary:

$$\frac{dr(h)}{dh}=\frac{M}{2}-\left(\frac{\alpha}{\sqrt{2\pi}}\exp\left(\frac{-h^2(1-\rho)^2}{2\left(\sigma^M\right)^2}\right)\frac{1-\rho}{\sigma^M}\right). \tag{3.9}$$

If $\alpha<M\sqrt{2\pi}\sigma^M/2(1-\rho)$ then the first derivative is always positive and hence the optimizer is at lower boundary 0. Otherwise, there exists a zero of the derivative h_s^* as below. One can easily verify that $d^2r(h)/dh^2>0$ for all h and $\alpha>M\sqrt{2\pi}\sigma^M/2(1-\rho)$, hence that h_s^* *is the unique minimizer.*

$$h_s^*=\begin{cases}\sqrt{\frac{-\log(C_1)}{C_2}}, \text{ with } C_1=\frac{M\sqrt{2\pi}\sigma^M}{2\alpha(1-\rho)} \text{ and } C_2=\frac{(1-\rho)^2}{2(\sigma^M)^2}, & \text{if } \alpha\geq M\sqrt{2\pi}\sigma^M/2(1-\rho)\\ 0 & \text{else.}\end{cases} \tag{3.10}$$

We use sub-script h_s^* to indicate that this is the 'stationary optimal policy'.

Remarks: It is immediately clear that the optimal bus frequency (inverse of h_s^*) decreases as the number of stops increase. Similarly, the optimal frequency decreases with increase in traffic variability factor β (see (3.10)).

3.1.4 Monte-Carlo Simulations

In this section, we verify the derived performance measures of the proposed model through Monte-Carlo (MC) simulations. We emulate the buses travelling on a single route with 8 bus stops, boarding a random number of passengers using gated service and avoiding parallel boarding (into two or more buses simultaneously) of passengers at any stop. The travelling times between stops are perturbed by normally distributed noise with zero mean and variance β^2. Passenger arrivals are either due to fluid arrivals or due to Poisson arrivals, and we consider

[1] Note that the bunching probability is low at initial stops and increases with stop number; as seen from Lemma 2.2 of Chapter 2 (with constant headway policy), the bunching probability $\Phi_k^i(h(1-\rho))$ is increasing with stop, i. This is true for any trip k. Hence forth, we consider only bunching probability of last stop.

fluid boarding. We conduct the simulations with $M = 8, \lambda = 200,\ \rho = 0.3, s = 50$ and $\alpha = 150$ in Figure 3.1.

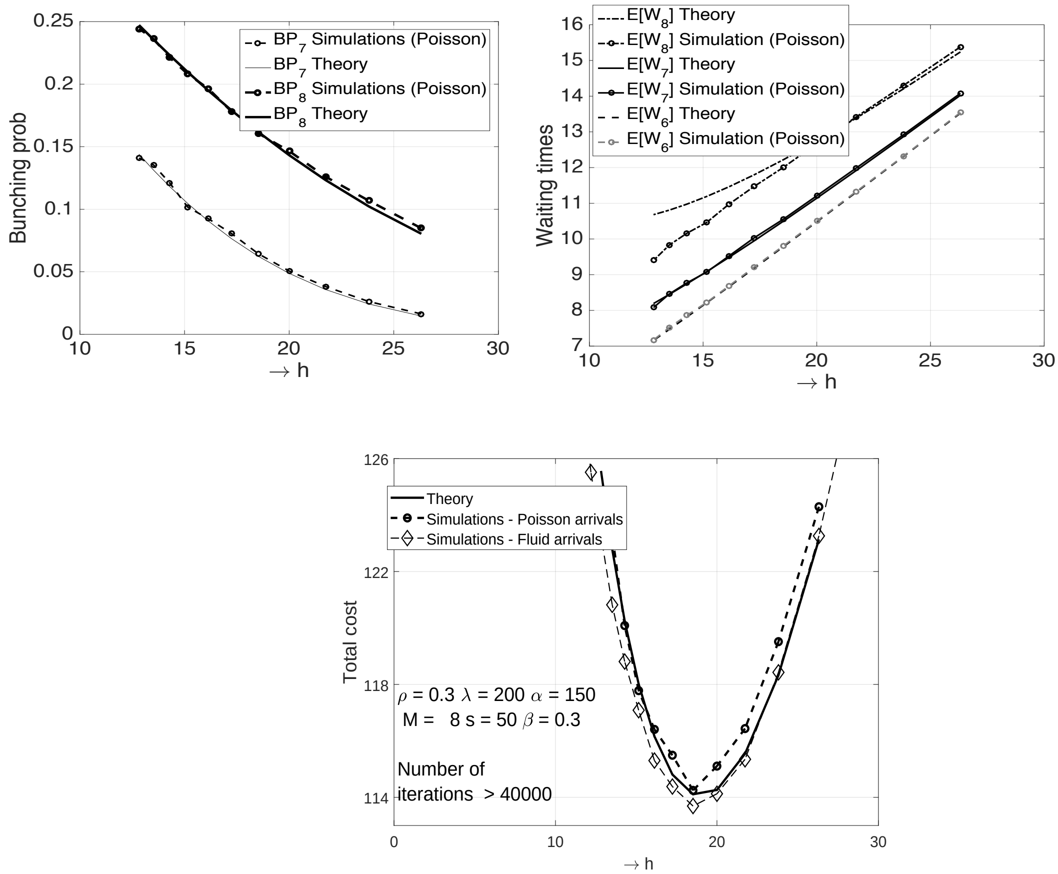

Figure 3.1: Bunching probability, Waiting time and Total cost vs headway

In Figure 3.1, we compared the theoretical quantities with the ones estimated through simulations. We plot bunching probability, average passenger waiting times and total cost respectively as a function of headway. We find a good match between theory (curves without markers) and simulations considering Poisson arrivals (curves with circular markers). We also conducted simulations using fluid arrivals. The simulation results with fluid arrivals better match the theoretical counterparts. We included the simulation based results for fluid arrivals (curves with diamond markers) in the third sub-figure, i.e., for total cost. We notice a good match between (both) the simulated quantities and the theoretical expressions in majority of the cases. These observations affirm the theory derived.

From the sub-figures of Figure 3.1, we observe that the bunching probability improves (decreases) with depot-headway (h), while, the passenger waiting times degrade (increase). This is the *inherent trade-off that needs to be considered to design an efficient system*. We plot optimal depot-headways for some examples in Figures 3.2-3.3, which are estimated using numerical simulations (dashed curves with circular markers). We also plot h_s^* given by (3.10) in the same figures (solid curves). We observe that theoretically derived h_s^* well matches the optimizers estimated using numerical simulations, for small load factors (ρ) and or small traffic variability (β). When variability increases the theoretical h_s^* (3.10) is not a very good approximation (for load factors bigger than 0.5 in Figure 3.2 and traffic variance $\beta > 5$ in Figure 3.3). Nevertheless, the error is significant only for $\beta > 10$ and approximation error is not very big for all ρ plotted in Figure 3.2.

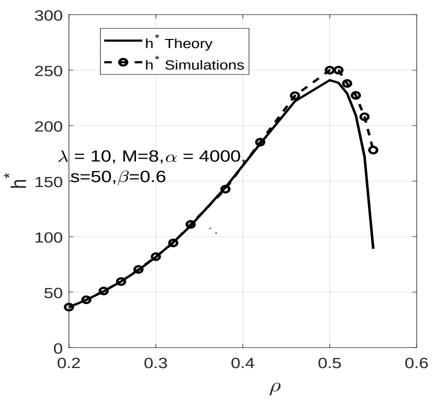

Figure 3.2: Optimal h^* versus load

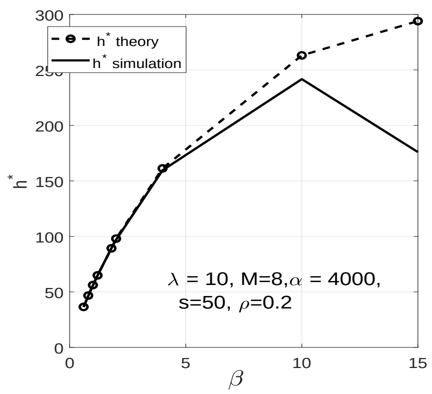

Figure 3.3: h^* versus traffic variability

Recall the passenger/customer average of waiting times corresponding to all the T trips is given by (2.1) from Chapter 2 and its expected value is given by (3.4). The MC simulations capture this average. However, as already discussed, we instead consider the two level average defined using (3.5). The difference between the two types of averages for the last stop equals $\varrho_k^M/(2h)$ (see Lemma 3.1), which becomes significant when the load factor/traffic variability is high. This is the reason for large approximation errors of Figures 3.2-3.3.

As the load factor increases, or equivalently when the customer arrival rates increase, one would anticipate an increased bus-frequency to be optimal. On the contrary we notice that the optimal headway increases (initially) with increase in load-factor in Figure 3.2. This is because the passenger waiting times for any given headway h, approximately equal $h/2$ when the variability components $(\{\varrho_k^i\}_i)$ due to traffic and or load conditions) are negligible (see

(3.4)). Thus with increase in load factor, the bunching probabilities increase sharply, while waiting times are less influenced and hence an increase in optimal depot-headway. However as seen in the same figure, when load increases beyond 0.5, the variability components in waiting times also become significant and now we notice that the optimal depot-headways are smaller. *To summarize, the optimal frequency of the buses decreases initially with increase in load, but for higher range of load factors it increases with load.*

3.2 Summary

We modelled the bus bunching problem with Gaussian bus travel times and fluid arrivals. We studied the related performance measures under stationary policies. Using numerical simulations, we showed that the performance of the system with Poisson arrivals can be well approximated with the derived theoretical expressions, when the arrival rates are large. We obtained the optimal depot headway time, i.e., the optimal bus frequency as a function of parameters like load conditions (passenger arrival rates), number of bus stops, traffic variability conditions (variance of the travel times) etc. We made the following observations using the theoretical as well as numerical study: a) When bus frequency decreases, bunching probability decreases and passenger waiting times increase; b) Optimal bus frequency decreases with increase in traffic variability and load conditions; and c) If the load is significantly larger, then the optimal bus frequency actually increases with load.

Chapter 4

Bus bunching: partially dynamic policies

We now consider non-stationary policies. In particular, we consider partially dynamic policies in this chapter[1]; the policies that depend upon the headways of the previous trips. Further we consider independent (as in previous chapter) as well as the correlated travel times. The rest of the details of the system model are the same as that in the previous chapters. we compare the Pareto frontier of both stationary policies (derived in last chapter) and the partially dynamic policies of this chapter. We observed huge improvement with partially dynamic policies in comparison with the stationary policies.

4.1 Finite Horizon (Transient) Problem

As opposed to the case study in the previous Chapter 3, we now consider the case when the number of trips T is not very large in comparison with M, the number of stops. As discussed previously, even with stationary policies, system is in transience behaviour for the first $M+1$ trips and stationary performance is reached only after $M+1$ trips. Further it may not be sufficient to consider stationary policies, when T is comparable with M. We actually notice, using numerical simulations, that the stationary optimal policy performs significantly inferior to the dynamic optimal policy (under nominal traffic variabilities and or when the load factors are significant) in section 4.3. Thus we consider headway policies that vary with time, however these do not depend upon (any) system state. Thus in a way these are dynamic policies that

Results of this Chapter and Chapter 3 are submitted to Transportaion Research:B (Koppisetti and Kavitha (2019)).

[1] Fully dynamic policies are in the next chapter.

For all $0 \leq l \leq M, 0 \leq r \leq M, 1 \leq i \leq M, 0 \leq k \leq T-1$
$\gamma_l^i \triangleq (-1)^l \binom{i-1}{l} \rho^l (1+\rho)^{i-1-l}$, $\mu_l^r \triangleq (-1)^l \sum_{u=0}^{r} \binom{r}{u+l-1} \binom{u+l}{l} \rho^{u+l-1}$, with $\binom{n}{r} := 0$ when $n < r$.
$\bar{\gamma}_l := \frac{1}{2} \sum_{i=l+1}^{M} \gamma_l^i$ and $\vartheta_k^M := \sum_{i=1}^{M} \frac{1}{2} \left(\mathbb{1}_{i \geq k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \right)$,
$\tilde{\gamma}_l^i \triangleq \gamma_l^i - \rho \gamma_{l-1}^i \mathbb{1}_{l>0}$ and $\varpi_k^i = \begin{cases} \mathbb{1}_{i \geq k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \\ -\mathbb{1}_{i \geq k-1} \rho \sum_{j=1}^{i-k+2} \gamma_{k-2}^{i-j+1} s^{(j)} & \text{if } k \leq M \\ \rho \sum_{j=1}^{i-M+1} \gamma_{M-1}^{i-j+1} s^{(j)} & \text{if } k = M+1 \\ 0 & \text{if } k > M+1. \end{cases}$
$\eta_l^{T-k} = \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1} \frac{\tilde{\gamma}_{l+1}^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_l, \quad a_{T-k} = \sigma_{T-k}^M \sqrt{-2 \log \left(\frac{\eta_0^{T-k} \sqrt{2\pi} \sigma_{T-k}^M}{\tilde{\gamma}_0^M \alpha} \right)}$, $\left(\sigma_{T-k}^M \right)^2 = \beta^2 \sum_{r=0}^{M-1} \left\{ (1+\rho)^{2r} + \sum_{l=1}^{\min\{r+2, T-k-1\}} \left(\tilde{\mu}_l^r \right)^2 \right\}$
$\delta^{T-k} = \alpha \left(1 - \Phi_k^M \left(a_{T-k} \right) \right) + \vartheta_k^M + \frac{\eta_0^{T-k} \left(a_{T-k} - \varpi_{T-k}^M \right)}{\tilde{\gamma}_0^M} + \delta^{T-k+1}$.

Table 4.1: Notations and coefficients summarized

depend upon economic/cheap/readily available information, the headways corresponding to the previous buses/trips.

In all, we have time varying policies $\phi := \{h_k\}_{1 \leq k \leq T}$ with h_k representing the depot headway between trip $(k-1)$ and k, and, we seek the optimal among such policies that minimizes a total cost, similar to the one defined in the previous chapter.

We require optimal policies among non-stationary (but state independent) policies, thus we pose it as a finite horizon (T) sequential control problem. We begin with derivation of the performance measures under these non-stationary (and finite horizon) policies. That is, we define the running cost corresponding to each trip and eventually the cost corresponding to all the trips. The main consideration of this chapter is to obtain optimal policies that depend just upon the headways of the previous trips and nothing else. *Thus these are not stationary (the headway is not the same for all trips), nor, are they fully dynamic (the headway of a trip does not depend upon any system state other than the previous trip headways).*

4.1.1 Passenger waiting times

The customer average (and the corresponding expected value) of the waiting times of the passengers belonging to Q_i and corresponding to the T trips equals:

$$\frac{\sum_{k=1}^{T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k=1}^{T} X_k^i} \text{ and } E^\phi\left[\frac{\sum_{k=1}^{T}\sum_{n=1}^{X_k^i} W_{n,k}^i}{\sum_{k=1}^{T} X_k^i}\right],$$

where E^ϕ is the expectation with respect to probability measure P^ϕ that is resultant when policy ϕ is used. In view of the Remarks following Lemma 3.1 from Chapter 3, we consider the following modified (approximate) average which also makes it mathematically tractable:

$$\frac{E^\phi\left[\sum_{k=1}^{T}\left(\frac{\sum_{n=1}^{X_k^i} W_{n,k}^i}{X_k^i}\right)\right]}{T} = \frac{1}{T}\sum_{k=1}^{T} E^\phi\left[\bar{w}_k^i\right].$$

Thus the waiting time-component of the running cost corresponding to trip k (corresponding to all stops) equals, $\sum_{i=1}^{M} E\left[\bar{w}_k^i\right]$. From Lemma 2.1 and equation (2.13) from Chapter 2, this component equals,

$$\sum_{i=1}^{M} E^\phi\left[\bar{w}_k^i\right] = \sum_{i=1}^{M}\frac{1}{2}E^\phi\left[I_k^i\right] = \sum_{i=1}^{M}\frac{1}{2}\left(\sum_{l=0}^{\min\{i-1,k-1\}}\gamma_l^i h_{k-l} + \mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right). \quad (4.1)$$

4.1.2 Bunching probability

The other component of running cost, under any given policy ϕ, is related to bunching. The average bunching cost across all the T trips at stop Q_i can be defined as the expected value of the fraction of trips that resulted in bunching at stop Q_i, i.e., as the following:

$$E^\phi\left[\frac{\sum_{k=1}^{T}\mathbb{1}_{V_{k-1}^i > I_k^i}}{T}\right] = \frac{\sum_{k=1}^{T} P^\phi\left(I_k^i - \rho I_{k-1}^i < 0\right)}{T}.$$

Thus the bunching component of the running cost corresponding to trip k (and stop Q_i) equals the corresponding bunching probability, $P\left(I_k^i - \rho I_{k-1}^i < 0\right)$ (see (2.14 from Chapter 2)). From Lemma 2.2 and equation (2.18) from Chapter 2, this component equals,

$$P^\phi\left(I_k^i - \rho I_{k-1}^i < 0\right) = 1 - \Phi_k^i\left(\sum_{l=0}^{\min\{i,k-1\}}\tilde{\gamma}_l^i h_{k-l} + \varpi_k^i\right). \quad (4.2)$$

As discussed in the previous section, the chances of bunching at initial stops is low, while that at later stops is significant. Thus the running cost component corresponding to bunching, for trip k is given by:

$$1-\Phi_k^M\left(\sum_{l=0}^{\min\{M,k-1\}}\tilde{\gamma}_l^M h_{k-l}+\varpi_k^M\right). \tag{4.3}$$

4.1.3 Pareto frontier and overall running cost

We thus have multiple objective functions, which are time averages of the bunching probability of last stop,

$$P_B(\phi):=\frac{1}{T}\sum_{k=1}^{T}P^{\phi}\left(I_k^M-\rho I_{k-1}^M<0\right),$$

and passenger average waiting times at all stops is equal to:

$$E_W(\phi):=\frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M}E^{\phi}\left[\bar{w}_k^i\right].$$

We are naturally interested in the Pareto frontier, the space of all 'efficient' points.

Pareto frontier is the efficient sub-region of any achievable region which consists of dominating performance vectors. In our context we say $(P_B(\phi^*), E_W^*(\phi^*))$, corresponding to policy ϕ^*, is a dominating pair of performance measures, if there exists no other policy ϕ which achieves a strictly better performance pair, i.e., such that

$$P_B(\phi)<P_B(\phi^*),\text{ and }E_W(\phi)\leq E_W(\phi^*)\text{ or vice versa.}$$

One of the common technique to obtain Pareto frontier is to solve a weighted combination of the two costs. Thus we consider the equivalent optimization of the following weighted combination, parametrized by α (from equations (4.1)-(4.3)):

$$\begin{aligned}P_B(\phi)+\alpha E_W(\phi) &= \frac{1}{T}\sum_{k=1}^{T}r_k\text{ with}\\ r_k &:= \sum_{i=1}^{M}\frac{1}{2}\left(\sum_{l=0}^{\min\{i-1,k-1\}}\gamma_l^i h_{k-l}+\mathbb{1}_{i\geq k}\sum_{j=1}^{i-k+1}\gamma_{k-1}^{i-j+1}s^{(j)}\right)\\ &\quad+\alpha\left(1-\Phi_k^M\left(\sum_{l=0}^{\min\{M,k-1\}}\tilde{\gamma}_l^M h_{k-l}+\varpi_k^M\right)\right),\end{aligned}$$

where r_k represents the overall running cost of trip k. This running cost depends upon the headway policy $\phi = \{h_k\}_{k \le T}$ and can be re-written as,

$$
\begin{aligned}
r_k(\phi;\alpha) &= \sum_{i=1}^{M} \frac{1}{2} \left(\sum_{l=0}^{\min\{i-1,k-1\}} \gamma_l^i h_{k-l} + \mathbb{1}_{i \ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \right) \\
&\quad + \alpha \left(1 - \Phi_k^M \left(\sum_{l=0}^{\min\{M,k-1\}} \tilde{\gamma}_l^M h_{k-l} + \varpi_k^M \right) \right) \\
&= \sum_{l=0}^{\min\{k,M\}-1} \bar{\gamma}_l h_{k-l} + \vartheta_k^M + \alpha \left(1 - \Phi_k^M \left(\sum_{l=0}^{\min\{k-1,M\}} \tilde{\gamma}_l^M h_{k-l} + \varpi_k^M \right) \right) \quad \text{with} \qquad (4.4)
\end{aligned}
$$

$$
\bar{\gamma}_l := \frac{1}{2} \sum_{i=l+1}^{M} \gamma_l^i \quad \text{and} \qquad \vartheta_k^M := \sum_{i=1}^{M} \frac{1}{2} \left(\mathbb{1}_{i \ge k} \sum_{j=1}^{i-k+1} \gamma_{k-1}^{i-j+1} s^{(j)} \right). \qquad (4.5)
$$

The coefficients are summarized in Table 4.1. Our objective is to equivalently optimize the summation of running costs (as T is a fixed number) corresponding to all trips in consideration (recall h_1 is set to 0, as the first bus starts immediately by convention):

$$
\min_{\phi = \{h_2, \cdots, h_T\}} \sum_{k=1}^{T} r_k(\phi;\alpha). \qquad (4.6)
$$

This is like the well known sequential control problem and the most convenient solution concept for this is the Dynamic Programming equations. We follow the same approach to solve this problem.

4.1.4 Dynamic Programming equations, solved by backward induction

We are interested to find the optimal policy i.e., the optimal headway times between the buses at the depot for all trips considered, which optimizes (4.6). As already mentioned, the optimal policy can be obtained by solving Dynamic Programming (DP) equations using backward induction. Towards this we discuss the necessary ingredients, like running cost, transition probabilities etc.

From (4.4), the running cost of any trip depends (at maximum) only upon the headways of the last M trips, to be specific for trip k it depends upon $\mathbf{h}_k := [h_{k-M}, \cdots h_k]$ when $k > M$ and $\mathbf{h}_k := [0, \cdots, 0, h_1, h_2, \cdots, h_k]$ when $k \le M$ (required number of zeros are inserted to make them same length vectors). Note that $\mathbf{h}_1 = [0, \cdots, 0, h_1] = [0, \cdots, 0, 0]$ (as h_1 set to 0 without loss of generality), $\mathbf{h_2} = [0, \cdots, 0, h_2]$ and $\mathbf{h_3} = [0, \cdots, 0, h_2, h_3]$ and so on. The dynamic programming equations, for any $k \le T$ are given by (Puterman (2014)):

$$
v_k(\mathbf{h}_{k-1}) = \min_{h_k} \Big(r_k(\mathbf{h}_k) + v_{k+1}(\mathbf{h}_k) \Big) \text{ and } v_{T+1}(\mathbf{h}_T) = 0. \qquad (4.7)
$$

In the above $v_k(\mathbf{h}_{k-1})$ represents the optimal cost from trip k till the last trip (T), when the previous trip depot-headways are given by $\mathbf{h}_{k-1}$. The running/trip-wise costs are given by (4.4) and hence these equations can be rewritten as (for any $k \leq M+1$):

$$v_k(\mathbf{h}_{k-1}) = \min_{h_k} \left\{ \sum_{l=0}^{k-1} h_{k-l}\bar{\gamma}_l + \vartheta_k^M + \alpha \left(1 - \Phi_k^M \left(\sum_{l=0}^{k-1} \tilde{\gamma}_l^M h_{k-l} + \varpi_k^M \right) \right) + v_{k+1}(\mathbf{h}_k) \right\} \quad (4.8)$$

We solved these optimality equations for a special case (mentioned in the hypothesis) and derived the optimal policy (see Table 4.1 for coefficients):

Theorem 4.1. *Assume $T > M+1$. Define the following constants backward recursively for all $0 \leq k < T-1$: first set $\delta^{T+1} = 0$, $\eta_l^{T+1} = 0$ for all $0 \leq l \leq M$ and then set*

$$\eta_l^{T-k} = \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{\tilde{\gamma}_{l+1}^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_l, \qquad a_{T-k} = \sigma_{T-k}^M \sqrt{-2\log\left(\frac{\eta_0^{T-k}\sqrt{2\pi}\sigma_{T-k}^M}{\tilde{\gamma}_0^M \alpha}\right)}, \quad (4.9)$$

$$\delta^{T-k} = \alpha\left(1 - \Phi_k^M(a_{T-k})\right) + \vartheta_k^M + \frac{\eta_0^{T-k}(a_{T-k} - \varpi_{T-k}^M)}{\tilde{\gamma}_0^M} + \delta^{T-k+1}.$$

If for any trip k, the following condition is satisfied

$$\alpha > e^{\left(\sum_{l=1}^M h_{T-j-l}\tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}\sigma_{T-j}^M}{\tilde{\gamma}_0^M} \text{ for all } 0 \leq j < k \text{ and } \eta_0^{T-k} > 0, \quad (4.10)$$

then the optimal policy and the value function are respectively given by:

$$h_{T-k}^*(\mathbf{h}_{T-k-1}) = \frac{1}{\tilde{\gamma}_0^M}\left[-\sum_{l=1}^M h_{T-k-l}\tilde{\gamma}_l^M - \varpi_{T-k}^M + a_{T-k} \right], \text{ and} \quad (4.11)$$

$$v_{T-k}^*(\mathbf{h}_{T-k-1}) = \sum_{l=1}^M h_{T-k-l}\left(\frac{\tilde{\gamma}_0^M \eta_l^{T-k} - \eta_0^{T-k}\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \right) + \delta^{T-k}. \quad (4.12)$$

Proof: is in Appendix A. ■

From (4.11), the optimal policy (once the hypothesis is satisfied) is linear in previous trip depot-headways and depends only on values of previous M trips. The coefficients of this linear dependence $\{\tilde{\gamma}_l^M/\tilde{\gamma}_0^M\}_l$ are the same for all trips. The affine component ϖ_{T-k}^M disappears for later trips (i.e., for trips $k > M+1$). Interestingly the only component that depends upon the trip number are the constants $\{a_{T-k}\}_k$, which is the main element of non-stationarity in the optimal policy.

Low load factors

We will now prove that the hypothesis of the above theorem are satisfied for low load factors, under some simple assumptions. Typical operating conditions work under such low load factors.

We obtained the optimal depot headway policy (see (4.11)) under the condition (4.10), and the following lemma proves that the condition (4.10) is satisfied under low load factors.

Lemma 4.2. *Assume $\alpha > \beta M\sqrt{2\pi}/2$. Then there exists an upper bound, $\bar{\rho}_\eta \in [0,1]$, such that condition (4.10) is satisfied (for all k) for all load factors $\rho \leq \bar{\rho}_\eta$. For all such ρ the optimal policy is completely defined by Theorem 4.1, i.e., by equations (4.11)-(4.12) with $h_1^* = 0$.*

Proof: is in Appendix A. ■

Algorithm 1 Defined using Theorem 4.1

---

1. Input parameters: M (number of stops), T (number of trips), ρ (load factor), α (trade-off factor) , β (traffic variability) and $\{s^i\}$ (inter-stop walking times)

2. Compute coefficients $\gamma_k^i, \tilde{\gamma}_k^i, \bar{\gamma}_k^i, \varpi_k^i$ for all k, i, using equations (2.7), (2.16) from Chapter 2 and (4.5) also given in Table 4.1.

3. Compute iterative coefficients (backward recursion)

    - Initial values: Set $\eta_l^{T+1} = 0$ for each l
    - for k = 0 to T-1
        - Compute η_l^{T-k} for each l, using (4.9) of Theorem 4.1
        - Compute a_{T-k} using η_0^{T-k} and the other coefficients

4. Computation of Optimal policy (forward manner)

    - Set initial trip bus headways to zero, i.e., $h_1 = 0$. By notation set even $h_l = 0$ for all $l \leq 0$.
    - for k = 2 to T-1
        - Compute trip k headway, h_k from (4.11), using the previous trip headways $\mathbf{h}_{k-1}$ and coefficients calculated above

---

4.2 Algorithms

The Lemma 4.2 guarantees the applicability of Theorem 4.1, for low load factors which are more relevant for practical scenarios. And then, one can obtain the optimal policy using the algorithm defined in Algorithm 1, constructed using Theorem 4.1.

Using DP equations directly :

For general conditions, one needs to solve the DP equations (4.8) using well known numerical techniques (see e.g., Puterman (2014)). We will need to consider discrete choices for the headways.

4.2.1 Asymptotic ($T \to \infty$) analysis and Simplified algorithm

From Theorem 4.1, it is a complicated procedure to compute the trip-wise-constants, $\{\eta_0^t\}_t$ and $\{a_t\}_t$ required for Algorithm 1. One can have a much simplified (approximate) algorithm if there is a possibility to (well) approximate these by appropriate constants. We attempt this using large trips ($T \to \infty$) approximation: compute the required coefficients under this limit.

Proposition 4.3. *As $T \to \infty$ then,*

$$\lim_{T\to\infty} \eta_0^T = \eta_0^* = \frac{M(1+\rho)^{M-1}}{2(1-\rho)}, \tag{4.13}$$

$$\lim_{t\to\infty} a_T = a_* = \sigma_M^M \sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma_M^M}{2(1-\rho)\alpha}\right)}. \tag{4.14}$$

Proof: is in Appendix A. ■

One can propose an approximate algorithm using this limit, which is much more simplified than Algorithm 1. The approximate algorithm is provided as Algorithm 2. We verify the accuracy of this approximation, later using numerical examples.

4.2.2 Comparison with Stationary analysis

In section 3.1 from Chapter 3, we derived optimal stationary policy, wherein the same depot-headway is applicable for all trips. This policy is optimal when the number of trips tends to infinity; the system reaches stationarity, and hence the system performance is close to the stationary performance. That is, the performance with huge number of trips does not depend upon initial trips (i.e., on transient behaviour). In this chapter, we are working with finite/small number of trips; here, the performance of all trips (including initial transient trips) is significant. As anticipated we have an optimal policy that is non-stationary, the headway depends upon the trip number and the headways of the previous trips (equation (4.11) of Theorem 4.1).

We now investigate if the optimal partially dynamic policy of this chapter approaches the stationary policy of Chapter 3 in some suitable sense. To be more precise, we study the

Algorithm 2 Using approximate iterative co-efficients

1. Input parameters: $M, T, \rho, \alpha, \sigma_M^M$

2. Approximate iterative coefficients $\eta_0^{T-j} \approx \eta_0^*$ and $a^{T-j} \approx a_*$ for all j, using η_0^*, a_* defined in (4.13)-(4.14)

3. Computation of Optimal policy (forward manner)

   - Set initial trip bus headways to zero, i.e., $h_1 = 0$. By notation set even $h_l = 0$ for all $l \leq 0$.

   - for k = 2 to T-1

     – Compute trip k headway, h_k from (4.11), using the previous trip headways $\mathbf{h}_{k-1}$ and coefficients calculated above

headways related to the last trips (i.e., h^*_{T-k}, for k small) given by Theorem 4.1 as $T \to \infty$. Our aim is to verify if these headways approximately coincide with the stationary headway h_s^* (3.10) obtained from Chapter 3.

In the previous subsection, we computed the limits of the iterative coefficients defining the optimal policy (4.11), and the idea is to use these asymptotic constants for the above mentioned verification. From (4.11), the headway policy update equation is a difference equation (with time varying co-efficients) and our question is related to the settling/limit point of this equation. It is well known that the stationary points of the difference equation can become the required limits. We will first show that the stationary point of this equation is h_s^* given by equation (3.10) from Chapter 3. One requires extra technical arguments to prove that the updates h_k indeed converge to this stationary point, however, this aspect is not considered in the thesis as of now.

Stationary point of (4.11): We would now verify that h_s^* defined in (3.10) of Chapter 3 as the stationary point, by direct substitution; we reproduce h_s^* here for ease of reference:

$$h_s^* \;=\; \frac{a_*}{1-\rho} \;=\; \frac{\sigma_M^M}{1-\rho}\sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma_M^M}{2(1-\rho)\alpha}\right)}. \tag{4.15}$$

By substituting $h_{T-k} = h_s^*$ (for all k), and after replacing iterative coefficients with their approximations given in (4.13)-(4.14) one can easily verify the following:

$$h_T \;=\; \frac{1}{\tilde{\gamma}_0^M}\left(-\sum_{l=1}^{M} h_s^* \tilde{\gamma}_l^M + a_*\right) = h_s^*. \tag{4.16}$$

From the above, one can observe that the optimal stationary policy h_s^* of Chapter 3 is a stationary point of the iterative (non-stationary) partial dynamic optimal policy of this chapter; we would further show that this stationary point is indeed the limit of the partial-dynamic policy using some numerical examples in the next section.

4.3 Numerical Analysis

We now consider few numerical examples to study and compare the policies derived in previous sections. We study various aspects related to the optimal policies like, the structure of the policy, the performance comparison of different policies etc. In Figure 3.1 from Chapter 3 we observed that the theoretical performance (derived after simplifying assumptions) well approximates the MC estimates for practical values of load factors (even upto $\rho = 0.5$). The approximation is good even for Poisson (passenger) arrivals, however only when the arrival rate is large. Thus we anticipate the optimal policies (derived using those theoretical expressions) to improve the performance even for real systems. Towards this, in all the experiments described in this section, we have *compared the performance improvement obtained using MC simulation based estimates.* These simulations are carried using a similar procedure as explained in subsection 3.1.4 from Chapter 3. In all these experiments, we observed that the theoretical estimates under the optimal stationary policy (when computed) well approximates (well within 10% error) the MC based estimates. Further all these experiments are for nominal values of λ in the range of $[10, 20]$, which again reinforces that the theoretical expressions well approximate that of the actual system (with fluid arrivals) under nominal values of λ.

4.3.1 Structure of optimal policy

In Figure 4.1, we plot the optimal policy for small load factors, which is computed using Algorithm 1 (and given by Theorem 4.1). Basically we plot the optimal depot headway h_k^* (time gap between starting instances of k and $(k-1)$ buses at depot), as a function of the trip number k. As observed from the figure, the structure of the optimal policy is different for different load factors. For very small loads ($\rho = 0.1$ in Figure 4.1) the depot headway is decreasing and

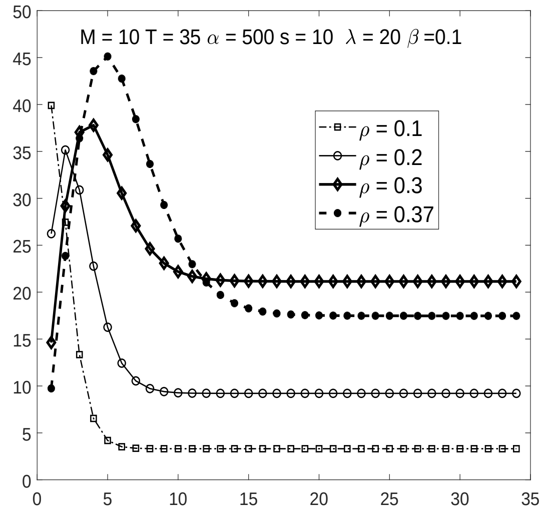

Figure 4.1: Trip number versus optimal depot headway

finally converges to the corresponding stationary optimal headway (h_s^*). And for larger load factors ($\rho \geq 0.2$), the depot headway is increasing initially and then starts decreasing to optimal h_s^*. For low load factors, the number of passengers waiting for the initial trip may not be significant, hence it starts with smaller frequency (higher headway) of operation. However when the load factors are high, the people waiting for the first bus might be significant, which is probably the reason for increased frequency of operation for these initial trips (see Figure 4.1).

The transient behaviour is different for different regimes of load factors, however in all cases the headway settles down (as the trip number increases) to the corresponding stationary optimal value h_s^* given by (3.10) from Chapter 3. Recall that this stationary optimal value is also the limit of non-stationary policy as derived in (4.16), thus the observation of the Figure 4.1 reinforces the discussion of subsection 4.2.2.

The policy might settle down eventually to h_s^*, however as noticed we have significant (load factor dependent) transience prior to this and this leads to substantial improvement. We will observe in subsequent examples a significant improvement in both the performance measures (bunching probability as well as expected waiting times), when one uses optimal non-stationary policies in place of optimal stationary policies.

4.3.2 Comparison of the Algorithms

In this subsection, we compare the performance measures obtained under various algorithms; Algorithm 1, Algorithm 2. We show that the much simpler (approximate) Algorithm 2 works almost as good as the exact Algorithm 1.

ρ	Exact DP		Approximate DP		Error(%)	
	BP	WT	BP	WT	BP	WT
0.05	0.00066	1.459	0.00068	1.459	3.03	0
0.1	0.00124	2.230	0.00123	2.235	0.81	0.22
0.2	0.00479	6.038	0.00483	6.078	0.84	0.66
0.3	0.01942	16.399	0.01953	16.494	0.57	0.58
0.35	0.03910	25.730	0.03981	25.783	1.82	0.21
0.4	0.07976	37.930	0.08262	37.668	3.59	0.69

Table 4.2: Solutions and Performance of Exact and Approximate DP equations, $\beta = 0.1$

ρ	Exact DP		Approximate DP		Error(%)	
	BP	WT	BP	WT	BP	WT
0.05	0.00141	2.377	0.00142	2.377	0.71	0
0.1	0.00265	3.790	0.00262	3.794	1.13	0.11
0.2	0.01042	10.596	0.01045	10.631	0.29	0.33
0.3	0.04352	27.542	0.04427	27.538	1.72	0.01
0.35	0.09274	40.423	0.09445	40.078	1.84	0.85
0.4	0.20972	50.542	0.21544	48.417	2.73	4.2

Table 4.3: Solutions and Performance of Exact and Approximate DP equations, $\beta = 0.2$

In Tables 4.2 and 4.3 we compare the performance measures corresponding to Algorithm 1 and Algorithm 2 through numerical simulations for different load factors (ρ), which is tabulated in the first column. The performance measures obtained using exact dynamic programming equations (Algorithm 1) are tabulated in the next two columns, while that with the approximate algorithm (Algorithm 2) are in the fourth and fifth columns. The percentage errors are calculated and tabulated in the last two columns. For both the tables, we set $M = 10, T = 35, \lambda = 20, \alpha = 2000$. We consider different traffic variabilities in the two tables; we set $\beta = 0.1$ in Table 4.2

and $\beta = 0.2$ in Table 4.3. We notice from both the Tables that the performance measures with approximate algorithm well matches that obtained by solving exact DP equations. By Lemma 4.2 and related discussions, we proved that the approximation is good for low load factors. However we notice from the tables that the approximation is good even for load factors upto $\rho = 0.4$. In fact the approximation error is well within 5%, where the percentage error is calculated as below:

$$\text{percentage error} = \frac{|\text{approximate value} - \text{exact value}|}{\text{exact value}} \times 100.$$

Thus the approximate algorithm is a good substitute for sufficiently large range of load factors. Further these simulations also indicate that the DP solutions given by Theorem 4.1 are good for a large range of load factors.

4.3.3 Pareto Frontier and comparison with stationary policies

We now compare the two dynamic policies with the optimal stationary policy h_s^*. Towards this, we plot the Pareto frontier of performance measures in Figures 4.2 and 4.3. The Pareto frontier is obtained by the MC estimates of both the performance measures, basically using the optimal policies given by (4.6), for different α. The figure also shows the performance under various optimal stationary policies (3.10 from Chapter 3), corresponding to the same set of trade-off factors α. We conduct the experiment with $M = 5, T = 30, s = 1, \beta = 0.1, \lambda = 20$ and $\rho = 0.1$ in Figure 4.2 and $\rho = 0.01$ and $T = 200$ in Figure 4.3. We first observe that the approximate algorithm (Algorithm 2, given by straight line) once again well matches the Algorithm 1 (given by stars) in all examples.

We can also observe that the performance corresponding to both the algorithms is significantly better than the corresponding stationary optimal performance in Figure 4.2. However in Figure 4.3 the performance under stationary policy is almost similar to that under dynamic policies. For this example case, the number of trips $T = 200$, which is significantly high and $\rho = 0.01$ the load factor is significantly small. We conducted many more such experiments and found similar observations. Some more examples are discussed below.

In Table 4.4 we compare the optimal average waiting time of passengers corresponding to stationary and partially dynamic policies (Algorithm 1), while maintaining the same bunching probability. Basically we choose/tune different α for the two cases such that their bunching probabilities are (almost) the same and then compare the respective passenger average waiting

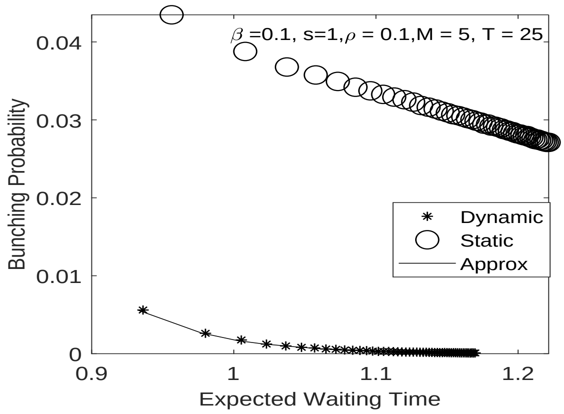

Figure 4.2: Expected waiting time versus bunching probability

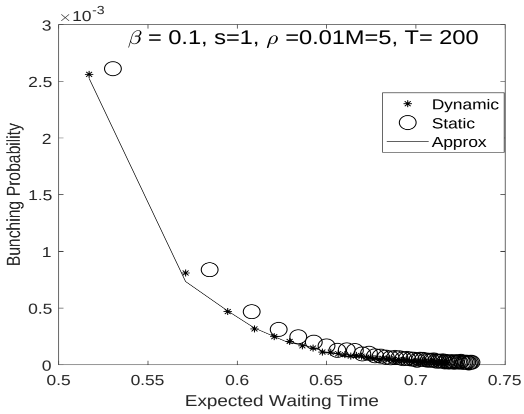

Figure 4.3: Expected waiting time versus bunching probability

times. We conduct the experiment with $M = 10, T = 35, \lambda = 20$ and $s = 5$. The parameters traffic variability (β) and load factor (ρ) are tabulated in the first column. As already mentioned, we choose different values of α for the two policies such that the bunching probabilities are almost equal (under both the policies) and these values are reported in the next two columns. We then tabulate the corresponding average passenger waiting times in the fourth and fifth columns. We compare any two policies using an index call N.P.I (Normalised Performance Improvement) of appropriate performance measures (e.g., waiting in this case study), which is defined as below:

$$\text{N.P.I.} = \frac{|\text{dynamic performance -stationary performance }|}{\text{dynamic performance}} \times 100.$$

The performance index N.P.I. is tabulated in the last column and is a large positive value for all

β, ρ	Bunching probability		Waiting time		N.P.I (%)
	Stationary	Dynamic	Stationary	Dynamic	WT
4,0.05	0.0418	0.0418	30.41	28.76	5.7
4,0.1	0.1133	0.1129	44.5	38.18	16.5
4,0.15	0.2584	0.2562	60.03	46.04	30.4
2,0.1	0.0741	0.0738	28.19	22.03	27.9
2,0.2	0.3255	0.326	63.95	36.07	77.3
1,0.1	0.0735	0.0731	17.46	12	45.5
1,0.2	0.2427	0.244	47.23	23.51	100

Table 4.4: Performance of Dynamic and stationary analysis

row, quantifying the (huge) improvement provided by the dynamic policies. When the traffic variability is high and the load is small (see first two rows of Table 4.4), the stationary performance is comparable (N.P.I. is within 16%). But at high loads, the dynamic policies are far superior.

4.3.4 Performance at high load factors

As already discussed, our performance analysis of the transportation system is accurate when the bunching probabilities are small. Further Theorem 4.1, providing the optimal dynamic policy, is accurate under low load conditions. We now consider the sub-cases/examples in which the above mentioned assumptions are not valid. For example we consider the cases with moderate/high load factors. One can easily extend the policy defined by Theorem 4.1 (or given by Algorithm 1) to this case, but we are not clear if this policy would perform well? Basically, in this sub-section we would like to verify that our dynamic policies improve (in comparison with the 'best' static policies), even for the case studies for which the policy may not be optimal.

As shown in Figure 3.2 or 3.3 from Chapter 3, as the load/traffic variability increases, the stationary policy (3.10) from Chapter 3 is no more close to the optimal one obtained using MC simulations. Hence one can not use stationary policy (3.10) from Chapter 3 for computing the N.P.I. (with the purpose of comparing dynamic policy with a 'good' stationary policy). We instead estimate the optimal stationary policy using the MC based estimates (of the performance) exactly as in Figure 3.1 from Chapter 3, and compare its performance with that under the dynamic policy given by Algorithm 1. The following is the exact procedure of comparison.

- Choose a value of α and obtain the MC estimates of the performances under the dynamic (Algorithm 1) policy, $P_{B_{dy}}$ and $E_{W_{dy}}$.
- Consider a fine grid of [0, 1000], where each value in the grid can represent a stationary headway policy. Compute MC performance when the successive buses depart from the depot after fixed headway given by one of the grid values. Choose the $\hat{h}^*$ for which the MC estimate of P_B is close to $P_{B_{dy}}$.
- Find the MC based estimate of the expected passenger waiting time, $E_{W_{st}}$ at $\hat{h}^*$.
- Compare the dynamic waiting time $E_{W_{dy}}$ with the best MC-based expected waiting time $E_{W_{st}}$ by computing the corresponding N.P.I.

By this procedure we obtained (an estimate of) the best average passenger waiting time under a static policy, further under the constraint that the bunching probability is within $P_{B_{dy}}$.

ρ	Bunching probability		Waiting time		N.P.I. (%)
	MC Stationary	Dynamic	MC Stationary	Dynamic	WT
0.4	0.2076	0.2152	24.93	13.52	84.4
0.5	0.2795	0.2799	54.05	17.34	211.7
0.6	0.4049	0.4054	187.35	23.94	682.6

Table 4.5: Performance of Dynamic and stationary analysis, the optimal stationary policy obtained using MC based estimates

In Table 4.5, we tabulate the N.P.I.s as obtained by the procedure given above for different case studies with high load factors. We conduct the experiment with $M = 5, T = 35, \lambda = 20$, $\beta = 0.2$ and $s = 10$. The load factor is tabulated in the first column. All the relevant performance measures are tabulated in various columns as indicated in the Table. The N.P.I. comparing the average waiting times under the dynamic policy and the MC based optimal stationary policy is tabulated in the last column. The N.P.I is a large positive quantity (above 84%) demonstrating that our dynamic policy performs significantly better than any 'good' stationary/static policy, even under large road factors.

We conclude that the dynamic policies perform significantly better than the stationary policies under normal operating conditions. However analysis/study of the stationary policies is still important: a) because they are much simpler to implement; and b) because these policies are widely used in practice as of now.

4.4 Correlated Travel times

Previously we considered independent travel times, i.e., we had the following model:

$$S_{k+1}^j = s^j + N_{k+1}^j,$$

where s^j was fixed across various trips. In certain scenarios, especially when the frequency of buses is high, it is more appropriate to consider correlated travel times as below: recall $\{N_{k+1}^j\}_{k,\, j}$ are IID random variables which are normally distributed with mean 0 and variance β^2 and S_k^j is the k-th bus travel time to stop j,

$$S_{k+1}^j = S_k^j + N_{k+1}^j.$$

One can easily extend the analysis to this case and we mention the required changes. Firstly we have the following result regarding the inter-arrival times:

Lemma 4.4. *For correlated travel times, we have:*

$$\begin{aligned}
I_k^i &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma_l^i h_{k-l} + \sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W_{k-l}^j \\
I_k^i - \rho I_{k-1}^i &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma_l^i h_{k-l} + \sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W_{k-l}^j - \rho \sum_{l=0}^{\min\{k-2,i-1\}} \gamma_l^i h_{k-1-l} \\
&\quad -\rho \sum_{l=0}^{\min\{k-2,i-1\}} \sum_{j=1}^{i-l} \gamma_l^{i-j+1} W_{k-1-l}^j \\
E[I_k^i] &= \sum_{l=0}^{\min\{k-1,i-1\}} \gamma_l^i h_{k-l} \\
(\sigma_{c,k}^i)^2 &= variance\left(I_k^i - \rho I_{k-1}^i\right) = \beta^2 \left(\sum_{l=0}^{\min\{k-1,i-1\}} \sum_{j=1}^{i-l} \left(\tilde{\gamma}_l^{i-j+1}\right)^2 + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2} \right).
\end{aligned}$$

Proof: *is in Appendix A.* ■

As with independent travel times, the inter-arrival times and the difference $\{I_k^i - \rho I_{k-1}^i\}_k$ become stationary with constant headways (i.e., when $h_k \equiv h$ for some $h > 0$), once $k > M$.

Further the dependency of various terms on the history is almost as in Lemmas 2.1-2.2 from Chapter 2 for independent travel times, except for the variances. One can notice that $(\sigma_k^i)^2$ ($variance\left(I_k^i - \rho I_{k-1}^i\right)$ corresponding to independent travel times given in Lemma 2.2 from Chapter 2) is different from $(\sigma_{c,k}^i)^2$ given in the above lemma, but the rest of the coefficients are exactly the same.

4.4.1 Stationary policies

One can obtain the optimal head way policy among stationary policies as in section 3.1 from Chapter 3:

$$h_s^* = \begin{cases} \sqrt{\frac{-\log(C_1)}{C_2}}, \text{ with } C_1 = \frac{M\sqrt{2\pi}\sigma_c^M}{2\alpha(1-\rho)} \text{ and } C_2 = \frac{(1-\rho)^2}{2(\sigma_c^M)^2}, & \text{if } \alpha \geq M\sqrt{2\pi}\sigma_c^M/2(1-\rho) \\ 0 & \text{else.} \end{cases} \tag{4.17}$$

The above optimal policy is structurally the same as the optimal policy (3.10) of section 3.1.3 from Chapter 3, the only difference being in variance.

4.4.2 Partially dynamic policies

As in section 4.1, one can consider the analysis of first few trips and derive optimal control when one starts with the first trip. In this sub-section we present a slightly different alternative which is of independent importance. We consider control of some T-trips after some t_0 initial uncontrolled trips with $t_0 > M$. There could be more turbulences after some initial period and control may become important after t_0 trips. For this case the optimal control policy can be computed as in the previous case and we have the following:

Theorem 4.5. *Assume $t_0 > M$ and $T \geq 1$. Define the following constants using the $variance$ $\left(I_k^i - \rho I_{k-1}^i\right)$ corresponding to correlated travel times (with remaining coefficients as in Table 4.1) for all $T - t_0 - 1 \geq k \geq 1$*

$$\begin{aligned} a_{c,T-k} &= \sigma_{c,T-k}^M \sqrt{-2\log\left(\frac{\eta_0^{T-k}\sqrt{2\pi}\sigma_{c,T-k}^M}{\tilde{\gamma}_0^M \alpha}\right)}, \\ \delta_c^{T-k} &= \alpha\left(1 - \Phi_k^M\left(a_{c,T-k}\right)\right) + \vartheta_k^M + \frac{\eta_0^{T-k}(a_{c,T-k} - \varpi_{T-k}^M)}{\tilde{\gamma}_0^M} + \delta_c^{T-k+1}. \end{aligned} \tag{4.18}$$

If for any trip $T - k$ (with $T - t_0 - 1 \geq k \geq 1$), the following condition is satisfied

$$\alpha > e^{\left(\sum_{l=1}^M h_{T-k-l}\tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}\sigma_{c,T-j}^M}{\tilde{\gamma}_0^M} \textit{ for all } 0 \leq j < k \textit{ and } \eta_0^{T-k} > 0, \tag{4.19}$$

then the optimal policy and the value function are respectively given by:

$$h^*_{T-k}(\mathbf{h}_{T-k-1}) = \frac{1}{\tilde{\gamma}_0^M}\left[-\sum_{l=1}^{M} h_{T-k-l}\tilde{\gamma}_l^M - \varpi^M_{T-k} + a_{c,T-k}\right], \text{ and} \tag{4.20}$$

$$v^*_{T-k}(\mathbf{h}_{T-k-1}) = \sum_{l=1}^{M} h_{T-k-l}\left(\frac{\tilde{\gamma}_0^M \eta_l^{T-k} - \eta_0^{T-k}\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M}\right) + \delta_c^{T-k}. \tag{4.21}$$

Proof: The constants and the hypothesis is similar to Theorem 4.1. The only difference is variance. Hence the proof is in similar lines as with the proof of Theorem 4.1. ■

One can again prove the validation of the hypothesis of this theorem for low-load factors exactly as in Lemma 4.2. We finally conclude that the optimal policy for the case with correlated travel times is similar to the ones in Algorithms 1-2, except for the variance term and its influence on the coefficients $\{a_{c,T-k}\}_k$. It is trivial to realize that Algorithms 1-2 can also work with intermediate trips control (i.e., for the case $t_0 > M$), for which the terms like $\{\varpi_{T-k}\}_k$ are excluded in (4.11).

ρ	Bunching probability		Waiting time		N.P.I. (%)
	Stationary	Dynamic	Stationary	Dynamic	
0.1	0.0011	0.0009	1.14	1.14	22.2
0.2	0.0044	0.0039	3.16	3.13	12.8
0.3	0.1065	0.0923	5.45	5.4	15.4
0.4	0.2676	0.2164	20.19	20.08	23.7

Table 4.6: Performance of Dynamic and stationary analysis under correlated travel times

4.4.3 Numerical analysis

In Table 4.6, we compare the bunching probabilities corresponding to stationary and dynamic policies while maintaining the same average waiting time. We set $M = 10$, $T = 40, \lambda = 20$, $\beta = 0.1$ and $s = 1$. The load factor (ρ) is tabulated in the first column. The bunching probabilities corresponding to both the policies are tabulated in the next two columns. We choose different values of α for the two cases such that the corresponding passenger average waiting times are almost equal and these values are tabulated in the fourth and fifth columns. The N.P.I. (in bunching probability) of dynamic policy when compared to stationary policy is tabulated in the last column. The performance is always better with the dynamic policy. The N.P.I. is less when compared to independent travel times, however this is because of low traffic variability (β). We will observe in the next example (Figure 4.4) that the N.P.I. is a huge positive quantity, and this example has large β (traffic variability). In fact we considered many more case studies and found huge improvement, for all cases with medium/high traffic variability. For the cases with low traffic variability, we still have improvement, in the range of 13% to 23%.

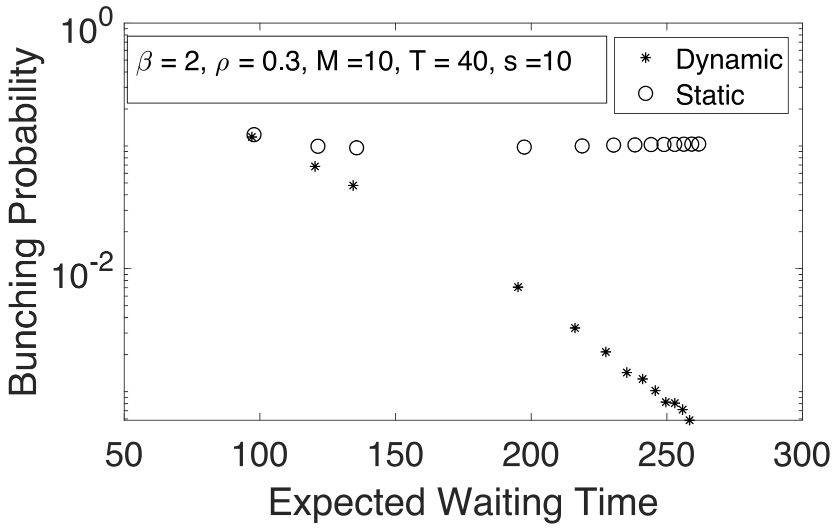

Figure 4.4: Expected waiting time versus bunching probability under correlated travel times

In Figure 4.4, we plot the Pareto frontier of both the performance measures, under both the type of policies. The Pareto frontier is obtained by MC estimates of both the performance measures for different α. We conduct the experiment with $M = 10, T = 40, s = 10, \lambda = 20, \beta = 2$ and $\rho = 0.3$. One can observe from the Figure 4.4 that the dynamic policies significantly outperform the stationary policies.

4.5 Summary

We considered a bus transport system, where the buses start their journey from the depot and traverse along a cyclic route repeatedly. We considered fluid arrivals and boarding times, Gaussian (bus) travel times and derived two important performance measures, bunching probability and passenger-average waiting time. We derive these performance measures 'partially' dynamic policies (headway of any trip depends upon some information related to previous trips). In these policies, the headway decision depends upon the headways of the previous trips; it is independent of other details of previous trips (e.g., trip times, number of waiting passengers at various stops etc). Thus our partially dynamic policies depend upon easily available (cheap) information, nevertheless, they perform significantly better than (commonly used) stationary policies discussed in Chapter 3.

We obtained (stationary) Pareto frontier, the set of all efficient points; each efficient point is a pair of performance measures obtained by a stationary policy, such that there exists no other stationary policy which can simultaneously improve both the performances. The Pareto frontier is obtained by solving several optimization problems: a) each problem optimizes a weighted combination of the two performance measures; and b) the complete frontier is obtained by considering all possible trade-off parameters/weights.

We also derived partially dynamic Pareto frontier, by optimizing over the partially dynamic policies. We derived this dynamic frontier by minimizing the weighted-combined costs using corresponding dynamic programming equations. We obtained closed form expressions for the optimal policies under low load conditions; the optimal policies depend linearly upon the previous headway times. Using Monte-Carlo based simulations we demonstrated that our linear (dynamic) policies significantly outperform the best static/stationary policy under all load conditions; the stationary Pareto frontier is significantly away from the dynamic Pareto frontier. By using asymptotic analysis, we further proposed a simplified algorithm to implement the optimal policy. The simplified algorithm has almost similar performance as that of the exact algorithm implementing the optimal policy.

We derived the optimal policies under correlated as well as independent (bus) travel times. The optimal policies are structurally (linearly depends upon previous headways) the same, for both the types of travel times; the only difference is in (some) constants/coefficients defining the policy.

Chapter 5

Bus bunching: dynamic policies

In this chapter, we discuss the fully dynamic policies. These are the actual closed loop policies, while the previous two types of policies are open loop policies. In Chapter 4 we viewed the previous (trip) decisions as state and derived the optimal policies, as result we referred them as partially dynamic policies. In reality these are again open-loop, (but) time varying policies.

The system model and related performance measures are similar to previous chapters except for the following: we consider Markovian travel times (as in previous chapter 4) and further extend the results to the case with different passenger arrival rates at different bus-stops. Because of these two significant differences, for ease of clarity, we choose to describe the system completely in this chapter (also repeating some of the modelling details of Chapter 2).

We obtain near optimal policies; these depend linearly upon the previous trip headways and the bus-inter arrival times at various stops corresponding to the latest trip whose information is available.

5.1 Problem description

We consider a bus-transport system with M bus stops; one of them is a depot at which all the buses start. The journey of any bus starts at the depot, it traverses through the entire path boarding and de-boarding the passengers at various stops en-route and finally stops at the depot. In most of the systems, the successive buses depart from the depot at equal intervals of time; these time gaps are referred to as depot-headways. These (equal/constant) headways are designed based on the operating time of the day, the expected demand for the service at the given time,

Results of this chapter are ready to be submitted.

the traffic conditions etc, according to some practice rules.

As already mentioned, our aim is to design the depot-headways according to the sense of optimality (as in previous chapters), but now the 'optimal' headways also depend upon 'true system state'. To be more precise, we consider that the system has to choose T headways, which determine the starting time points of T number of consequent trips. The design of these headways can depend upon the details of some of the previous trips (if any), trips that occured just before the controllable part of the trips, and these are discussed in the coming sections. Thus to summarize, we would like to design dynamic headway policies, that decide the headway/time-gap before the next bus departure, based on the information available at the departure time of the previous bus.

We now (re)describe the notion of optimality considered in this Chapter. We are interested in the two performance measures; a) *Bunching probability:* We consider bunching probability, the probability that two buses start travelling together somewhere along the path; b) *Passenger-average of the waiting times:* We consider the average of waiting times of all the passengers.

Thus we have a multi-objective problem and hence consider the well known Pareto optimal solutions, by optimizing a weighted sum of the two performance measures. To begin with, we describe the precise details of the system model and the required assumptions.

5.1.1 System model

Let S_k^i be the time taken by the k-th bus to travel between the stops $(i-1)$ and i. We consider Markovian (correlated) travel times, between any two stops. To be precise we assume that

$$S_k^i = S_{k-1}^i + W_k^i, \text{ for any, } k \geq 0, \text{ and } S_0^i = s^i,$$

where W_k^i is the random difference between two successive travel times and $\{s^i\}_i$ are the sojourn times of the first trip. We assume $\{W_k^i\}_k$ are IID (Independent and Identically distributed) *Gaussian random variables with mean* 0 *and variance* β^2 and this is true for all stops i. Further these are independent across the stops.

To analyse this system, one needs to study the dwell/visit times of various buses at various stops, inter-arrival times between successive buses at different stops etc. Towards this we make some assumptions **R.**1 to **R.**5 from Chapter 2 along with the following assumption.

A. Fluid arrivals and boarding: *The number of passengers arrived to a stop, during a period* t *equals* λt*, where* $\lambda > 0$ *is the arrival rate.* More details about this modelling is provided in the next section. *The time taken to board X number of passengers equals bX, where $b > 0$ is the boarding rate.*

These assumptions are not very restrictive, and are satisfied by most of the commonly used practices in bus transport systems. The fluid arrivals can be justified owing to Elementary Renewal theorem, and because typical (bus) inter-arrival times at any stop would be significant; passenger arrival (e.g., Poisson) process can be modelled as renewal process with rate λ and then the number of passenger arrivals in a large time interval $[0, t]$ (during one bus inter-arrival time), approximately equal λt.

5.1.2 Bus (stop) inter-arrival times

Because of the above assumptions, the number of passengers boarding a bus in any trip k and at any stop i equals the ones that arrived during the bus-inter arrival time $I_k^i := A_k^i - A_{k-1}^i$, where A_k^i is the arrival instance of k-bus at stop i. Thus the total number of passengers X_k^i waiting at stop i, at bus arrival instance, equals $\lambda^i I_k^i$. Thus the dwell time of $k-$th bus at stop i equals[1]:

$$V_k^i = X_k^i b = b\lambda^i I_k^i = \rho^{(i)} I_k^i \text{ with } \rho^{(i)} := \lambda^i b. \tag{5.1}$$

In the above, $\rho^{(i)}$ represents the load factor of the stop i, for all $1 \leq i \leq M$.

Let h_k be the headway between $(k-1)$-th and k-th bus at depot. Then the inter-arrival times are given by:

$$\begin{aligned} I_k^1 &= \left(h_k + S_k^1\right) - S_{k-1}^1 = h_k + W_k^1 \text{ for first stop, similarly} \\ I_k^i &= h_k + \sum_{1 \leq j \leq i} S_k^j + \sum_{1 \leq j < i} V_k^j - \left(\sum_{j \leq i} S_{k-1}^j + \sum_{j<i} V_{k-1}^j \right) \\ &= h_k + \sum_{1 \leq j \leq i} W_k^j + \sum_{1 \leq j < i} \rho^{(j)} \left(I_k^j - I_{k-1}^j\right) \text{ for any stop } i. \end{aligned} \tag{5.2}$$

The last equality follows by fluid arrival and gated service assumptions as in (5.1).

The analysis of these inter-arrival times are instrumental in obtaining the results of this Chapter. In Lemmas 9.1-9.2 provided in Appendix B, it is shown that the inter-arrival times are Gaussian and their expectations, variances are computed.

We now describe the Markov decision process based problem formulation that optimizes a given weighted combination of the two performance measures, the bunching probability and the passenger-average waiting times.

[1] Since bunching is a rare event we neglect the affects of **A**.5 in this part of the modelling. Further this is a common practice in many transportation systems like Trams, metros, local train etc

5.2 Markov Decision Process (MDP)

5.2.1 Decision epochs, State and Action spaces

When the $(k-1)$-th bus leaves the depot, the system needs to determine the headway for the k-th bus. A decision at this epoch, can depend upon the available system state. One will have access only to delayed information: the headway decision for the current bus has to be made immediately after the previous bus departs the depot, the information related to the previous bus trajectory (no delay) is obviously not available. Further one may not even have the information about some of the trips, previous to the trip that just started.

We assume the availability of d-delayed (with $d \geq 1$) information; the bus-inter-arrival times $\{I_{k-d-1}^{j}\}_{1\leq j\leq M-1-d}$ at various stops related to $(k-d-1)$-th trip are known at k-th decision epoch. As in previous chapter, one can also have access to the information about the headways of all the previous trips $\{h_{k-l}\}_{l\geq 1}$; there is no delay in this component of the information. We will observe later that one requires only the headways of the unobserved trips, that of d-previous trips. Thus, in all, at $(k-1)$-th bus departure (i.e., at k-th decision epoch), we have access to the following state (see equation (5.2)):

$$Y_k = (h_{k-1}, \ldots, h_{k-d}, \{I_{k-d-1}^{j}\}_{1\leq j\leq M-1-d}). \tag{5.3}$$

By fluid arrivals and gated boarding, this state is equivalent to the number of passengers boarding the bus at various stops corresponding to the latest available trip (see equation (5.1)).

Remarks: In (5.3), it is interesting to observe that we require only the information related to first $(M-1-d)$ stops of the $(k-d-1)$-th trip; it becomes evident from Theorem 5.1 (given below) that such a state is sufficient if one has access (at maximum) to d-delay information. From Lemmas 9.1-9.2 of Appendix B the random components (I_k^i and $I_k^i - \rho^{(i)}I$), that define the required performance measures related to the k-th trip, depend at maximum upon the information related to stops $1, 2\cdots i-1-d$, when d-delay information is available.

It is easy to observe that the random vector sequence $\{Y_k\}_k$ forms a Markov Chain, whose evolution depends upon the headway (of the current trip that needs to be decided) and the previous state and hence is a controlled chain.

Trips prior to the controlled trips

Initial trips may have light load conditions (passenger arrival rates) and can be subjected to small variations in traffic, load conditions. Alternatively, in some cases the initial trips can be subjected to high load conditions. Further, one may consider controlling some of the trips and not all, probably ones that have maximum fluctuations. It is clear that some of the trips (just prior) to the controlled ones also influence the performance.

By abuse of notation, we call all the previous trips that influence the controlled trips (the one whose headways are to be controlled) as initial trips. As in Chapter 3, one can show that only some t_0 (to be more precise (at maximum) previous $(M+1)$-trips can influence) previous trips would influence. We assume that the buses operate during these initial trips (say t_0 of them) at some fixed headway h_0. We consider controlling the depot-headway starting from trip t_0+2, and to keep the notations simple, we refer (t_0+2+k)-th trip by index k. Alternatively one can consider controlling the buses starting from the first trip as in our previous Chapter 4. More details about these two varieties of initial trips are provided in Remarks after Theorem 5.1.

5.2.2 Performance measures

Passenger waiting times

We compute the passenger waiting times as in Chapter 2. Recall, *the waiting time of a typical passenger is the time gap between its arrival instance at the stop and the arrival instance of its bus (to the same stop).* Let $W^i_{n,k}$ be the waiting time of the n-th passenger that boards the k-th bus at stop i. The customer average of the waiting times corresponding to trip k and stop i equals (e.g., Chapters 3,4):

$$\bar{w}^i_k \triangleq \frac{\bar{W}^i_k}{X^i_k} \text{ with } \bar{W}^i_k := \sum_{n=1}^{X^i_k} W^i_{n,k}.$$

Fluid approximation/arrivals: The passengers are assumed to arrive at regular intervals (of length $1/\lambda$), with λ large. The waiting time of the first passenger during bus inter-arrival period (I^i_k) is approximately[2] $W^i_{1,k} \approx I^i_k$, that of the second passenger is approximately $W^i_{2,k} \approx I^i_k - 1/\lambda^i$ and so on. Thus as $\lambda^i \to \infty$, $1 \le i \le M$, the following (observe it is a Riemann sum)

[2] The residual passenger inter-arrival times at bus-arrival epochs get negligible as $\lambda \to \infty$.

$\binom{i}{l} = 0$ when $i < l$, $\quad$ for all $1 \le i \le M$, $\quad 0 \le l \le d$, for all $1 \le j \le M$, $\quad 0 \le d \le M$
$\gamma_l^{i-1} = (-1)^l \Bigg(\mathbb{1}_{l=0} + \binom{1}{l} \sum_{j_1=1}^{i-1} \rho^{(j_1)} + \binom{2}{l} \sum_{j_1<j_2}^{i-1} \rho^{(j_1)}\rho^{(j_2)} + \dots$ $\qquad + \binom{i-1}{l} \sum_{j_1<j_2<\dots j_{i-1}}^{i-1} \rho^{(j_1)}\rho^{(j_2)} \dots \rho^{(j_{i-1})} \Bigg),$ $\bar{\gamma}_l = \frac{1}{2} \sum_{i=1}^{M} \gamma_l^{i-1}, \qquad \tilde{\gamma}_l^{i-1} = \gamma_l^{i-1} - \rho^{(i)} \gamma_{l-1}^{i-1} \mathbb{1}_{l>0},$
$\varpi_{l,r}^{i-1} = (-1)^l \Bigg(\mathbb{1}_{l=0} + \binom{1}{l} \sum_{j_1=r}^{i-1} \rho^{(j_1)} + \binom{2}{l} \sum_{j_1<j_2}^{i-1} \rho^{(j_1)}\rho^{(j_2)} + \dots$ $\qquad + \binom{i-1}{l} \sum_{j_1<j_2<\dots j_{i-1}}^{i-1} \rho^{(j_1)}\rho^{(j_2)} \dots \rho^{(j_{i-1})} \Bigg),$ $\bar{\varpi}_{d,j+1} = \sum_{i=j+1}^{M} \varpi_{d,j+1}^{i-1},$ $\tilde{\varpi}_{d,j+1}^{i-1} = \rho^{(j)} \varpi_{d,j+1}^{i-1} \mathbb{1}_{j<i-d} - \rho^{(i)}\rho^{(j)} \varpi_{d-1,j+1}^{i-1},$
$\tilde{\mu}_{l,r}^{i-1} = \varpi_{l,r}^{i-1} \mathbb{1}_{r<i-l+1} - \rho^{(i)} \varpi_{l-1,r}^{i-1} \mathbb{1}_{l>0},$ $\omega^2 = \beta^2 \left(\sum_{l=0}^{d} \sum_{r=1}^{M-l+1} \left(\tilde{\mu}_{l,r}^{M-1} \right)^2 \right)$
$\eta_{T-k}^{l} = \frac{1}{\tilde{\gamma}_0^{M-1}} \Bigg(\left(\bar{\gamma}_l + \mathbb{1}_{l<d} \eta_{T-k+1}^{l+1} - \mathbb{1}_{l=d} \sum_{j=1}^{M-d} \psi_{T-k+1}^{j} \gamma_0^{j-1} \right) \tilde{\gamma}_0^{M-1}$ $\qquad - \left(\bar{\gamma}_0 + \eta_{T-k+1}^{1} \right) \tilde{\gamma}_l^{M-1} \Bigg)$ and $\eta_{T+1}^{l} = 0$ for all $k \ge 0$
$\psi_{T-k}^{j} = \left(\rho^{(j)} \bar{\varpi}_{d,j+1} + \sum_{r=j}^{M-1-d} \psi_{T-k+1}^{r+1} \rho^{(j)} \varpi_{0,j+1}^{r} \right) \mathbb{1}_{j<M-d} - \tilde{\varpi}_{d,j+1}^{M-1} \frac{\left(\bar{\gamma}_0 + \eta_{T-k+1}^{1}\right)}{\tilde{\gamma}_0^{M-1}}$ and $\psi_{T+1}^{j} = 0$ for all $k \ge 0$,
$a_{T-k} = \omega \sqrt{-2 \log \left(\frac{\left(\bar{\gamma}_0 + \eta_{T-k+1}^{1}\right)\sqrt{2\pi}\omega}{\tilde{\gamma}_0^{M-1} \alpha} \right)},$ $\delta_{T-k} \frac{\left(\bar{\gamma}_0 + \eta_{T-k+1}^{1}\right) a_{T-k}}{\tilde{\gamma}_0^{M-1}} + \alpha \left[1 - \Phi\left(a_{T-k}\right) \right] + \delta_{T-k+1},$ and $\delta_{T+1} = 0$ for all $k \ge 0$.

Table 5.1: Notations and constants

converges:

$$\frac{\bar{W}_k^i}{\lambda^i} = \frac{1}{\lambda^i} \sum_{n=0}^{\lambda^i I_k^i} \left(I_k^i - \frac{n}{\lambda^i} \right) \to \int_0^{I_k^i} (I_k^i - x) dx = \frac{(I_k^i)^2}{2}. \tag{5.4}$$

Thus for large λ^i,

$$\bar{w}_k^i \approx \frac{I_k^i}{2} \text{ because } \quad \bar{W}_k^i \approx \lambda^i \frac{(I_k^i)^2}{2} \quad \text{and } X_k^i \approx \lambda^i I_k^i. \tag{5.5}$$

We *refer this approximation throughout, as the fluid approximation.* One gets the same expressions for the average waiting times even with Poisson arrivals and hence these conditional expressions (conditional on bus inter arrival times) are valid for Poisson arrivals (see Chapter 4) also.

The trip average of the waiting times is given by:

$$\frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M} E[\bar{w}_k^i] = \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M}\frac{E[I_k^i]}{2},$$

which is one of the components to be optimized. By Lemma 9.1 (Appendix B), the conditional expectation given the state Y_k and the depot-headway decision h_k equals (see (5.3)):

$$E\left[I_k^i \middle| Y_k, h_k\right] = \sum_{l=0}^{d} h_{k-l}\gamma_l^{i-1} - \sum_{j=1}^{i-1-d} \rho^{(j)} I_{k-1-d}^j \varpi_{d,j+1}^{i-1} \text{ for any } k \geq 1, \tag{5.6}$$

where the coefficients are tabulated in Table 5.1.

Bunching probability

Starting from the depot, the buses travel on a single route with some headway (time gap between successive arrivals to the same location) between successive buses. If these headways were maintained constant thought their journey, the successive buses would not meet each other. However, because of variability in load/traffic conditions, the above is not always true. A bus can get delayed (to some stop) significantly because of the random fluctuations. The delayed bus has larger number of passengers to board and hence is further delayed for the next stop. The trailing bus has lesser number of passengers and hence departs early from the stop. This continues in the subsequent stops, and there is a possibility of the headway between the two buses becoming zero. This is called *bus bunching*.

Bus bunching increases the waiting times of passengers, further and more importantly wastes the capacity of the trailing buses. Thus the system becomes inefficient. The larger depot headway times decreases the chances of bunching but, however, increases the passenger waiting times. Thus one needs an optimal trade-off.

The bunching probability *is the probability that a bus arrives to a stop before the departure of the previous bus.* Recall this is the probability that the dwell time of $(k-1)$-th bus, V_{k-1}^i given by (5.1) is greater than the inter arrival time between $(k-1)$ and k-th buses, I_k^i given by (5.2):

$$P_{B_k}^i = P(V_{k-1}^i > I_k^i) = P(I_k^i - \rho^{(i)} I_{k-1}^i < 0). \tag{5.7}$$

We consider optimizing the bunching probability of the last stop, as this stop experiences maximum variations. Conditioned on Y_k, h_k the value $I_k^M - \rho^{(M)} I_{k-1}^M$ is Gaussian distributed (see

(5.2)) and from Lemma 9.2 of Appendix B, we have for any $k \geq 1$ (constants are given in Table 5.1):

$$\begin{aligned}
P^\phi \left(I_k^M - \rho^{(M)} I_{k-1}^M < 0 \middle| Y_k, h_k \right) &= 1 - \Phi \left(E \left[I_k^M \middle| Y_k, h_k \right] - \rho^{(M)} E \left[I_{k-1}^M \middle| Y_k, h_k \right] \right) \\
&= 1 - \Phi \left(\sum_{l=0}^{d} h_{k-l} \tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I_{k-1-d}^j \tilde{\varpi}_{d,j+1}^{M-1} \right), \quad (5.8) \\
\Phi(x) &:= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\omega^2}} \exp\left(\frac{-t^2}{2\omega^2} \right) dt,
\end{aligned}$$

where Φ is the cdf of a normal random variable with mean 0 and variance ω^2. Observe that the variance ω^2 does not depend upon headway policy, rather only depend upon the load factors of various stops and traffic variability. However, also observe that the bunching probability depend upon the headway policy.

5.2.3 The MDP problem

Let $\phi = (d_1 \cdots, d_T)$ be any Markov policy, in that $d_k(y)$ represents the depot headway for the k-th bus if the system observes the state y. *We choose a headway in the range* $[0, \bar{h}]$ *for some* $\bar{h} < \infty$. The expected values of the above two cost components depend upon the policy ϕ and the initial trajectories specified by (t_0, h_0) (see sub-section 5.2.1). To be more specific, given (t_0, h_0) one has probabilistic description of the system state Y_1. Let $E^\phi_{t_0,h_0}$ represent the expectation given the policy and the initial conditions, at times we omit the subscript and superscript to keep notations simple. We have multi-objective (two) optimization and a natural way is to optimize the following weighted combination of the two costs (5.6), (5.8):

$$\begin{aligned}
J(\phi; h_0, t_0) &= \sum_{k=1}^{T} E^\phi \left[\bar{w}_k^i \right] + \alpha P^\phi \left(I_k^M - \rho^{(M)} I_{k-1}^M < 0 \right) = \sum_{k=1}^{T} E^\phi_{t_0,h_0} \left[r(Y_k, h_k) \right] \text{ with} \\
r(Y_k, h_k) &= \sum_{i=1}^{M} E[I_k^i | Y_k, h_k] + \alpha P^\phi \left(I_k^M - \rho^{(M)} I_{k-1}^M < 0 \middle| Y_k, h_k \right) \\
&= \sum_{l=0}^{d} h_{k-l} \bar{\gamma}_l - \sum_{j=1}^{M-1-d} \rho^{(j)} I_{k-1-d}^j \bar{\varpi}_{d,j+1} + \alpha \left[1 - \Phi \left(\sum_{l=0}^{d} h_{k-l} \tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I_{k-1-d}^j \tilde{\varpi}_{d,j+1}^{M-1} \right) \right],
\end{aligned}$$

where $\alpha > 0$ is the trade-off factor and the constants are in Table 5.1. Our objective is to obtain a policy that optimizes the following for any given (t_0, h_0):

$$v(t_0, h_0) := \inf_\phi J(\phi; t_0, h_0).$$

It is easy to verify that the above value function equals:

$$v(t_0, h_0) = E_{t_0,h_0}\left[v(Y_d)\right],$$

and this can be solved by solving the MDP problem for any given initial condition $y_d = (h_{d-1}, h_{d-2}, \cdots, h_0, \{I^j_{-1}\}_j)$, i.e., by deriving the value function $v(y_d)$ for any y_d (e.g., Puterman (2014)).

5.3 Optimal Policies

The optimal policy is obtained by solving dynamic programming (DP) equations using backward induction. The DP equations, for any $k < T$ are given by (Puterman (2014)):

$$\begin{aligned} v_k(Y_k) &= \inf_{h_k \in [0,\bar{h}]} \left\{ r_k(Y_k, h_k) + E\left[v_{k+1}(Y_{k+1})|Y_k, h_k\right]\right\}, \\ v_{T+1}(Y_{T+1}) &= 0. \end{aligned}$$

From the trip wise running costs (5.6)-(5.8), these equations are rewritten as (constants are given in Table 5.1):

$$\begin{aligned} v_k(Y_k) = \inf_{h_k \in [0,\bar{h}]} \Bigg\{ & \sum_{l=0}^{d} h_{k-l}\bar{\gamma}_l - \sum_{j=1}^{M-1-d} \rho^{(j)} I^j_{k-1-d}\bar{\varpi}_{d,j+1} \\ & + \alpha\left[1 - \Phi\left(\sum_{l=0}^{d} h_{k-l}\tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I^j_{k-1-d}\tilde{\varpi}^{M-1}_{d,j+1}\right) + E\left[v_{k+1}(Y_{k+1})|Y_k, h_k\right]\right]\Bigg\}. \end{aligned} \tag{5.9}$$

One can derive optimal policies by solving these DP equations and there are many known numerical techniques to do the same (e.g., Puterman (2014)). In the following we derive the structure of near optimal policies (closed form expressions) for the case with small load factors:

Theorem 5.1. *Assume $T > M+1$ and assume $\alpha > (1+\tilde{\phi})\beta M\sqrt{2\pi}/2$ for some $\tilde{\phi} > 0$. We define the coefficients $\{\eta^l_k\}_{k,l}$, $\{\psi^j_k\}_{k,j}$ and $\{a_k\}_k$ backward recursively: first set $\eta^l_{T+1} = 0$ for all $0 \le l \le d$, $\delta_{T+1} = 0$, $\psi^j_{T+1} = 0$ for all $1 \le j \le M$ and then set the rest of them as in Table 5.1. There exists a $\bar{\rho} > 0$, such that for all $\rho \le \bar{\rho}$ with $\rho := \max\{\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(j)}\}$ for all $1 \le j \le M$:*

$$\alpha > \frac{(\bar{\gamma}_0 + \eta^1_{T-k+1})\sqrt{2\pi}\omega}{\tilde{\gamma}_0^{M-1}} \text{ for all } k \ge 0. \tag{5.10}$$

Further for all such ρ, the following is an ε-optimal policy[3] *with*[4] $\varepsilon = O(\rho)$*:*

$$\begin{aligned} h^*_{T-k}(Y_{T-k}) &= \max\left\{0,\ \min\left\{\bar{h},\ h^{uc}_{T-k}(Y_{T-k})\right\}\right\},\ \textit{where} \qquad (5.11)\\ h^{uc}_{T-k}(Y_{T-k}) &:= \frac{1}{\tilde{\gamma}_0^{M-1}}\left[-\sum_{l=1}^{d} h_{T-k-l}\tilde{\gamma}_l^{M-1} + \sum_{j=1}^{M-d} I^j_{T-k-1-d}\tilde{\varpi}^{M-1}_{d,j+1} + a_{T-k}\right]. \end{aligned}$$

The expected value function (for any k, Y_{T-k-1}, h_{T-k-1}) equals:

$$\begin{aligned} & E[v_{T-k}(Y_{T-k})|Y_{T-k-1}, h_{T-k-1}] \\ =\ & E\left[\sum_{l=1}^{d} h_{T-k-l}\eta^l_{T-k} - \sum_{j=1}^{M-d} I^j_{T-k-1-d}\psi^j_{T-k} + \delta_{T-k}\middle| Y_{T-k-1}, h_{T-k-1}\right] + O(\rho). \qquad (5.12) \end{aligned}$$

Proof: is in Appendix B. ■

Remarks: i) Thus the ε-optimal policy is affine linear in the previous trip headways and the bus-inter-arrival times. By the above theorem, the policy well-approximates the optimal one, as ρ the load factor reduces. We will notice that the policy works well even for nominal load factors (in some examples upto $\rho = 0.5$) in the next section.

ii) When $d > M$, one can notice from (5.11) that the optimal policy depends only upon the previous trip headways. These are precisely the partially dynamic policies of the previous chapter. These policies have exactly the same structure as in the previous chapter; the policies are now for the case with varying load factors at various stops. Further we would like to point an important difference between these two set of optimal (partially dynamic) policies (which is also discussed in Chapter 4) :

- The partially dynamic policy of Chapter 4 is applicable when we apply the (headway) control from the start of the system. In this case the first bus has to cater for all the customers that arrived from the start, while the later buses serve only the customers that arrived during a bus-inter-arrival period. Thus the nature of dynamics is different in these two phases and as seen from policy (4.11) of previous chapter the effects of the initial phase disappear after initial $(M + 1)$ trips.

- Policy (5.11) (with $d > M$) is the partially dynamic policy, when the buses were operating for more than M trips before the control part started.

- Both types of initial dynamics could be of independent interest.

[3] The cost under this policy is within ε radius of the optimal cost.

[4] Big O notation: $f = O(\rho)$, as $\rho \to 0$, implies $f(\rho) \leq C\rho$ for some constant $C > 0$, for all ρ sufficiently small.

- One can derive fully dynamic policies for both kinds of initial phases in a similar way.

5.4 Numerical analysis

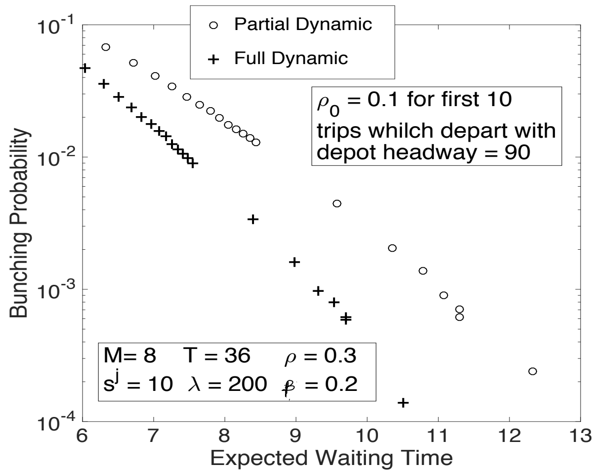

Figure 5.1: Comparison between Partially dynamic and Dynamic policies.

We consider a particular case study of the optimal policy (5.11) to derive numerical analysis; the load factor is the same for all stops ($\rho^{(i)} = \rho$ for all i), and we have one-delay information, i.e., $d = 1$. We conduct Monte-Carlo based simulations to estimate the performance of the system, under various 'optimal' policies.

5.4.1 Partially Dynamic policies (Chapter 4)

In Chapter 4, we derived policies that depend only on previous headways. We refer them as 'partially-dynamic' policies, as they do not consider the random component of the system state Y_k. It is obvious that one can improve with the 'fully' dynamic policies of Theorem 5.1. In this section, we study the extent of improvement provided by the extra information. Towards this we reproduce the optimal policies of Chapter 4, for the purpose of easy comparison. For all load factor $\rho \leq \bar{\rho}$, where $\bar{\rho} > 0$ is an appropriate constant, the optimal policy is given by ($\mathbf{h}_{T-k} := [h_{T-k-1}, h_{T-k-2} \cdots h_{T-k-M}]$):

$$
\begin{aligned}
h^*_{T-k}(\mathbf{h}_{T-k}) &= \left[-\sum_{l=1}^{M} h_{T-k-l}\psi^p_l + a^p_* \right], \text{ with} && (5.13)\\
a^p_* &= \frac{\sigma^M_M}{(1+\rho)^{M-1}} \sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma^M_M}{2(1-\rho)\alpha}\right)}. && (5.14)\\
\psi^p_l &= \frac{1}{(1+\rho)^{M-1}} \left((-1)^l \binom{M-1}{l} \rho^l (1+\rho)^{M-1-l} - (-1)^{l-1} \binom{M-1}{l-1} \rho^l (1+\rho)^{M-l} \mathbb{1}_{l>0} \right).
\end{aligned}
$$

The constant σ^M_M is from Lemma 4.4 of Chapter 4. Recall these are for the special case with equal load factors in each stop.

5.4.2 Experiments

We conduct many Monte Carlo based simulations to compare the proposed dynamic policies with the partially dynamic policies of Chapter 4. We basically generate several sample paths of transport system trajectories, where each sample path is generated using a sample of the random walking times between the stops and the random passenger arrivals at various stops for all the T-trips. We dispatch the buses according to one of the two policies for different values of trade-off factors α and obtain the estimates of the bunching probability and the average passenger waiting times using the sample means.

In Figure 5.1, we plot the estimates of average passenger waiting times versus the estimates of the bunching probability for different values of α and for both the policies. The details of the experiment are mentioned in the figure itself. We notice a significant improvement with fully dynamic policies. The curve of bunching probability versus expected waiting time obtained with fully dynamic policy is placed well below that with partially dynamic policies. This implies that one can simultaneously improve both the performance measures, when one has access to more information about the system state. We conducted many more experiments and the observations are similar.

In Table 5.2 we consider various system configurations, which is described in the first column. We choose different values of α for the two policies such that the bunching probabilities are almost equal (under both the policies) and these values are reported in next two columns. We then tabulate the corresponding average passenger waiting times in the last two columns. These are the estimates averaged across all the T trips. The different configurations span across different levels of traffic variability (β), different load factors during controllable trips (ρ) and or different level of α/trade-off factors. In all the configurations, we notice a good improvement with fully dynamic policies. Since α were chosen such that the bunching probabilities of both the policies are almost equal, one can study the improvement via the improvement in average passenger waiting times. We observe that improvements are in the range of 21% to 44%.

Configuration	Bunching probability		Waiting times	
	Dynamic	Partial	Dynamic	Partial
$\beta = 0.3$, $\rho = 0.3$, α big	1.95e-02	2.1e-02	18.42	25.29
$\beta = 0.3$, $\rho = 0.3$, α small	1.52e-01	1.51e-01	11.33	14.01
$\beta = 0.4$, $\rho = 0.3$, α big	1.78e-02	1.77e-02	24.41	34.15
$\beta = 0.4$, $\rho = 0.3$, α small	1.36e-01	1.37e-01	14.85	18.95
$\beta = 0.2$, $\rho = 0.5$, α big	2.64e-01	2.66e-01	144.23	185.81
$\beta = 0.2$, $\rho = 0.5$, α small	6.54e-02	6.51e-02	242.68	380.01

Table 5.2: Performance for various configurations with Initial trip details: $\rho_0 = 0.2$, $h_0 = 100$, $t_0 = 12$ and the controlled trip details: $M = 10$ $T = 36$ $s^j = 10$ $\lambda = 200$.

5.5 Summary

Unlike the popular models considered in literature, we directly studied the inherent trade-off between the two most important aspects of any bus transport system, the bunching possibilities and the passenger waiting times. Further, we formulated a Markov decision processes based problem to derive optimal (depot) dispatch (i.e., headway) policies that depend upon the random state observed at various bus stops of the previous trips. The observation is that of the time gaps between arrivals of the successive buses at the same stop.

We consider systems with Markovian travel times, fluid passenger arrivals and with d-delayed information. The objective function optimized is the sum of a weighted combination of the two performance measures, corresponding to all the trips of the given session. We obtained a near-optimal dynamic policy for small load factors by solving the corresponding finite horizon dynamic programming equations, using backward induction. This policy is linear in previous trip headways and the bus-inter-arrival times corresponding to a previous trip (the latest trip whose information is available at the decision epoch).

We conducted Monte-Carlo based simulations to plot the estimates of the average passenger waiting times and the bunching probability for various trade off factors. We also observed that the proposed dynamic policy performs significantly better than the previously proposed partially dynamic policies. These partially dynamic policies depend only upon the headways of the previous trips.

Chapter 6

Serve on the Move Wireless LANs

In previous chapters, we discussed the bus transportation system and issues related to bunching. In this chapter, we study an economic wireless based data network exploit constructed to the repeated journey of buses in a public transportation system (see Figure 6.1). Wireless LANs are fitted into a public transport system (ferry) to serve the wireless users in the background, i.e., while the vehicle is on move. We obtained the stability conditions of both intersecting and non intersecting queues. We obtain workload analysis of the system for some scheduling policies and also discuss optimality.

6.1 System Model

A Ferry Based Wireless LAN (FWLAN) is moving along a closed tour $\mathcal{C}$, of length $|\mathcal{C}|$, repeatedly with constant speed α and is serving static users that are waiting along/near its path. It can detect the waiting users, i.e., senses the existence of their signal, once the user is sufficiently close. If a favorable scheduling decision is made, it starts serving the encountered user. It does not stop to serve, rather it serves the user while on move. Thus the received signal starts fading away, as the server moves away from the user and service would be stopped once out of visibility zone. Thus for every user there exists an interval on the path, which we refer to as *service interval*, moving along which the server can potentially serve the user. Let L_p represent the *service interval* of point p on the server path, whose length equals $l_p := |L_p|$. The users in the visible vicinity of the path can be projected on to the path.

The server attends the detected user, either when the previous user completes its service or when

Results of this chapter have been published in the proceedings of WiOpt-2015 (Kavitha and Rao (2015)).

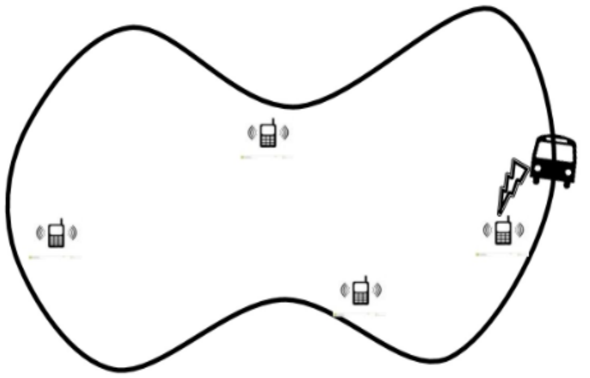

Figure 6.1: Serve on Move FWLAN

it moves out of the visibility zone of the previous user. It may also switch over to the new user (the next visible user) either if the signal strength is significantly higher or if the load is higher, depending upon the *scheduling policy.* Thus there is a possibility that the service of a user might be postponed to the next cycle. In all, server attends: a) some customers from previous cycles; b) some from the present cycle.

Arrivals: Every arrival requests for job requirements of size B and arrives at a random position Q. We have N queues, and the probability that an arrival lands in queue $\mathcal{Q}_i$ is given by:

$$Prob(Q \in \mathcal{Q}_i) = q_i \text{ with } \sum_{i=1}^{N} q_i = 1. \tag{6.1}$$

The service requests arrive at rate λ per cycle and these arrivals are modeled either by a Poisson process or as Bernoulli arrivals. In either case, the rate of arrival (per cycle) into a queue $\mathcal{Q}_i$ equals $\lambda_i = \lambda q_i$. In case of Bernoulli arrivals, at maximum one customer arrives in the queue in any cycle and this happens with probability λ_i. Alternatively, the number of arrivals per cycle into queue $\mathcal{Q}_i$ can be Poisson distributed with parameter λ_i. If $\mathcal{C}$ is the length of the path and if the FWLAN traverses the route at speed α and $\bar{\lambda}$ is rate of the Poisson arrivals, then rate per cycle is given by $\lambda = \bar{\lambda}|\mathcal{C}|/\alpha$. Most of our results can handle random speeds, however we consider fixed speeds for ease of explanation. For example IID (identically and independently distributed) speed variations across cycles, can be modeled as IID service intervals/visit times. This represents a very commonly prevalent scenario. For example, the location of points with clustered arrivals can represent a queue. If there is a set of nearby (almost inseparable) points with high arrivals, one can again group them as a queue. Because of 'Serve on the Move' (SoM) nature, the arrivals elsewhere in the path would not alter the performance of the entire system. Either they are served immediately (in the same cycle) or the left overs are completed in the subsequent cycles. This is because, for sparse arrivals the probability of arrivals at interfering points (with intersecting service intervals) is negligible. Thus we concentrate only on clustered arrivals. We discretize the path (parts of the path with clustered arrivals) into finite intervals, model each interval as one queue (see Kavitha and Altman (2009) for similar details), which results in arrivals as given by (6.1).

Service times: Let μ be the service rate per distance per time. If service is offered for l length while the server moves at α speed, then $(l/\alpha)\mu$ amount of job is completed. We assume without loss of generality that $\mu/\alpha = 1$, i.e., the amount of job completed in one cycle equals the length of the service interval. The job requirements B can depend upon the queue into which they arrive. The conditional moments (first and second) of the job requirements, given that the arrival belongs to $\mathcal{Q}_i$, are given by b_i and $b_i^{(2)}$.

Notations: The cycle indices are usually referred by subscripts k or n while the queue indices by subscripts i or j. The quantities that belong to queue i and for cycle k are referred by double subscripts $_{k,i}$ (e.g., $V_{k,i}$ for workload in $\mathcal{Q}_i$ at k-th departure epoch) while the corresponding stationary quantities are referred by single subscript $_i$ (e.g., V_i for stationary workload in $\mathcal{Q}_i$ at departure epoch).

6.2 Stability Analysis

We call *the system is stable, if there exists a scheduling policy which renders all the queues stable.* We begin with introducing the required notations. Let $\mathcal{N}_{k,i}$ represent the number of arrivals into $\mathcal{Q}_i$ in its k-th cycle time (time duration between $(k-1)$-th and k-th departures of the same queue) and let $\xi_{k,i}$ be the corresponding workload:

$$\xi_{k,i} = \sum_{j=1}^{\mathcal{N}_{k,i}} B_{j,i}, \tag{6.2}$$

where $\{B_{j,i}\}_j$ are IID service times of $\mathcal{Q}_i$ with b_i, $b_i^{(2)}$ as first and second moments. If there are no intersecting intervals, then the departure workload $V_{k,i}$ of any queue $\mathcal{Q}_i$ evolves independently of others according to (with $l_i := |L_i|$):

$$V_{k,i} = (V_{k-1,i} + \xi_{k,i} - l_i)^+ \text{ for all } i. \tag{6.3}$$

Then the system is stable if and only if $E[\xi_i] < l_i$ for each i. This is a well known stability condition of the random walks on half line (see for example Meyn and Tweedie (2009)). With Poisson arrivals $P(\mathcal{N}_{k,i} = n) = e^{-\lambda_i}\frac{(\lambda_i)^n}{n!}$, and apply Wald's lemma for any $n \geq 0$ to Equation (6.2) then we have

$$E[\xi_i] = E\left[\sum_{j=1}^{\mathcal{N}_{k,i}} B_{j,i}\right] = \lambda_i b_i.$$

With Bernoulli arrivals we again have $E[\xi_i] = \lambda_i b_i$, because:

$$P(\mathcal{N}_{k,i} = 1) = \lambda_i = 1 - P(\mathcal{N}_{k,i} = 0).$$

The system is stable if and only if all the queues are stable. In all, the condition for stability with no intersecting intervals is:

$$\max_i \frac{\lambda_i b_i}{l_i} < 1. \tag{6.4}$$

We now consider two queues, say $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where the service intervals L_1 and L_2 intersect for a length δ as in Figure 6.3. Recall that the system is stable, if there exists a scheduling policy which renders all the queues stable and the stability is ensured under the following conditions:

Theorem 6.1. *The two queues are stable if and only if $E[\xi_{k,1} + \xi_{k,2}] < l_1 + l_2 - \delta$ in addition to (6.4).*

Proof: If $E[\xi_i] > l_i$ for any $i = 1, 2$, clearly the corresponding queue is unstable (Meyn and Tweedie (2009)) and hence so is the FWLAN. Two queues share the server capacity, only while the server is traversing δ length. Consider a combined system in which the server can serve any customer of the two queues, while it is traversing anywhere in the interval of the length $l_1+l_2-\delta$. The total workload process in the combined system, is clearly lower than the sum workload $V_{k,1} + V_{k,2}$ of the FWLAN[1]. And the combined system is unstable if $E[\xi_{k,1} + \xi_{k,2}] > l_1 + l_2 - \delta$, and hence so is FWLAN. Thus the FWLAN is unstable if any of the conditions is not true.

[1] One needs to use common cycle time for both the queues in this case.

Now assume all the conditions are true. We consider a scheduling policy in which $\mathcal{Q}_1$ is served independently of $\mathcal{Q}_2$, after dividing the shared service interval as below:

$$\begin{aligned} l_i^{\gamma} &= E[\xi_i] + \gamma\delta \text{ for } i = 1,2 \text{ where} \\ \gamma &:= \min\left\{\frac{-c_e}{2\delta}, \frac{l_1 - E[\xi_1]}{2\delta}, \frac{l_2 - E[\xi_2]}{2\delta}\right\} \text{ with} \\ c_e &:= E[\xi_{k,1} + \xi_{k,2}] - (l_1 + l_2 - \delta) < 0. \end{aligned}$$

The server attends $\mathcal{Q}_1$ while it is traversing interval of length l_1^{γ} starting from first end of L_1 and $\mathcal{Q}_2$ while traversing an interval of length l_2^{γ} whose end coincides with the end of L_2. Because of their definitions and by the given hypothesis, the serving intervals of the two queues do not intersect with each other and thus the two queues evolve independently. Further, both of them are stable again by the well known results of random walk on half line (Meyn and Tweedie (2009)). ■

Thus the stability of two intersecting queues is guaranteed, once the rate of the combined arrivals into the two queues is less than the length of the joint service interval. This is true irrespective of the length δ of intersection. Of course it also demands for individual stability conditions. One can easily extend the above result to the following:

Theorem 6.2. *Consider M intersecting queues with δ_i, for any $i < M$, representing the length of intersection between the service intervals L_i and L_{i+1}. The M queues are jointly stable if and only if the following are satisfied along with (6.4):*

(i) $E[\xi_i + \xi_{i+1}] < l_i + l_{i+1} - \delta_i$ *for any* $i < M$ *and*

(ii) $E[\sum_{j=0}^{2} \xi_{i+j}] < \sum_{j=0}^{2} l_{i+j} - \sum_{j=0}^{1} \delta_{i+j}$ *for any* $i < M - 1$ *and so on till*

(iii) $E\left[\sum_{i \le M} \xi_i\right] < \sum_{j=0}^{M-1} l_{1+j} - \delta$ *with* $\delta := \sum_{i<M} \delta_i$. ■

Proof: is direct from Theorem 6.2.

Henceforth, the stability conditions are assumed to be satisfied. We will now deviate from the SoM FWLAN and obtain the analysis of G-time limited polling systems, which would be instrumental in obtaining the workload performance of the FWLAN.

6.3 G-Time limited polling systems

We consider a time-limited finite polling system, in which the visit time at any queue is pre-decided and is independent of the awaiting workload. So far in the literature, these systems are studied only under exponentially distributed visit times. While we require the time limits/visit times with any arbitrary distribution. Hence we call them as G-time limited polling systems, to indicate that the distribution of the visit times is general.

We have N-queues, the server visits them periodically in the same order, every cycle. In the k-cycle it spends time $G_{k,i}$ at queue $\mathcal{Q}_i$, irrespective of whether or not the later is empty. This system is similar to the autonomous system, studied previously in the context of exponential visit times. The service of any interrupted customer is resumed exactly at the point left, when the server next visits its queue. We consider IID visit times $\{G_{k,i}\}_k$, for all the queues. The server spends time $S_{k,i}$ to walk between the queues $\mathcal{Q}_i$ and $\mathcal{Q}_{i+1}$ (modulo addition) during the k-th cycle.

In all the cases we assume that the cycle time is independent of the (fractions of the) times spent in individual queues. For any queue (say $\mathcal{Q}_i$) the sequence of workloads $\{\xi_{i,k}\}_k$, arrived in successive cycle times, is assumed to be an independent sequence and this is true for each i. Here we consider queue specific cycle times, the time period between successive exits by the server, of the same queue.

Each queue is processed independently of the other queues and hence can be analyzed independently. Let $\Phi_{k,i}$ represent the cycle time (time between two consecutive departures) with respect to queue $\mathcal{Q}_i$:

$$\Phi_{k,i} = \sum_{j=1}^{i-1}(G_{k,j} + S_{k,j}) + \sum_{j\geq i}^{N}(G_{k+1,j} + S_{k+1,j}).$$

The cycle times $\{\Phi_{k,i}\}_k$ form IID sequence for each i. The remaining details of the polling system are similar to that of the FWLAN. With $V_{k,i}$ representing the total workload left at $\mathcal{Q}_i$ when the server leaves the queue for the k-th time, the system evolution can be written using the following equation:

$$V_{k,i} = (V_{k-1,i} + \xi_{k,i} - G_{k,i})^{+}. \tag{6.5}$$

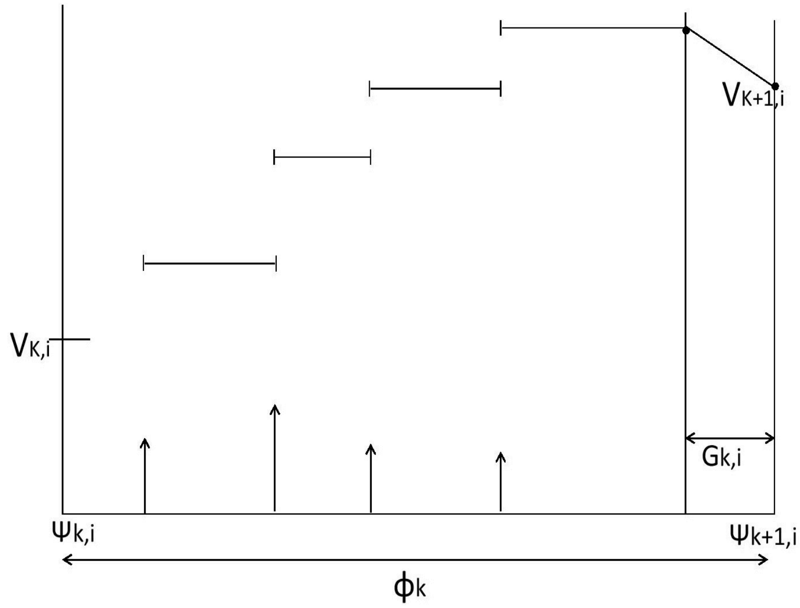

Figure 6.2: Workload process in a cycle

6.3.1 Analysis of workload using GI/G/1 queue

The waiting time of the k-th customer of a GI/G/1 queue can be written in terms of the service time $\mathcal{B}_k$ and the inter arrival times $\mathcal{A}_k$ as below (Blanc (2004-2009)):

$$\mathcal{W}_k = (\mathcal{W}_{k-1} + \mathcal{B}_k - \mathcal{A}_k)^+.$$

From (Blanc, 2004-2009, equation (6.105)) the average waiting time equals:

$$E[\mathcal{W}] = \frac{E[\mathcal{B}^2] - 2E[\mathcal{B}]E[\mathcal{A}] + E[\mathcal{A}^2]}{2(E[\mathcal{A}] - E[\mathcal{B}])} - \frac{E[\mathcal{I}^2]}{2E[\mathcal{I}]}, \tag{6.6}$$

where $\mathcal{I}$ is the idle period and the quantities in the above equation are stationary averages. Equation (6.5) resembles the evolution of GI/G/1 queue, and hence the stationary average of the departure-workload process $\{V_k\}$ can be obtained using the equation (6.6). Further for large load factors (FWLANs usually operate in this regime), we approximate the idle times with inter arrival times, the queue visit times ($\{G_i\}$) in our context; in the coming sections, we would either use this approximation after sufficient justification or directly estimate moments of idle times. Thus the stationary average of the workload at departure epochs of queue $\mathcal{Q}_i$ is given by (see (6.5)-(6.6))

$$E[V_i] = \frac{E[\xi_i^2]-2E[G_i]E[\xi_i]+E[G_i{}^2]}{2(E[G_i]-E[\xi_i])} - \frac{E[\mathcal{I}_i^2]}{2E[\mathcal{I}_i]} \tag{6.7}$$
$$\approx \frac{E[\xi_i^2]-2E[G_i]E[\xi_i]+E[G_i{}^2]}{2(E[G_i]-E[\xi_i])} - \frac{E[G_i^2]}{2E[G_i]}$$
$$= \frac{E[\xi_i^2]E[G_i]-2E[\xi_i](E[G_i])^2+E[G_i{}^2]E[\xi_i]}{2E[G_i](E[G_i]-E[\xi_i])}. \tag{6.8}$$

As already mentioned, the last equation has to be used after sufficient justification.

6.3.2 Time average of workload process

Let $V_i(t)$ represent the workload of queue i at time t. We already obtained the stationary analysis of the workload at departure instances, $\{V_{k,i}\}$ as given by (6.5) , where $V_{k,i} = V_i(\Psi_{k,i})$ with the cycle end instances $\Psi_{k,i} := \sum_{n\leq k} \Phi_{k,i}$. This is a Markov process, in particular a random walk in half line, which is stable and for which the law of large numbers (Meyn and Tweedie, 2009, Proposition 17.6.1, pp. 447) is applicable, as the time limits $\{G_{k,i}\}$ are bounded with probability one.

The area under the workload process during k-th cycle majorly depends upon $V_{k,i}$, new arrivals $\xi_{k+1,i}$ and their arrival instances $\{U_n\}_{n=1}^{\mathcal{N}_{k+1,i}}$ as shown in the Figure 6.2. For Poisson arrivals the arrival instances $\{U_n\}$, conditioned on the cycle length $\Phi_{k+1,i}$, are uniformly distributed over $\Phi_{k+1,i}$, while for Bernoulli arrivals these are assumed to be uniformly distributed.

The computations below can be inaccurate when one or more arrivals occur while the server is at the queue. Further the workload decreases linearly when the server is at the queue, however we neglect this effect too. Given that the cycle times $\Phi_{k,i}$ are large compared to visit times $G_{k,i}$, these inaccuracies lead to negligible errors. Barring these inaccuracies, the required area in k-th cycle equals (area under the curve of Fig. 6.2, after approximating the slanted line during the visit time $G_{k,i}$ with a straight line):

$$\bar{V}_{k,i} := \int_{\Psi_{k,i}}^{\Psi_{k+1,i}} V_i(t)dt \approx V_{k,i}\Phi_{k,i} + \sum_{n-1}^{\mathcal{N}_{k+1,i}} B_{n,i}(\Phi_{k,i} - U_n). \tag{6.9}$$

The above area of the workload, is a function of workload at departures (6.5) and new arriving workload ξ_{k+1} and their arrival instances. The time average of the workload process can be obtained by considering the following limit:

$$\bar{v}_i := \lim_{k\to\infty} \frac{1}{\Psi_{k,i}} \int_0^{\Psi_{k,i}} V_i(t)dt = \lim_{k\to\infty} \frac{\sum_{n=0}^{k-1} \bar{V}_{k,i}}{\Psi_{k,i}}.$$

Using law of large numbers, (Meyn and Tweedie, 2009, Proposition 17.6.1, pp 447), and Wald's lemma the above limit is almost surely a constant given by:

$$\begin{aligned}\bar{v}_i &= \frac{E[V_i]E[\Phi_i] + \lambda_i b_i(E[\Phi_i] - E[\Phi_i/2])}{E[\Phi_i]} \\ &= E[V_i] + \frac{\lambda_i b_i}{2} \text{ a.s.}\end{aligned} \tag{6.10}$$

The above is obtained by computing the stationary average of the terms on the right hand side of (6.9), which in turn is obtained by conditioning on the cycle lengths $\Phi_{k,i}$ and the number of arrivals $\mathcal{N}_{k,i}$. In case of Poisson arrivals, by PASTA property $\bar{v}_i$ represents the stationary average workload of the system in queue i. We notice from (6.10) that optimizing the stationary average workload $\bar{v}_i$ is equivalent to optimizing the stationary average workload at departure instances $E[V_i]$ (given by (6.7)).

The total stationary average workload of the polling system equals:

$$\bar{v} = \sum_i \bar{v}_i = \sum_i \left(E[V_i] + \frac{\lambda_i b_i}{2} \right).$$

Using the results of this section, we obtain the workload analysis for various configurations of SoM FWLAN in the coming sections.

6.4 Workload Analysis

6.4.1 Queues without intersecting intervals

The workload analysis of any queue is independent of the others. For every i, the server drains out maximum possible workload of $\mathcal{Q}_i$ while traversing its service interval, L_i. The departure workload evolution is given by equation (6.3).

FWLAN with N non intersecting queues can be modeled by N-queue G-time limited polling system of the previous section. The walking/switching times of the polling system are given by the physical time taken by the server to move from one queue to other, while the cycle times of all the queues are deterministic and equal $|\mathcal{C}|/\alpha$. From the previous section (see equation (6.7)), the workload analysis of FWLAN depends upon the idle times of a fictitious GI/G/1 queue, whose waiting time evolution resembles the evolution given by (6.3). The stationary analysis of workload at departures (6.3) can be obtained once the first two moments of the idle period of the corresponding fictitious GI/G/1 queue is computed.

An idle period (in fictitious queue) by definition is the time interval between the exit of the last customer and the next arrival, given there is a positive time difference between the two. In other words, an idle period is a fraction of those interval arrival periods, in which the exit of the last customer occurs before the next arrival. The idle periods in the fictitious queue occurs under the following two conditions of the original FWLAN:

a) when the newly arrived workload $\xi_{k,i}$ is finished before the next arrival (i.e., if $l_i - \xi_{k,i} > 0$) and when previous epoch workload, $V_{k-1,i} = 0$;

b) or because the newly arrived workload as well as the left over workload is finished before the next arrival, i.e., if $l_i - V_{k-1,i} - \xi_{k,i} > 0$.

Obviously the possibility of the former event is higher. When the load factor increases (by decreasing l_i) the busy period increases. However the nature of subsequent idle periods, once the queue is empty, remains the same. Hence as load factor increases, the fraction of idle periods resulting out of event (a) increases. Thus, our conjuncture is that the idle period, with sufficient load factor, can be well approximated by the events of the type[2] (a). So we approximate the stationary idle periods of fictitious queue by

$$\mathcal{I}_i = l_i - \xi_i \text{ conditioned that } l_i > \xi_i.$$

Thus with $p_i := Prob(\xi_i > 0) = Prob(\mathcal{N}_i > 0)$,

$$\begin{aligned} E[\mathcal{I}_i] &\approx \frac{l_i(1-p_i) + p_i E[(l_i - \xi_i); \xi_i \leq l_i]}{(1-p_i) + p_i P(\xi_i \leq l_i)}, \\ E[\mathcal{I}_i^2] &\approx \frac{l_i^2(1-p_i) + p_i E[(l_i - \xi_i)^2; \xi_i \leq l_i]}{(1-p_i) + p_i P(\xi_i \leq l_i)}. \end{aligned} \tag{6.11}$$

Substituting the above directly into equation (6.7) we obtain the stationary average of the departure epoch workload.

The arrivals in any queue of FWLAN occur while the server is traversing a long path of length $|\mathcal{C}|$, while they are served only when the server passes through the corresponding service interval, which is much smaller. Under stability conditions (6.4), the expected value of the arrived workload in one such cycle is smaller than the workload cleared while traversing the much smaller service intervals. Thus the load factors are usually large and hence idle period moments can be well approximated by (6.11). Further there is a positive probability $(1 - p_i)$ of zero arrivals in any cycle and this probability can also be significant. If the probability of zero arrivals $(1 - p_i)$ is significant, one can in fact neglect the second terms in the numerators of (6.11) and

[2] This conjuncture needs explicit proof and we are working towards it.

approximate idle period by inter-arrival time. We consider this approximation for the rest of the analysis which results in equation (6.8) of section 6.3 and corresponding formula for the stationary workload at departure epoch of $\mathcal{Q}_i$ in the FWLAN equals:

$$E[V_i] \approx \frac{E[\xi_i^2] - E[\xi_i]l_i}{2(l_i - E[\xi_i])} \quad \text{for any} \quad i. \tag{6.12}$$

The accuracy of this approximation is demonstrated via Monte-Carlo (MC) simulations in Table 6.2.

Simulations

We consider a typical queue $\mathcal{Q}_i$ and generate a sufficiently long random sequence $\{\xi_{k,i}\}_{k \leq N}$ (with $N > 5000000$). Using this, we generate the sample path of the departure workloads $\{V_{k,i}\}_{k \leq N}$ as given by (6.3), and estimate the stationary average departure epoch workload using the sample mean

$$\frac{1}{N} \sum_{k \leq N} V_{k,i}.$$

We compare it with two approximate formulae: one obtained using (6.11) and the other given by (6.12). We consider both Bernoulli and Poisson arrivals and various terms related to the computations are in Table 6.1. The Poisson arrivals with probability $P(\mathcal{N}_i > 1)$ is close to zero, will have similar terms as that for Bernoulli arrivals and we use the same approximation. We consider uniformly distributed service requirements $\{B_i\}$.

We notice from Table 6.2 that both the formulae (columns 3 and 4) approximate well the stationary workloads estimated via MC simulations (column 2). Majority of cases, the approximation error is well within 10%. When the probability of arrival p_i in a cycle is small (.05), the approximation is good even for small values of ρ (until 0.083 as seen from the first 4 rows). With moderate values of p_i we have good approximation for larger ρ: e.g., when (p_i, ρ) equals one of $(0.4, 0.5)$, $(0.6, 0.9)$, $(0.864, 0.95)$ for Poisson arrivals or the pair equals one of $(0.5, 0.5)$, $(0.9, 0.9)$ for Bernoulli arrivals.

In all, the formula (6.12) is a good approximation for stationary departure workloads (with significant load factors).

6.4.2 Queues with intersecting service intervals

We begin with the scenario of Figure 6.3, that of two queues with intersecting service intervals. The dots represent the queues, while the curves above the dots represent the signal strength of

Configuration	Terms related to $\mathcal{Q}_i$
Bernoulli	$E[\xi_i] = p_i b_i$, $E[\xi_i^2] = p_i b_i^{(2)}$
Poisson	$E[\xi_i] = p_i b_i$, $E[\xi_i^2] = \lambda_i b_i^{(2)} + \lambda_i^2 (b_i)^2$
Bernoulli Uniform$([0,b])$	$E[(l_i - \xi_i); 0 < \xi_i \le l_i] = p_i \frac{l_i^2}{2b}$, for $b > l_i$ and $E[(l_i - \xi_i)^2; 0 < \xi_i \le l_i] = p_i \frac{l_i^3}{3b}$

Table 6.1: Some terms related to the formulae

l_i, λ_i, P/B	MC	$E[V_i]$=(6.12)	$E[V_i]$ =(6.11) in (6.7)	ρ	p_i
.3, .05, P	16.12	16.54	16.54	.83	.05
.3, .05, B	15.76	15.92	15.92	.83	.05
3, .05, P	.191	.178	.186	.083	.05
3, .05, B	.179	.167	.175	.083	.05
5, .5, B	1.11	.833	1.17	.5	.5
3.5, .5, P	7.18	7.08	7.2	.714	.39
3.5, .5, B	4.16	3.96	4.13	.714	.5
4, .7, P	21.76	21.58	21.81	.875	.5
4, .7, B	9.73	9.33	9.76	.875	.7
5, .9, P	28.25	27.75	28.2	.9	.59
5, .9, B	8.20	7.5	8.65	.9	.9
10.5, 2, P	63.5	61.67	64.35	.95	.864

Table 6.2: Monte Carlo Simulations with $B_i \sim$ Uniform ([0,10])

the corresponding queue with distance. Service intervals are determined by the signal strength up to which the FWLAN can identify the customer, and these intersect.

The evolution of the two queues depends upon the scheduling policies used. The two queues evolve as below:

$$\begin{aligned} V_{k,1} &= (V_{k-1,1} + \xi_{k,1} - G_{k,1})^+ \\ V_{k,2} &= (V_{k-1,2} + \xi_{k,2} - G_{k,2})^+, \end{aligned} \tag{6.13}$$

where the sequences $\{G_{k,1}\}_k$, $\{G_{k,2}\}_k$ are determined by the scheduling policy used. By *scheduling decision, we mean the allocation of the shared interval to the individual queues,*

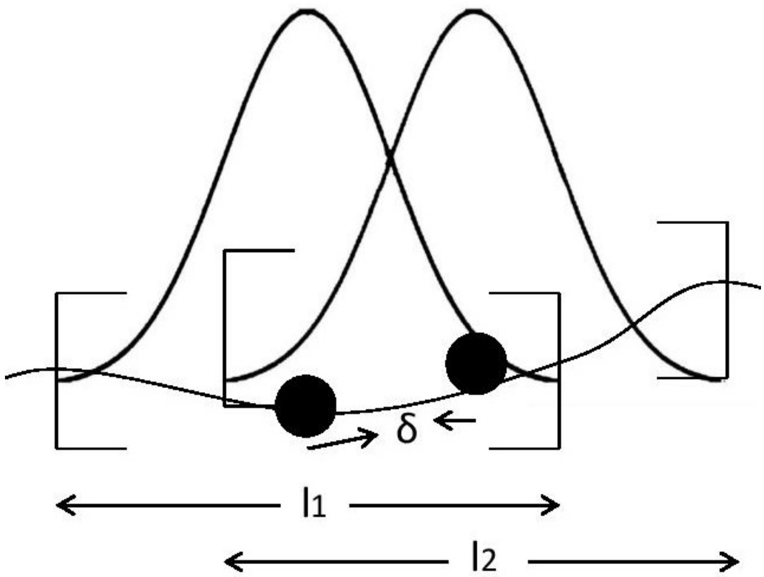

Figure 6.3: Two intersecting queues

either dynamic or static. Note that $G_{k,1}$, $G_{k,2}$ and $G_{k,1}$+ $G_{k,2}$ will be upper bounded respectively by l_1, l_2 and l_1+ l_2 - δ, at any time k.

In *a naturally used policy,* the server first senses the users of the first queue in the orbit, say $\mathcal{Q}_1$, completes their work requirements to the maximum extent possible (while traversing L_1). It then looks for the users of $\mathcal{Q}_2$. Thus the workload evolution at $\mathcal{Q}_1$ is independent of the second one (as in the case with no intersecting intervals) while that of the second one depends upon the left overs of the first one:

$$
\begin{aligned}
G_{k,1} &= l_1 \text{ for all } k \quad \text{and} \\
G_{k,2} &= (l_2 - \delta) + \min\left\{\delta,\ (l_1 - (V_{k-1,1} + \xi_{k,1}))^+\right\}. \qquad (6.14)
\end{aligned}
$$

With natural scheduling policy the first queue in the orbit is the favoured one, while the second one can suffer, more so if its load requirements are larger, i.e., if $E[\xi_2]/l_2 > E[\xi_1]/l_1$. We thus study another set of scheduling policies. We continue with the independent policies of Theorem 6.2 and obtain optimal shared interval division. That is, we set

$$G_{k,1} = l_1 - \gamma\delta \text{ and } G_{k,2} = l_2 - (1-\gamma)\delta \text{ for all } k, \qquad (6.15)$$

and chose a γ^* that minimizes the sum of the workloads. We compare the optimal policy with the natural policy.

We are interested in choosing the optimal policies that minimize the stationary average workload given by equation (6.10) of section 6.3. And as observed in section 6.3 the only controllable part of this workload is the stationary average workloads of the various queues at the departure instances, $E[V_1]$ and $E[V_2]$ and hence work directly with these.

Natural Policy (6.14)

In a natural policy the server attends the users, as and when it senses one and keeps attending them as long as they are in its visibility zone. Hence it first senses the users of the first queue in orbit and attends that of the next one only when the users of first queue are away. Because of preference, the shared service interval is used by the first queue whenever it has load. This kind of a policy could be good as long as the load factor of the first queue is high. On the contrary if the second queue has much higher load factor, it might be better to dedicate the shared interval to the second one. The partial load of the first one, if required, can be postponed to the next cycle to be served utilizing the service interval unavailable to the second queue. This helps drain out the higher load-second queue at faster rate, while the lower load of the first queue is drained out sufficiently faster, just using the private service interval.

We *analyze the policy (6.14) under an extra assumption that* $\xi_1 \leq l_1$ *almost surely.* Under this assumption, the workload in $\mathcal{Q}_1$ is never carried forward to the next cycle, i.e., $V_{k,1} = 0$ almost surely and hence with $\tilde{l}_i := l_i - \delta$ for $i = 1, 2$

$$\begin{aligned} G_{k,2} &= l_2 - \delta + \min\{\delta, (l_1 - \xi_{k,1})^+\} \\ &= l_2 1_{\{\xi_{k,1} \leq \tilde{l}_1\}} + (\tilde{l}_2 + (l_1 - \xi_{k,1})) 1_{\{\xi_{k,1} > \tilde{l}_1\}}. \end{aligned}$$

Thus the first scheduling sequence $\{G_{k,1}\}_k$ is a constant sequence while $\{G_{k,2}\}_k$ is an IID sequence and hence the results of section 6.3 can be applied. It is easy to see that $E[G_1] = l_1$, $E[G_1^2] = l_1^2$ while with $\tilde{l}_i := l_i - \delta$

$$\begin{aligned} E[G_2] &= l_2 P(\xi_1 \leq \tilde{l}_1) + E[\tilde{l}_2 + l_1 - \xi_1; \xi_1 > \tilde{l}_1] \\ E[G_2^2] &= l_2^2 P(\xi_1 \leq \tilde{l}_1) + E[(\tilde{l}_2 + l_1 - \xi_1)^2; \xi_1 > \tilde{l}_1]. \end{aligned}$$

We have $E[V_1] = 0$ and the stationary workload of the second queue at the departure epoch $E[V_2]$ is obtained by substituting the above into equation (6.8) of section 6.3.

This assumption is satisfied by Bernoulli arrivals (at maximum one arrival) with $B_{k,1} \leq l_1$ almost surely for all k. It is also satisfied approximately by Poisson arrivals when the arrival rate λ_1, is small.

Independent policies (6.15)

From (6.15), we have constant scheduling sequences which satisfy the assumptions of section 6.3. For any fraction γ, with $l_1^\gamma := l_1 - \gamma\delta$ and $l_2^\gamma := l_2 - (1-\gamma)\delta)$

$$E[G_i] = l_i^\gamma \text{ and } E[G_i^2] = (l_i^\gamma)^2, \text{ for } i = 1, 2.$$

The stationary workloads $E[V_1]$, $E[V_2]$ can be computed using (6.8) as before, and exactly resemble equation (6.12) with l_i replaced by l_i^γ. Let $\bar{V}^I(\gamma) = E^I[V_1] + E^I[V_2]$ represent the total workload and let $\sigma_{\xi_i}^2 := E[\xi_i^2] - (E[\xi_i])^2$. We have the following

Lemma 6.3. *The total departure epochs workload $\bar{V}^I(\gamma)$ is optimized by the division threshold*

$$\begin{aligned} \gamma^* &= \max\{0, \min\{1, \tilde{\gamma}\}\} \text{ where} \\ \tilde{\gamma} &:= \frac{l_1\sigma_{\xi_2} - l_2\sigma_{\xi_1} - E[\xi_1]\sigma_{\xi_2} + E[\xi_2]\sigma_{\xi_1} + \delta\sigma_{\xi_1}}{\delta(\sigma_{\xi_2} + \sigma_{\xi_1})}. \end{aligned}$$

Proof: Let $g(\gamma)$ represent the derivative of $\bar{V}^I(\gamma)$:

$$\begin{aligned} g(\gamma) &= \delta\left(\frac{E[\xi_1^2] - (E[\xi_1])^2}{2(l_1^{\gamma^*} - E[\xi_1])^2} - \frac{E[\xi_2^2] - E[\xi_2])^2}{2(l_2^{\gamma^*} - E[\xi_2])^2}\right) \\ &= \delta\left(\frac{\sigma_{\xi_1}^2}{2(l_1^{\gamma^*} - E[\xi_1])^2} - \frac{\sigma_{\xi_2}^2}{2(l_2^{\gamma^*} - E[\xi_2])^2}\right) \end{aligned}$$

If $\tilde{\gamma} < 0$, $\sigma_{\xi_2}(l_1 - E[\xi_1]) - \sigma_{\xi_2}(l_2\sigma_{\xi_1} - \delta - E[\xi_1]) < 0$, and this implies $g(0) > 0$. The derivative g is an increasing function of γ, when $l_1^\gamma > 0$ and $l_2^\gamma > 0$. These γ render both the queues stable and hence one needs chose optimal among these gamma. Thus $g(0) > 0$ implies $g(\gamma) > 0$ for all γ. Hence $\bar{V}^I(\gamma)$ is increasing with $\gamma \uparrow$ and therefore $\gamma^* = 0$. Similarly if $\tilde{\gamma} > 1$, then $g(1) < 0$ as well as $g(0) < 0$. Thus $\bar{V}^I(\gamma)$ decreases with $\gamma \uparrow$ and hence $\gamma^* = 1$.
When $0 < \tilde{\gamma} < 1$, then $g(0) > 0$ and $g(1) < 0$ and $g(\tilde{\gamma}) = 0$. By differentiating g with respect to γ, one can see that the second derivative of workload $\bar{V}^I(\gamma)$ is positive at $\tilde{\gamma}$. Thus we have the minimizer at the interior $\tilde{\gamma}$. ■

M intersecting queues

Consider M intersecting queues in cascade, as in Theorem 6.2, with δ_i representing the intersecting length between L_i and L_{i+1} for any $i < M$. For each i, we divide the common service interval δ_i optimally between $\mathcal{Q}_i$ and $\mathcal{Q}_{i+1}$ and consider independent processing as before. Let $\Gamma = [\gamma_1, \cdots, \gamma_{M-1}]$ represent the corresponding fractions. The lengths of the service intervals dedicated for individual queues are:

$$
\begin{aligned}
l_1^{\Gamma} &:= l_1 - \gamma_1 \delta_1, \qquad l_M^{\Gamma} = l_M - (1-\gamma_{M-1})\delta_{M-1} \\
l_i^{\Gamma} &:= l_i - (1-\gamma_{i-1})\delta_{i-1} - \gamma_i \delta_i, \text{ for all } 1 < i < M.
\end{aligned}
$$

The stationary workload for queue $\mathcal{Q}_i$ is given again by (6.12) after replacing l_i by l_i^{Γ}. Computing the derivative and equating it to zero, as done in previous section for case with interior optimizer, we obtain the optimal division ratios as $\Gamma^* = D^{-1}\mathbf{e}$, where the matrix D and the vector $\mathbf{e}$ are defined in the equation (6.16).

$$
D = \begin{bmatrix}
-\delta_1(\sigma_{\xi_2}+\sigma_{\xi_1}) & \delta_2\sigma_{\xi_2} & 0 & \cdots & 0 \\
\delta_1\sigma_{\xi_3} & -\delta_2(\sigma_{\xi_2}+\sigma_{\xi_3}) & \delta_3\sigma_{\xi_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \delta_{M-3}\sigma_{\xi_{M-1}} & -\delta_{\xi_{M-2}}(\sigma_{\xi_{M-1}}+\sigma_{\xi_{M-2}}) & \delta_{M-1}\sigma_{\xi_{M-1}} \\
0 & 0 & \cdots & \delta_{M-2}\sigma_{\xi_M} & -\delta_{M-1}(\sigma_{\xi_M}+\sigma_{\xi_{M-1}})
\end{bmatrix}
$$

$$
\mathbf{e} = \begin{pmatrix}
l_2\sigma_{\xi_1} - l_1\sigma_{\xi_2} - \delta_1\sigma_{\xi_1} + E[\xi_1]\sigma_{\xi_2} - E[\xi_2]\sigma_{\xi_1} \\
l_3\sigma_{\xi_2} - l_2\sigma_{\xi_3} + \delta_1\sigma_{\xi_3} - \delta_2\sigma_{\xi_2} + E[\xi_2]\sigma_{\xi_3} - E[\xi_3]\sigma_{\xi_2} \\
\vdots \\
l_{M-1}\sigma_{\xi_{M-2}} - l_{M-2}\sigma_{\xi_{M-1}} + \delta_{M-3}\sigma_{\xi_{M-1}} - \delta_{M-2}\sigma_{\xi_{M-2}} + E[\xi_{M-2}]\sigma_{\xi_{M-1}} - E[\xi_{M-1}]\sigma_{\xi_{M-2}} \\
l_M\sigma_{\xi_{M-1}} - l_{M-1}\sigma_{\xi_M} + \delta_{M-2}\sigma_{\xi_M} - \delta_{M-1}\sigma_{\xi_{M-1}} + E[\xi_{M-1}]\sigma_{\xi_M} - E[\xi_M]\sigma_{\xi_{M-1}}
\end{pmatrix}. \qquad (6.16)
$$

The boundary optimal points can be obtained as done in Lemma 6.3.

Comparison of the two policies

We now compare the two policies, natural and optimal independent policy. The sum workloads are computed for the example scenario with $B_2 \sim$ Uniform([0,1]) and $B_1 \equiv l_1$ and the results are tabulated in Table 6.3. We consider Bernoulli arrivals at both the queues. When load factor of the first queue is larger than that of the second, the natural policy performs better (first three rows of Table 6.3). Under the converse conditions, the independent policy performs better (last two rows of Table 6.3). In fact the performance of the natural policy is significantly inferior (33% in the last row), calling in the need for better scheduling policies.

Coupled Policies and Future work

We have seen that the natural policy can be inefficient, especially when the queues later visited by the server, have higher load factor. This calls for a policy which forces the server to stop serving the customers of the closer queues, even before it moves out of the visibility zone. We

$\lambda_1, \lambda_2, l_1 = l_2$	γ^*	$E^N[V]$, $E^I[V]$	ρ_1, ρ_2
0.97, 0.1, 0.12	0	0.41, 0.43,	.97, .56
0.9, 0.1, 0.12	0.116	0.39, 0.41	.93, .57
0.8, 0.16, 0.2	0.217	0.29, 0.32	.86, 52
0.2, 0.23, 0.15	1	1.17 , 0.86	.29, .77
0.24, 0.28, 0.18	1	1.30, 0.87	.34, .78

Table 6.3: Natural policy versus Optimal Independent policy

found the optimal switching threshold, while using independent policies. It would be advantageous to consider dynamic switching (based on the workload status of the second queue), in case there is a mechanism to sense the workload of the second, a priori. We would like to study these policies in future. The results of section 6.3 can be used as before, however, only if the scheduling sequences $\{G_{k,1}\}_k$, $\{G_{k,2}\}_k$ are IID. The study of non IID scheduling policies require the results of Markovian queueing models, and would be a topic of future research.

In the following we present few preliminary discussions, while a detailed analysis would be considered in future. We consider the policies, that depend only upon the new workload arrived in the latest cycle:

$$
\begin{aligned}
G_{k,1} &= l_1 - \delta + \delta 1_{\{\xi_{k,1}>0\}} 1_{\{\xi_{k,2}=0\}} + \gamma\delta \left(1_{\{\xi_{k,1}=0\}} 1_{\{\xi_{k,2}=0\}} + 1_{\{\xi_{k,1}>0\}} 1_{\{\xi_{k,2}>0\}} \right) \\
G_{k,2} &= l_2 - \delta + \delta 1_{\{\xi_{k,1}=0\}} 1_{\{\xi_{k,2}>0\}} + (1-\gamma)\delta \left(1_{\{\xi_{k,1}=0\}} 1_{\{\xi_{k,2}=0\}} + 1_{\{\xi_{k,1}>0\}} 1_{\{\xi_{k,2}>0\}} \right).
\end{aligned}
$$

The corresponding moments are given by:

$$
\begin{aligned}
E[G_1] &= l_1 - \delta + \delta p_1(1-p_2) + \gamma\delta((1-p_1)(1-p_2) + p_1 p_2), \\
E[G_2] &= l_2 - \delta + \delta p_2(1-p_1) + \delta(1-\gamma)(1-p_1-p_2+2p_1p_2).
\end{aligned}
$$

The stationary workload is obtained by substituting the above into (6.8). Computing the derivative and equating it to zero, we obtain the optimal division ratio for two intersecting queues, when the optimizer is an interior point:

$$
\begin{aligned}
\gamma^* &= \frac{-l_1\sigma_{\xi_2} + l_2\sigma_{\xi_1} + \delta\sigma_{\xi_2} - \delta p_1(\sigma_{\xi_2} + \sigma_{\xi_1})}{(\sigma_{\xi_2} + \sigma_{\xi_1})(1-p_1-p_2+2p_1p_2)} \\
&\quad + \frac{\delta p_1 p_2(\sigma_{\xi_2} + \sigma_{\xi_1}) + E[\xi_1]\sigma_{\xi_2} - E[\xi_2]\sigma_{\xi_1}}{(\sigma_{\xi_2} + \sigma_{\xi_1})(1-p_1-p_2+2p_1p_2)}.
\end{aligned}
$$

6.5 Summary

Ferry based WLANs offer the service while on move, where as the existing polling models do not consider this feature. We obtain approximate analysis of such general time limited polling systems, and demonstrate that the approximation is good under high load conditions using simulations. We show that the controllable part of the stationary average workload is given by the stationary average workload at departure epochs. The departure epoch stationary workload is obtained using the waiting time analysis of a fictitious GI/G/1 queue. The performance of SoM FWLAN majorly depends upon the service intervals of various queues and their intersections. We obtained the stability conditions, considering the intersecting service intervals. We also discussed some scheduling policies, which allocate/divide the shared service interval among the various intersecting queues. We also discussed optimal policy in a given family of scheduling policies.

As of now, we considered the policies that make independent decisions across cycles. We basically considered policies that depend upon the new workload arrived in the cycle just before the visit. While a more realistic dynamic policy would exhibit correlations, depends both upon the new workload as well as the workload left over at the previous departure epoch. The study of such policies would be a topic of future interest.

Chapter 7

Conclusions and Future directions

In this thesis, we derived optimal (open loop/closed loop) policies related to two important aspects in intelligent transportation systems; optimal bus-dispatch policies for public transportation systems and optimal scheduling policies for serve-on-the move wireless LANs mounted on public transportation systems.

Optimal bus dispatch policies: We considered a bus transport system, where the buses start their journey from the depot and traverse along a cyclic route repeatedly. We considered fluid arrivals and boarding times, Gaussian (bus) travel times and derived two important performance measures, bunching probability and passenger-average waiting time. We derive these performance measures under stationary policies (constant headway policies, i.e., the buses depart the depot at equal intervals of times, 'partially' dynamic policies (headway of any trip depends upon some information related to previous trips) and fully dynamic policies (headway also depends upon trip details like bus-inter-arrival times at various stops). The first two policies are open loop policies, while the third one is a closed loop policy. The last two policies are non-stationary policies while the first is the stationary policy.

We obtained (stationary) Pareto frontier, the set of all efficient points; each efficient point is a pair of performance measures obtained by a stationary policy, such that there exists no other stationary policy which can simultaneously improve both the performances. The Pareto frontier is obtained by solving several optimization problems: a) each problem optimizes a weighted combination of the two performance measures; and b) the complete frontier is obtained by considering all possible trade-off parameters/weights.

In the second type of policies (referred by us as partially dynamic policies), the headway decision depends upon the headways of the previous trips; it is independent of other details of previous trips (e.g., trip times, number of waiting passengers at various stops etc). Thus our partially dynamic policies depend upon easily available (cheap) information, nevertheless, they perform significantly better than (commonly used) stationary policies.

We also derived partially dynamic Pareto frontier, by optimizing over the partially dynamic policies. We derived this frontier by minimizing the weighted-combined costs and solving the corresponding dynamic programming equations. We obtained closed form expressions for the optimal policies under low load conditions; the optimal policies depend linearly upon the previous headway times. Using Monte-Carlo based simulations we demonstrated that our linear (dynamic) policies significantly outperform the best static/stationary policy under all load conditions; the stationary Pareto frontier is significantly away from the dynamic Pareto frontier. By using asymptotic analysis, we further proposed a simplified algorithm to implement the optimal policy. The simplified algorithm has almost similar performance as that of the exact algorithm implementing the optimal policy.

We derived the optimal policies under correlated as well as independent (bus) travel times. The optimal policies are structurally (linearly depends upon previous headways) the same, for both the types of travel times; the only difference is in (some) constants/coefficients defining the policy.

We considered the system with correlated travel times and further derived fully dynamic policies; these are the actual closed loop policies, where the headway decision depends upon the (random) information related to a previous trip (the latest available information). Because of the nature of the problem, one has access only to d-delayed information (with d definitely bigger than one) and the fully dynamic policies depend upon this delayed information and the headways of those previous trips for which the information is not available.

The objective function is again the sum of a weighted combination of the two performance measures, corresponding to all the trips of the given session. We obtained a near-optimal dynamic policy for small load factors by solving the corresponding finite horizon dynamic programming equations, using backward induction. This policy is again linear in previous trip headways and the bus-inter-arrival times corresponding to the latest available trip. Interestingly with d-delay information, the optimal policy at decision epoch k depends only upon the information related to first $(M-1-d)$ stops of the $(k-1-d)$-th trip (M is the total number of stops) and upon headways used for $(k-1)$ to $(k-d)$ trips.

We estimated and plotted Pareto frontier under all the three sets of policies and observed that the dynamic policies significantly improve both the performances, followed by the partially dynamic policies. However it may not be easy to derive the information required for fully dynamic policies (every bus has to store the information related to various stops), while the partially dynamic policies depend upon readily available (at depot) information (that of previous trip headways). Thus we propose to replace more commonly used stationary policies with partially dynamic policies and with fully dynamic policies (if more information can be recorded).

Serve-on-move LANs: Ferry based WLANs offer the service while on move, where as the existing polling models do not consider this feature. We obtain approximate analysis of such general time limited polling systems, and demonstrate that the approximation is good under high load conditions using simulations. We show that the controllable part of the stationary average workload is given by the stationary average workload at departure epochs. The departure epoch stationary workload is obtained using the waiting time analysis of a fictitious GI/G/1 queue. The performance of SoM FWLAN predominantly depends upon the service intervals of various queues and their intersections. We obtained the stability conditions, considering the intersecting service intervals. We also discussed some scheduling policies, which allocate/divide the shared service interval among the various intersecting queues. We also discussed optimal policy in a given family of scheduling policies.

Future research directions

Bus bunching:

We studied the bus bunching problem on a single route with Gaussian bus travel times and fluid arrivals. We discussed some performance measures, bunching probability and passengers waiting times. We also considered variable arrival rates over the different stops. There are many other directions for this problem and some of them are the following:

- Operating cost: One can include the cost of operation (e.g., by including factors inversely proportional to the headways) in objective functions and derive more realistic/interesting optimal policies.

- Random arrivals: One extend the analysis to the case with Poisson passenger arrivals. We have some partial analysis with Poisson arrivals, however one may work to obtain complete analysis with Poisson or other type of arrival processes. Alternatively one can consider fluid arrivals with random arrival rates (e.g., Markov modulated arrival rates commonly used in queueing related literature).

- Sojourn times: One can consider sojourn time as a performance; sojourn time is the total time spent by a passenger in the system starting from its arrival to its de-boarding at its destined stop.

- One can think of multi route bus bunching problem. In this, some of the buses share the common path although their origin and destination are different.

- One can consider the round trip delay as the system state and obtain the optimal policies.

Ferry based wireless LANs: serve on move: As of now, we considered the policies that make independent decisions across cycles. We basically considered policies that depend upon the new workload arrived in the cycle just before the visit. While a more realistic dynamic policy would exhibit correlations, depends both upon the new workload as well as the workload left over at the previous departure epoch.

We discussed optimal policy in a given family of scheduling policies. One can study other varieties of service policies that optimize the workload among the stabilizing policies and also consider dynamic switching policies.

Performance of SoM FWLAN can be obtained via performance of some fictitious queues. The queues resulting in this context have some structural properties. For example, we might have D/G/1 or Geometric/M/1 queues. One can obtain waiting time/busy period analysis, exploiting such structural properties.

Chapter 8

Appendix A: stationary and partially dynamic headway policies

Proof of Lemma 2.1: The proof is based on mathematical induction. We begin with first trip, i.e., $k = 1$ and prove equation (2.6) by induction for all stops, i.e., for all $1 \leq i \leq M$. First note that (2.6) simplifies to the following when $k = 1$,

$$I_1^i = \gamma_0^i h_1 + \sum_{j=1}^{i} \gamma_0^{i-j+1} s^{(j)} + \sum_{r=0}^{i-1} \left[(1+\rho)^r N_1^{i-r} \right]. \tag{8.1}$$

We will prove the equation (8.1) again by mathematical induction. We begin with $i = 1$ then,

$$I_1^1 = h_1 + s^{(1)} + N_1^1 \text{ where } \gamma_0^1 = 1.$$

Hence (8.1) is true for $i = 1$. Assuming the result is true for $i = n < M$, i.e.,

$$I_1^n = \gamma_0^n h_1 + \sum_{j=1}^{n} \gamma_0^{n-j+1} s^{(j)} + \sum_{r=0}^{n-1} \left[(1+\rho)^r N_1^{n-r} \right].$$

Now we prove it for $i = n + 1$. From equation (2.4) is for first trip. One can rewrite I_1^{n+1} in terms of I_1^n as below and note from Table 4.1, $\gamma_0^n = (1 + \rho)^{n-1}$,

$$\begin{aligned}
I_1^{n+1} &= I_1^n + N_1^{n+1} + s^{(n+1)} + \rho I_1^n \\
&= (1+\rho) I_1^n + N_1^{n+1} + s^{(n+1)} \\
&= (1+\rho)\gamma_0^n h_1 + (1+\rho) \sum_{j=1}^{n} \gamma_0^{n-j+1} s^{(j)} + \sum_{r=0}^{n-1} \left[(1+\rho)^{r+1} N_1^{n-r} \right] + N_1^{n+1} + s^{(n+1)} \\
&= \gamma_0^{n+1} h_1 + \sum_{j=1}^{n+1} \gamma_0^{n+1-j+1} s^{(j)} + \sum_{r=0}^{n} \left[(1+\rho)^r N_1^{n+1-r} \right] \text{ because } \binom{m}{0} = 1 \;\; \forall m.
\end{aligned}$$

Hence equation (8.1) is verified for $i = n+1$. Hence lemma is true for $k = 1$ and $1 \leq i \leq M$. Assuming the equation (2.6) is true for $k = m$ and for any $1 \leq i \leq M$, i.e.,

$$I_m^i = \sum_{l=0}^{\min\{i-1,m-1\}} \gamma_l^i h_{m-l} + \mathbb{1}_{i\geq m} \sum_{j=1}^{i-m+1} \gamma_{m-1}^{i-j+1} s^{(j)} + \sum_{r=0}^{i-1} \left[(1+\rho)^r N_m^{i-r} + \sum_{l=1}^{r+1} \mu_l^r N_{m-l}^{i-r} \right] \tag{8.2}$$

Now we prove it for $k = m+1$ and any $1 \leq i \leq M$. We will prove this again by sub induction. The sub induction is true for $i = 1$ and $k = m+1$, because: using (2.4) we have:

$$I_{m+1}^1 = h_{m+1} + N_{m+1}^1 - N_m^1 \text{ where } \gamma_0^1 = 1.$$

Now, assuming the result is true for $i = n$ and $k = m+1$, i.e.,

$$\begin{aligned} I_{m+1}^n &= \sum_{l=0}^{\min\{n-1,m\}} h_{m+1-l}\gamma_l^n + \mathbb{1}_{n\geq m+1} \sum_{j=1}^{n-m} (-1)^m s^{(j)} \gamma_m^{n-j+1} \\ &\quad + \sum_{r=0}^{n-1} \left[(1+\rho)^r N_{m+1}^{n-r} + \sum_{l=1}^{r+1} \mu_l^r N_{m+1-l}^{n-r} \right]. \end{aligned}$$

We prove the result for $i = n+1$ and $k = m+1$. From above, equation (8.2) (with $i = n$) and equations (2.4)-(2.7) we have:

$$
\begin{aligned}
I_{m+1}^{n+1} &= I_{m+1}^{n} + N_{m+1}^{n+1} - N_{m}^{n+1} + \rho I_{m+1}^{n} - \rho I_{m}^{n} \quad \text{[from equation (2.4)]}\\
&= (1+\rho) I_{m+1}^{n} + N_{m+1}^{n+1} - N_{m}^{n+1} - \rho I_{m}^{n}\\
&= N_{m+1}^{n+1} + (1+\rho)\Bigg\{ \sum_{l=0}^{\min\{n-1,m\}} \gamma_l^n h_{m+1-l} + \mathbb{1}_{n\geq m+1} \sum_{j=1}^{n-m} \gamma_m^{n-j+1} s^{(j)}\\
&\quad + \sum_{r=0}^{n-1}\left[(1+\rho)^r N_{m+1}^{n-r} + \sum_{l=1}^{r+1} \mu_l^r N_{m+1-l}^{n-r}\right]\Bigg\} - N_m^{n+1} - \rho\Bigg(\sum_{l=0}^{\min\{n-1,m-1\}} \gamma_l^n h_{m-l}\\
&\quad + \mathbb{1}_{n\geq m} \sum_{j=1}^{n-m+1} \gamma_{m-1}^{n-j+1} s^{(j)} + \sum_{r=0}^{n-1}\left[(1+\rho)^r N_{m}^{n-r} + \sum_{l=1}^{r+1} \mu_l^r N_{m-l}^{n-r}\right]\Bigg)\\
&= \underbrace{\sum_{l=0}^{\min\{n-1,m\}} (1+\rho)\gamma_l^n h_{m+1-l} - \sum_{l=0}^{\min\{n-1,m-1\}} \rho\gamma_l^n h_{m-l}}\\
&\quad \underbrace{+ \mathbb{1}_{n\geq m+1} \sum_{j=1}^{n-m} (1+\rho)\gamma_m^{n-j+1} s^{(j)} - \mathbb{1}_{n\geq m} \sum_{j=1}^{n-m+1} \rho\gamma_{m-1}^{n-j+1} s^{(j)}}\\
&\quad \underbrace{+ N_{m+1}^{n+1} + \sum_{r=0}^{n-1} (1+\rho)^{r+1} N_{m+1}^{n-r}} \underbrace{- N_m^{n+1} + (1+\rho)\sum_{r=0}^{n-1} \mu_1^r N_m^{n-r} - \rho \sum_{r=0}^{n-1} (1+\rho)^r N_m^{n-r}}\\
&\quad +(1+\rho) \sum_{r=0}^{n-1}\sum_{l=2}^{r+1} \mu_l^r N_{m+1-l}^{n-r} - \rho \sum_{r=0}^{n-1}\sum_{l=1}^{r+1} \mu_l^r N_{m-l}^{n-r}\\
&= \sum_{l=0}^{\min\{n,m\}} \gamma_l^{n+1} h_{m+1-l} + \mathbb{1}_{n+1\geq m+1} \sum_{j=1}^{n-m+1} \gamma_m^{n+1-j+1} s^{(j)}\\
&\quad + \sum_{r=0}^{n} (1+\rho)^r N_{m+1}^{n+1-r} + \sum_{r=0}^{n} \mu_1^r N_m^{n+1-r} + \sum_{r=0}^{n}\sum_{l=2}^{r+1} \mu_l^r N_{m+1-l}^{n+1-r}\\
&= \sum_{l=0}^{\min\{n,m\}} \gamma_l^{n+1} h_{m+1-l} + \mathbb{1}_{n+1\geq m+1} \sum_{j=1}^{n-m+1} \gamma_m^{n+1-j+1} s^{(j)}\\
&\quad + \sum_{r=0}^{n}\left[(1+\rho)^r N_{m+1}^{n+1-r} + \sum_{l=1}^{r+1} \mu_l^r N_{m+1-l}^{n+1-r}\right].
\end{aligned}
$$

For the above, we need the following, which can easily be verified from equations (2.7) and (6.2):

$$
\begin{aligned}
(1+\rho)\gamma_l^n 1_{l<n} - 1_{l>0}\rho\gamma_{l-1}^n &= \gamma_l^{n+1} \text{ with } \gamma_l^n = 0 \text{ if } l \geq n,\\
(1+\rho)\mu_l^r 1_{l<r} - 1_{l>0}\rho\mu_{l-1}^r &= \mu_l^{r+1} \text{ with } \mu_l^n = 0 \text{ if } l > n.
\end{aligned}
$$

Hence the above equation is true for $i = n+1$ and hence the lemma is verified. ■

Lemma 8.1. *For Poisson arrivals,*

$$E\left[\bar{W}_k^i\right] = \lambda \frac{E\left[(I_k^i)^2\right]}{2} \text{ and } E\left[\bar{w}_k^i\right] = E\left[\frac{\bar{W}_k^i}{X_k^i}\right] \;=\; \frac{E[I_k^i]}{2} \text{ a.s.}$$

Proof: Note that:

$$E\left[\bar{W}_k^i\right] \;=\; E\left[\left[E[W_k^i \middle| I_k^i]\right]\right].$$

Now consider,

$$E\left[\bar{W}_k^i \middle| I_k^i\right] = E\left[\sum_{n=1}^{X_k^i} W_n^i \middle| I_k^i\right].$$

Passengers can arrive to the stop at any time epoch during the bus inter arrival times. The number of passenger arrivals during the bus inter arrival time I_k^i is X_k^i. First condition on I_k^i. Further conditioning on the number arrived X_k^i during I_k^i, the unordered arrivals are uniformly distributed in the interval I_k^i. i,e., the conditional waiting time of any passenger

$$W_n^i \sim I_k^i - Unif(0, I_k^i) \sim Unif(0, I_k^i).$$

The above is true for Poisson arrivals. Hence,

$$\begin{aligned}
E\left[\bar{W}_k^i \middle| I_k^i, X_k^i\right] &= X_k^i \frac{I_k^i}{2} \quad \text{a.s. and so} \\
E\left[\bar{W}_k^i \middle| I_k^i\right] &= E\left[E\left[\bar{W}_k^i \middle| I_k^i, X_k^i\right] \middle| I_k^i\right] = \lambda \frac{(I_k^i)^2}{2} \text{ and} \\
E\left[\bar{w}_k^i \middle| I_k^i\right] &= E\left[E\left[\frac{\bar{W}_k^i}{X_k^i} \middle| I_k^i, X_k^i\right] \middle| I_k^i\right] = \frac{I_k^i}{2} \text{ a.s.}
\end{aligned}$$

Finally:

$$E\left[\bar{W}_k^i\right] = \lambda \frac{E\left[(I_k^i)^2\right]}{2} \quad \text{and} \quad E\left[\bar{w}_k^i\right] = \frac{E\left[I_k^i\right]}{2}.$$

■

Theorem 8.2. *Let g be function of the following type:*

$$g(\mathbf{h}_M) = \sum_{l=0}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1 - \Phi\left(\sum_{l=0}^{M} \tilde{z}_l h_{M-l}\right)\right) + \delta$$

where $\{\tilde{z}\}_l, \alpha, \delta$ are real constants and $\bar{z}_0$ is positive constant and where $\mathbf{h}_M = [h_1, h_2, \ldots, h_M]$ is an M dimensional vector. Consider the following optimization problem:

$$g^*(\mathbf{h}_{M-1}) := \min_{h_M} g(\mathbf{h}_M) \text{ and } h_M^*(\mathbf{h}_{M-1}) := \arg\min_{h_M} g(\mathbf{h}).$$

Then there exists an unique optimizer to this problem. The optimizer is given by:

$$h_M^*(\mathbf{h}_{M-1}) = \frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right], \textit{ if } \alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2}\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}$$

$$\textit{with } a := \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}.$$

If the optimizer is an interior point then

$$g^*(\mathbf{h}_{M-1}) = \sum_{l=1}^{M-1} h_{M-l}\left(\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}\right) + \delta^* \textit{ with } \delta^* = \alpha\left[1-\Phi(a)\right] + \frac{\bar{z}_0 a}{\tilde{z}_0} + \delta.$$

Proof: Consider the objective function,

$$g(\mathbf{h}_M) = \min_{h_M}\sum_{r=0}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1-\Phi\left(\sum_{l=0}^{M}\tilde{z}_l h_{M-l}\right)\right) + \delta. \tag{8.3}$$

Let h_M^* be the optimal policy for equation (9.32), i.e.,

$$\begin{aligned} h_M^* &\in \arg\min(g(\mathbf{h}_M)), \text{ and either} \\ \Rightarrow \frac{d}{dh_M}\left(g(\mathbf{h}_M)\right)\Big|_{h_M=h_M^*} &= 0 \text{ or } h_M^* \text{ is on the boundary} \end{aligned}$$

The first derivative is given by:

$$\frac{d}{dh_M}\left(g(\mathbf{h}_M)\right) = \bar{z}_0 - \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{\left(\sum_{l=0}^{M}\tilde{z}_l h_{M-l}\right)^2}{2}\right)\tilde{z}_0.$$

By hypothesis $\bar{z}_0 > 0$. Under this condition, if $\alpha < \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}$, the first derivative is always positive and hence the optimizer is at lower boundary $\underline{h} = 0$. Otherwise, there exists a zero of the derivative as below:

$$\Rightarrow \sum_{l=0}^{M} h_{M-l}\tilde{z}_l = \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}.$$

This interior point is the minimizer (when unconstrained) as the second derivative is positive, and further we have unique minimizer for any given condition. This interior point is a positive quantity if

$$\frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right] > 0,$$

and it is easy to verify that the above is always true if $\sum_{l=1}^{M} h_{M-l}\tilde{z}_l < 0$ and in case $\sum_{l=1}^{M} h_{M-l}\tilde{z}_l > 0$, the above is true if

$$\alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2} \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}.$$

Thus if the above condition is satisfied, the interior point is always positive. Thus in all,

$$\begin{aligned} h_M^*(\mathbf{h}_{M-1}) &= \frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{M} h_{M-l}\tilde{z}_l + a\right], \text{ if } \alpha > e^{(\sum_{l=1}^{M} h_{M-l}\tilde{z}_l)^2} \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0} \\ & \text{with } a := \sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0\alpha}\right)}. \end{aligned}$$

Substituting the above, the value function equals:

$$\begin{aligned} g^*(\mathbf{h}_{M-1}) &= h_M^*\bar{z}_0 + \sum_{l=1}^{M-1} h_{M-l}\bar{z}_l + \alpha\left(1 - \Phi\left(h_M^*\tilde{z}_0 + \sum_{l=1}^{M} h_{M-l}\tilde{z}_l\right)\right) + \delta \\ &= \sum_{l=1}^{M} h_{M-l}\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0} + \frac{\bar{z}_0 a}{\tilde{z}_0} + \alpha\left(1 - \Phi\left(a\right)\right) + \delta \\ &= \sum_{l=1}^{M} h_{M-l}\left(\frac{\tilde{z}_0\bar{z}_l - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}\right) + \delta^*, \text{ where,} \end{aligned}$$

δ^* is defined in the hypothesis. ■

Proof of Theorem 4.1: The proof is based on mathematical induction. We begin with $i = 0$. Then the corresponding DP equations are given by (see equations (4.8) and Table 4.1),

$$\begin{aligned} v_T(\mathbf{h}_{T-1}) &= \min_{h_T} r_T(\mathbf{h}_T) \text{ and recall} \\ r_T(\mathbf{h}_T) &= \sum_{l=0}^{M-1} h_{T-l}\eta_l^T + \alpha\left(1 - \Phi_M\left(\sum_{l=0}^{M} \tilde{\gamma}_l^M h_{T-l}\right)\right). \end{aligned}$$

The above objective function is like the objective function of Theorem 9.6 with $\tilde{z}_l = \tilde{\gamma}_l^M, \bar{z}_l = \eta_l^T$ and note η_0^T is positive for all l and $\delta = 0$. By this theorem, the optimal policy and value function are respectively,

$$\begin{aligned} h_T^*(\mathbf{h}_{T-1}) &= \frac{1}{\tilde{\gamma}_0^M}\left[-\sum_{l=1}^{M} h_{T-l}\tilde{\gamma}_l^M + a_T\right], \quad a_T := \sigma_M^M\sqrt{-2\log\left(\frac{\eta_0^T\sqrt{2\pi}\sigma_M^M}{\tilde{\gamma}_0^M\alpha}\right)} \\ & \text{and} \\ v_T(\mathbf{h}_{T-1}) &= \sum_{l=1}^{M} h_{T-l}\left(\frac{\tilde{\gamma}_0^M\eta_l^T - \eta_0^T\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M}\right) + \delta^T, \text{ where } \delta^T = \alpha\left(1 - \Phi_M\left(a_T\right)\right) + \frac{\eta_0^T a_T}{\tilde{\gamma}_0^M}. \end{aligned}$$

Hence the theorem is verified for $i = 0$.

Assuming result is true for $i = n$, we prove it for $i = n + 1$.

The corresponding DP equation is given by,

$$\begin{aligned} v_{T-n-1} &= \min \left\{ \sum_{l=0}^{M-1} h_{T-n-1-l} \eta_l^{T-n-1} + \vartheta_{T-n-1}^M \right. \\ &\quad \left. + \alpha \left(1 - \Phi_{T-n-1}^M \left(\sum_{l=0}^{M} h_{T-n-1-l} \tilde{\gamma}_l^M + \varpi_{T-n-1}^M \right) \right) + \delta^{T-n} \right\}. \end{aligned}$$

The above objective function is like the objective function of Theorem 9.6 with $\tilde{z}_l = \tilde{\gamma}_l, \bar{z}_l = \eta_l^{T-n-1}$ for all l and $\delta = \delta^{T-n}$. Note that $\bar{z}_0 = \eta_0^{T-n-1} > 0$ by the given hypothesis. By this theorem, the optimal policy and value function are respectively,

$$\begin{aligned} h_{T-n-1}^*(\mathbf{h}_{T-n-2}) &= \frac{1}{\tilde{\gamma}_0^M} \left[- \sum_{l=1}^{M} h_{T-n-1-l} \tilde{\gamma}_l^M - \varpi_{T-n-1}^M + a_{T-n-1} \right], \\ & \qquad \text{where } a_{T-n-1} = \sigma_{T-n-1}^M \sqrt{-2 \log \left(\frac{\eta_0^{T-n-1} \sqrt{2\pi} \sigma_{T-n-1}^M}{\tilde{\gamma}_0^M \alpha} \right)}. \end{aligned}$$

Substituting the above, the value function equals,

$$\begin{aligned} v_{T-n-1}^*(\mathbf{h}_{T-n-2}) &= \sum_{l=1}^{M} h_{T-n-1-l} \left(\frac{\tilde{\gamma}_0^M \eta_l^{T-n-1} - \eta_0^{T-n-1} \tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \right) + \delta^{T-n-1} \text{ where} \\ \delta^{T-n-1} &= \alpha \left(1 - \Phi_{T-n-1}^M \left(a_{T-n-1} \right) \right) + \vartheta_{T-n-1}^M + \frac{\eta_0^{T-n-1} \left(a_{T-n-1} - \varpi_{T-n-1}^M \right)}{\tilde{\gamma}_0^M} \\ &\quad + \delta^{T-n}. \end{aligned}$$

Hence theorem is verified. ■

Proof of Lemma 4.2: Define the following function,

$$f_\eta : (0, 1] \to \mathbb{R}^{T+1}$$

i.e., $\rho \xrightarrow{f_\eta} \left(\{\eta_0^t\}_{t \geq 0} \right)$. For every $\rho > 0$, $f_\eta(\rho)$ is well defined and continuous (see Table 4.1). Now we define the continuous extension of f_η at $\rho = 0$, i.e.,

$$f_\eta(0) \triangleq \lim_{\rho \to 0} f_\eta(p).$$

Here $f_\eta(0)$ is not corresponding to any system. It is a continuous function on $[0, 1]$ by construction.

Before we proceed further one can easily verify the following (see Table 4.1):

$$\frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} \to 0 \text{ and for all } i > 0, \bar{\gamma}_i \to 0, \bar{\gamma}_0 \to \frac{M}{2}, \text{ as } \rho \to 0,$$

which would be instrumental in proving this lemma.

If one proves $f_\eta(0) = \lim_{\rho\to 0} f_\eta(\rho) > 0$ (i.e., all components are strictly positive), then by continuity there exists a $\bar{\rho}_\eta > 0$ such that $f_\eta(\rho) > 0$ for all $\rho \leq \bar{\rho}_\eta$. This proves the second part of assumption (4.10). Towards this we begin with $f_\eta(0)$. This part of the proof is based on backward induction. We begin with $i = 0$, i.e., with η_0^T and observe immediately that:

$$\lim_{\rho\to 0} \eta_0^T \;=\; \lim_{\rho\to 0} \bar{\gamma}_o = \lim_{\rho\to 0} \frac{(1+\rho)^M - 1}{2\rho} = \frac{M}{2} > 0, \text{ by using L'Hospital's rule.}$$

Further it is clear that

$$\lim_{\rho\to 0} \eta_l^T = 0 \text{ for all } l > 0.$$

In a similar way, for $i = 1$, i.e.,

$$\begin{aligned}
\lim_{\rho\to 0} \eta_0^{T-1} &= \lim_{\rho\to 0}\left(\eta_1^T - \eta_0^T \frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_0\right) = \lim_{\rho\to 0}\left(\bar{\gamma}_1 - \frac{M}{2}\frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \frac{M}{2}\right) \\
&= \lim_{\rho\to 0}\left(-((1+\rho)^{M-1} - 1) - \frac{M}{2}\frac{-\rho(1+\rho)^{M-2} - \rho(1+\rho)^{M-1}}{(1+\rho)^{M-1}}\right) + \frac{M}{2} \\
&= \frac{M}{2} > 0 \text{ and similarly } \lim_{\rho\to 0} \eta_l^{T-1} = 0 \text{ for all } l > 0.
\end{aligned}$$

Now assume the result is true for $i = n-1$ i.e., assume $\lim_{\rho\to 0} \eta_0^{T-n+1} = M/2 > 0$ and $\lim_{\rho\to 0} \eta_l^{T-n+1} = 0$ for all $l > 0$. Then for $i = n$:

$$\begin{aligned}
\lim_{\rho\to 0} \eta_0^{T-n} &= \lim_{\rho\to 0}\left(\eta_1^{T-n+1} - \eta_0^{T-n+1}\frac{\tilde{\gamma}_1^M}{\tilde{\gamma}_0^M} + \bar{\gamma}_0\right) \\
&= \lim_{\rho\to 0} \eta_1^{T-n+1} + \frac{M}{2} = \frac{M}{2} > 0 \text{ and similarly } \lim_{\rho\to 0} \eta_l^{T-n} = 0 \text{ for all } l > 0.
\end{aligned}$$

Hence, $\eta_0^t > 0$ for all $\rho \leq \bar{\rho}_\eta$, t and hence a_t (for all t) are well defined for all such ρ. Now define the following function,

$$f : (0, \bar{\rho}_\eta] \to \mathbb{R}^{2T+1}$$

i.e., $\bar{\rho}_\eta \xrightarrow{\text{f}} \left(\{h_t^*\}_{t\geq 1},\ \{\eta_0^t\}_{t\geq 0}\right)$, where $\{h_t^*\}_t$ are defined recursively in Theorem 4.1. Observed here that $\{h_t^*\}_{t\geq 1}$ is well defined using (4.11) (whether or not the condition (4.10) is true) and hence the function $f(\cdot)$ is well defined and continuous.

It remains to prove (recursively, i.e., using backward induction) that

$$\alpha > \lim_{\rho\to 0} e^{\left(\sum_{l=1}^M h_{T-i-l}^* \tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} \text{ for all } 0 \leq j < i,$$

as by continuity of f the result follows. And this is immediate since $\tilde{\gamma}_l^M \to 0$ for all $l \geq 1$, and, hence[1]

$$\lim_{\rho\to 0} e^{\left(\sum_{l=1}^M h^*_{T-i-l}\tilde{\gamma}_l^M\right)^2} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} = \lim_{\rho\to 0} \frac{\eta_0^{T-j}\sqrt{2\pi}}{\tilde{\gamma}_0^M} = \beta\frac{M\sqrt{2\pi}}{2},$$

by previous result and because α is greater than RHS by given hypothesis. ■

Proof of Proposition 4.3: Recall from Theorem 4.1,

$$\eta_l^{T-k} = \bar{\gamma}_l + \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{\tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \text{ for all } l \geq 0. \tag{8.4}$$

Summing over $l = 0\cdots M$ we get (from (4.5))

$$\begin{aligned}\sum_{l=0}^M \eta_l^{T-k} &= \sum_{l=0}^M \bar{\gamma}_l + \sum_{l=0}^M \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{\sum_{l=0}^M \tilde{\gamma}_l^M}{\tilde{\gamma}_0^M} \\ &= \frac{M}{2} + \sum_{l=0}^M \eta_{l+1}^{T-k+1} - \eta_0^{T-k+1}\frac{1-\rho}{(1+\rho)^{M-1}} \text{ for all } k \geq 0,\end{aligned} \tag{8.5}$$

and the above is true because, by a simple exchange of the indices of the two summations involved one can prove the following (see Table 4.1):

$$\begin{aligned}\sum_{l=0}^{i-1}\gamma_l^i &= 1 \text{ for all } 1 \leq i \leq M, \quad \sum_{l=0}^M \bar{\gamma}_l = \sum_{l=0}^M\sum_{i=l+1}^M \gamma_l^i = \sum_{i=1}^M\sum_{l=0}^{i-1}\gamma_l^i = \sum_{i=1}^M 1/2 = M/2 \text{ and} \\ \sum_{l=0}^M \tilde{\gamma}_l^M &= \sum_{l=0}^M\left(\gamma_l^M - \rho\gamma_{l-1}^M \mathbb{1}_{l>0}\right) = \sum_{l=0}^{M-1}(1-\rho)\gamma_l^M = 1-\rho, \qquad (\gamma_M^M = 0 \text{ from (2.7)}).\end{aligned}$$

If there exists a limit $\eta_l^{T-k} \to \eta_l^*$ (for each $0 \leq l \leq M$) as $T \to \infty$, then $\sum_{l=0}^M \eta_{l+1}^{T-k+1} - \sum_{l=0}^M \eta_{l+1}^{T-k}$ converges to 0 and then the limit η_0^* is given by the following (see equation (8.5)):

$$\begin{aligned}\eta_0^* &= \frac{M(1+\rho)^{M-1}}{2(1-\rho)}, \text{ and subsequently,} \\ a_* &= \sigma_M^M\sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma_M^M}{2(1-\rho)\alpha}\right)}.\end{aligned}$$

Substituting this limit in (8.4), by using induction, one can show that the limits $\eta_l^* = \lim_{T\to\infty}\eta_l^T$ for each l exists. ■

[1] Observe that $\{h_t^*\}$ are bounded for all $\rho \leq \bar{\rho}_\eta$ (forward) recursively as below (because of finite horizon problem):

$$\sup_{\rho\leq\bar{\rho}_\eta} h_2^* \leq \left(\max_l \sup_{\rho\leq\bar{\rho}_\eta}\tilde{\gamma}_l^M\right) h_1 + \sup_{\rho\leq\bar{\rho}_\eta}\sup_t \varpi_t + \sup_{\rho\leq\bar{\rho}_\eta}\sup_t a_t < \infty, \quad \sup_{\rho\leq\bar{\rho}_\eta} h_3^* \leq 2\sup_{\rho\leq\bar{\rho}_\eta}\sup_t \varpi_t + 2\sup_{\rho\leq\bar{\rho}_\eta}\sup_t a_t,$$
$$\cdots, \quad \sup_{\rho\leq\bar{\rho}_\eta} h_t^* \leq (t-1)\sup_{\rho\leq\bar{\rho}_\eta}\sup_t \varpi_t + (t-1)\sup_{\rho\leq\bar{\rho}_\eta}\sup_t a_t.$$

The above are bounded by continuity of the function defining coefficients $\{\varpi_t\}_t$ and $\{a_t\}_t$ with respect to ρ.

Proof of Lemma 4.4: The proof is based on mathematical induction. We begin with $i = 1$ then,

$$I_k^1 = h_k + N_k^1, \text{ where } \gamma_0^1 = 1.$$

Hence the hypothesis is true for $i = 1$. Assuming the result is true for $i = n$, i.e.,

$$I_k^n = \sum_{l=0}^{\min\{k-1,n-1\}} \gamma_l^n h_{k-l} + \sum_{l=0}^{\min\{k-1,n-1\}} \sum_{j=1}^{n-l} \gamma_l^{n-j+1} N_{k-l}^j.$$

Now we prove it for $i = n + 1$.

$$\begin{aligned}
I_k^{n+1} &= h_k + \sum_{1\le j\le n+1} N_k^j + \rho \sum_{1\le j<n+1} \left(I_k^j - I_{k-1}^j\right) \\
&= I_k^n + N_k^{n+1} + \rho\left(I_k^n - I_{k-1}^n\right) \\
&= (1+\rho) \sum_{l=0}^{\min\{k-1,n-1\}} \gamma_l^n h_{k-l} + (1+\rho) \sum_{l=0}^{\min\{k-1,n-1\}} \sum_{j=1}^{n-l} \gamma_l^{n-j+1} N_{k-l}^j + N_k^{n+1} \\
&\quad -\rho \sum_{l=0}^{\min\{k-1,n-1\}} \gamma_l^n h_{k-1-l} - \rho \sum_{l=0}^{\min\{k-1,n-1\}} \sum_{j=1}^{n-l} \gamma_l^{n-j+1} N_{k-1-l}^j \\
&= \sum_{l=0}^{\min\{k-1,n\}} \gamma_l^{n+1} h_{k-l} + \sum_{l=0}^{\min\{k-1,n\}} \sum_{j=1}^{n+1-l} \gamma_l^{n+1-j+1} N_{k-l}^j.
\end{aligned}$$

For the above, we need the following, which can easily be verified from equations (2.7) and (6.2):

$$(1+\rho)\gamma_l^n 1_{l<n} - 1_{l>0}\rho\gamma_{l-1}^n = \gamma_l^{n+1} \text{ with } \gamma_l^n = 0 \text{ if } l \ge n.$$

Hence the above equation is true for $i = n + 1$ and hence lemma is verified. ∎

Chapter 9

Appendix B: fully dynamic headway policies

Lemma 9.1. *The inter-arrival times can be expressed in terms of the relevant Markovian walking components as below, for any* $1 \leq i \leq M$ *,* $0 \leq s \leq d, 0 \leq l \leq d, 1 \leq r \leq M$ *and* $t_0 + 2 < k \leq T$*:*

$$I_{k-s}^{i} = \sum_{l=0}^{d-s} h_{k-s-l}\gamma_l^{i-1} + \sum_{l=0}^{d-s}\sum_{j=1}^{i-l} W_{k-s-l}^{j}\varpi_{l,j}^{i-1} - \sum_{j=1}^{i-1-d+s} \rho^{(j)} I_{k-1-d}^{j}\varpi_{d-s,j+1}^{i-1} \text{ where} \tag{9.1}$$

$$\gamma_l^{i-1} = (-1)^l\Bigg(\mathbb{1}_{l=0} + \binom{1}{l}\sum_{j_1=1}^{i-1}\rho^{(j_1)} + \binom{2}{l}\sum_{j_1<j_2}^{i-1}\rho^{(j_1)}\rho^{(j_2)} + \dots + \binom{i-1}{l}\sum_{j_1<j_2<\dots j_{i-1}}^{i-1}\rho^{(j_1)}\rho^{(j_2)}\dots\rho^{(j_{i-1})}\Bigg) \tag{9.2}$$

$$\varpi_{l,r}^{i-1} = (-1)^l\Bigg(\mathbb{1}_{l=0} + \binom{1}{l}\sum_{j_1=r}^{i-1}\rho^{(j_1)} + \binom{2}{l}\sum_{j_1<j_2}^{i-1}\rho^{(j_1)}\rho^{(j_2)} + \dots + \binom{i-1}{l}\sum_{j_1<j_2<\dots j_{i-1}}^{i-1}\rho^{(j_1)}\rho^{(j_2)}\dots\rho^{(j_{i-1})}\Bigg), \tag{9.3}$$

note that $\varpi_{l,1}^{i-1} = \gamma_l^{i-1}$.

The conditional expectation of inter arrival times (i.e., I_k^i*) given the state* Y_k *(from (5.3)) and* h_k *equals:*

$$E\left[I_k^i \middle| Y_k, h_k\right] = \sum_{l=0}^{d} h_{k-l}\gamma_l^{i-1} - \sum_{j=1}^{i-1-d}\rho^{(j)} I_{k-1-d}^{j}\varpi_{d,j+1}^{i-1}. \tag{9.4}$$

Proof: The proof is based on mathematical induction. We begin with $i = 1$ and prove equation (9.1) by backward induction for all $0 \le s \le d$. First note that (9.1) simplifies to the following when $i = 1$,

$$I^1_{k-s} = h_{k-s} + W^1_{k-s}. \tag{9.5}$$

From equation (5.2), the above equation (9.5) is true for all $0 \le s \le d$.

Hence lemma is true for $i = 1$ and $0 \le s \le d$. Assuming the equation (9.1) is true for $i \le n$ and for any $0 \le s \le d$, i.e.,

$$I^n_{k-s} = \sum_{l=0}^{d-s} h_{k-s-l}\gamma^{n-1}_l + \sum_{l=0}^{d-s}\sum_{j=1}^{n-l} W^j_{k-s-l}\varpi^{n-1}_{l,j} - \sum_{j=1}^{n-1-d+s} \rho^{(j)} I^j_{k-1-d}\varpi^{n-1}_{d-s,j+1}. \tag{9.6}$$

Now prove it for $i = n+1$ and any $0 \le s \le d$. We will prove this by backward induction on s. The backward induction is true for $s = d$ and $i = n+1$, because: using equations (5.2) and (5.3) we have,

$$I^{n+1}_{k-d} = h_{k-d}\gamma^n_0 + \sum_{j=1}^{n+1} W^j_{k-d}\varpi^n_{0,j} - \sum_{j=1}^{n}\rho^{(j)} I^j_{k-1-d}\varpi^n_{0,j+1}.$$

Now assume the result is true for $i = n+1$ and $m \le s \le d$, i.e.,

$$I^{n+1}_{k-s} = \sum_{l=0}^{d-s} h_{k-s-l}\gamma^n_l + \sum_{l=0}^{d-s}\sum_{j=1}^{n+1-l} W^j_{k-s-l}\varpi^n_{l,j} - \sum_{j=1}^{n-d+s} \rho^{(j)} I^j_{k-1-d}\varpi^n_{d-s,j+1} \text{ for all } m \le s \le d.$$

We prove the result is true for $s = m-1$ and $i = n+1$. From above, equation (9.6) and equation (5.2) we have:

$$\begin{aligned}
I^{n+1}_{k-m+1} &= h_{k-m+1} + \sum_{1\le j\le n+1} W^j_{k-m+1} + \sum_{1\le j<n+1} \rho^{(j)}\left(I^j_{k-m+1} - I^j_{k-m}\right) \\
&= W^{n+1}_{k-m+1} + (1+\rho^{(n)})I^n_{k-m+1} - \rho^{(n)} I^n_{k-m} \\
&= W^{n+1}_{k-m+1} + \left(1+\rho^{(n)}\right)\left[\sum_{l=0}^{d-m+1} h_{k-m+1-l}\gamma^{n-1}_l + \sum_{l=0}^{d-m+1}\sum_{j=1}^{n-l} W^j_{k-m+1-l}\varpi^{n-1}_{l,j}\right. \\
&\quad \left. - \sum_{j=1}^{n-d+m-2} \rho^{(j)} I^j_{k-1-d}\varpi^{n-1}_{d-m+1,j+1}\right] \\
&\quad -\rho^{(n)}\left[\sum_{l=0}^{d-m} h_{k-m-l}\gamma^{n-1}_l + \sum_{l=0}^{d-m}\sum_{j=1}^{n-l} W^j_{k-m-l}\varpi^{n-1}_{l,j} - \sum_{j=1}^{n-1-d+m} \rho^{(j)} I^j_{k-1-d}\varpi^{n-1}_{d-m,j+1}\right] \\
&= \sum_{l=0}^{d-m+1} h_{k-m+1-l}\gamma^n_l + \sum_{l=0}^{d-m+1}\sum_{j=1}^{n-l+1} W^j_{k-m+1-l}\varpi^n_{l,j} - \sum_{j=1}^{n-d+m-1} \rho^{(j)} I^j_{k-1-d}\varpi^n_{d-m+1,j+1}.
\end{aligned}$$

For the above, we need the following, which can easily be verified from equations (9.2) and (9.3):

$$\begin{aligned}(1+\rho^{(n)})\gamma_l^{n-1} - \rho^{(n)}\gamma_{l-1}^{n-1}\mathbb{1}_{l>0} &= \gamma_l^n,\\ (1+\rho^{(n)})\varpi_{l,j}^{n-1} - \rho^{(n)}\varpi_{l-1,j}^{n-1}\mathbb{1}_{l>0} &= \varpi_{l,j}^n, \text{ for all } j\\ (1+\rho^{(n)})\varpi_{d-m+1,j+1}^{n-1}\mathbb{1}_{j<n-d+m-1} - \rho^{(n)}\varpi_{d-m,j+1}^{n-1}\mathbb{1}_{d>0} &= \varpi_{d-m+1,j+1}^n \text{ for all } j.\end{aligned}$$

Hence the above is true for $s = m-1$ and hence the lemma is verified. ■

Lemma 9.2. *Recall the constants from equations (9.2)-(9.3) and for any* $1 \le i \le M$, $0 < d < i$ *and* $t_0 + 2 < k \le T$ *we have:*

$$I_k^i - \rho^{(i)} I_{k-1}^i = \sum_{l=0}^{d} h_{k-l}\tilde{\gamma}_l^{i-1} + \sum_{l=0}^{d}\sum_{r=1}^{i-l+1} W_{k-l}^r \tilde{\mu}_{l,r}^{i-1} - \sum_{j=1}^{i-d} I_{k-1-d}^j \tilde{\varpi}_{d,j+1}^{i-1}, \text{ where,} \tag{9.7}$$

$$\tilde{\gamma}_l^{i-1} = \gamma_l^{i-1} - \rho^{(i)}\gamma_{l-1}^{i-1}\mathbb{1}_{l>0}, \tag{9.8}$$

$$\tilde{\mu}_{l,r}^{i-1} = \varpi_{l,r}^{i-1}\mathbb{1}_{r<i-l+1} - \rho^{(i)}\varpi_{l-1,r}^{i-1}\mathbb{1}_{l>0}, \tag{9.9}$$

$$\tilde{\varpi}_{d,j+1}^{i-1} = \rho^{(j)}\varpi_{d,j+1}^{i-1}\mathbb{1}_{j<i-d} - \rho^{(i)}\rho^{(j)}\varpi_{d-1,j+1}^{i-1}. \tag{9.10}$$

The conditional mean and conditional variance given the state Y_k *(from (5.3))and* h_k *respectively given by,*

$$\begin{aligned}E\left[I_k^i \middle| Y_k, h_k\right] - \rho^{(i)} E\left[I_{k-1}^i \middle| Y_k, h_k\right] &= \sum_{l=0}^{d} h_{k-l}\tilde{\gamma}_l^{i-1} - \sum_{j=1}^{i-d} I_{k-1-d}^j \tilde{\varpi}_{d,j+1}^{i-1}\\ \textit{and } \omega^2 &= \beta^2\left(\sum_{l=0}^{d}\sum_{r=1}^{M-l+1}\left(\tilde{\mu}_{l,r}^{M-1}\right)^2\right).\end{aligned} \tag{9.11}$$

Proof: is direct from Lemma 9.1. ■

Proof of Theorem 5.1: By Lemma 9.5, there exists a $\bar{\rho} > 0$ such that condition (5.10) is satisfied for all $\rho \le \bar{\rho}$. We consider all such smaller values of ρ.

The rest of proof is based on backward mathematical induction. We begin with $k = 0$. Then the corresponding DP equations are (see Table 5.1, and equation (5.9)),

$$\begin{aligned}v_T(Y_T) &= \min_{h_T\in[0,\bar{h}]} r_T(h_T, Y_T),\\ &= \min_{h_T\in[0,\bar{h}]}\Bigg\{\sum_{l=0}^{d} h_{T-l}\bar{\gamma}_l - \sum_{j=1}^{M-1-d}\rho^{(j)} I_{T-1-d}^j \bar{\varpi}_{d,j+1}\\ &\quad +\alpha\left[1-\Phi\left(\sum_{l=0}^{d} h_{T-l}\tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I_{T-1-d}^j\tilde{\varpi}_{d,j+1}^{M-1}\right)\right]\Bigg\}.\end{aligned}$$

The above optimization of r_T is like the objective function of Theorem 9.6 and observe hypothesis (9.30) is satisfied by condition (5.10) with $k = 0$. By this theorem, the optimal policy and value function are respectively:

$$
\begin{aligned}
h_T^*(Y_T) &= \begin{cases} \bar{h} & \text{on } \bar{\mathcal{A}}_T \\ 0 & \text{on } \underline{\mathcal{A}}_T \\ \frac{1}{\tilde{\gamma}_0^{M-1}}\left[-\sum_{l=1}^{d} h_{T-l}\tilde{\gamma}_l^{M-1} + \sum_{j=1}^{M-d} I_{T-1-d}^{j}\tilde{\varpi}_{d,j+1}^{M-1} + a_T\right], & \text{on } \mathcal{A}_T^c \end{cases} \\
\text{with } \mathcal{A}_T &:= \underline{\mathcal{A}}_T \cup \bar{\mathcal{A}}_T, \\
\underline{\mathcal{A}}_T &= \left\{\sum_{j=1}^{M-d} I_{T-1-d}^{j}\tilde{\varpi}_{d,j+1}^{M-1} < -a_T + \sum_{l=1}^{d} h_{T-l}\tilde{\gamma}_l^{M-1}\right\}, \\
\bar{\mathcal{A}}_T &= \left\{\sum_{j=1}^{M-d} I_{T-1-d}^{j}\tilde{\varpi}_{d,j+1}^{M-1} > \bar{h}\tilde{\gamma}_0^{M-1} - a_T + \sum_{l=1}^{d} h_{T-l}\tilde{\gamma}_l^{M-1}\right\} \text{ and}
\end{aligned}
$$

with constants, e.g., a_T, η_T^l, ψ_T^j, as in Table 5.1 and

$$
v_T(Y_T) = \begin{cases} r_T(\bar{h}, Y_T) & \text{on } \bar{\mathcal{A}}_T \\ r_T(0, Y_T) & \text{on } \underline{\mathcal{A}}_T \\ \sum_{l=1}^{d} h_{T-l}\eta_T^l - \sum_{j=1}^{M-d} I_{T-1-d}^{j}\psi_T^j + \delta_T, & \text{on } \mathcal{A}_T^c, \end{cases} \tag{9.12}
$$

Further for any Y_{T-1}, and h_{T-1} the conditional expectation:

$$
\begin{aligned}
E_{Y_{T-1},h_{T-1}}\left[v_T(y_T)\right] &:= E\left[v_T(Y_T)\middle| Y_{T-1}, h_{T-1}\right] \\
&= E_{Y_{T-1},h_{T-1}}\left[\sum_{l=1}^{d} h_{T-l}\eta_T^l - \sum_{j=1}^{M-d} I_{T-1-d}^{j}\psi_T^j + \delta_T\right] + \Gamma_{T-1}, \text{ with,}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{T-1} := \; & E_{Y_{T-1},h_{T-1}}\Bigg[\bar{h}\bar{\gamma}_0 + \sum_{l=1}^{d} h_{T-l}\bar{\gamma}_l - \sum_{j=1}^{M-1-d} \rho^{(j)} I_{T-1-d}^{j}\bar{\varpi}_{d,j+1} \\
& + \alpha\left[1 - \Phi\left(\bar{h}\tilde{\gamma}_0^{M-1} + \sum_{l=1}^{d} h_{T-l}\tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I_{T-1-d}^{j}\tilde{\varpi}_{d,j+1}^{M-1}\right)\right] \; ; \bar{\mathcal{A}}_T\Bigg] \\
& + E_{Y_{T-1},h_{T-1}}\Bigg[\sum_{l=1}^{d} h_{T-l}\bar{\gamma}_l - \sum_{j=1}^{M-1-d} \rho^{(j)} I_{T-1-d}^{j}\bar{\varpi}_{d,j+1} \\
& + \alpha\left[1 - \Phi\left(\sum_{l=1}^{d} h_{T-l}\tilde{\gamma}_l^{M-1} - \sum_{j=1}^{M-d} I_{T-1-d}^{j}\tilde{\varpi}_{d,j+1}^{M-1}\right)\right] \; ; \underline{\mathcal{A}}_T\Bigg] \\
& - E_{Y_{T-1},h_{T-1}}\left[\sum_{l=1}^{d} h_{T-l}\eta_T^l - \sum_{j=1}^{M-d} I_{T-1-d}^{j}\psi_T^j + \delta_T; \; \mathcal{A}_T^c\right],
\end{aligned} \tag{9.13}
$$

where by Lemma 9.4 (from Table 5.1, ρ factors out from some coefficients $(\{\tilde{\varpi}^{M-1}_{d,j+1}\}_j, \{\psi^j_T\}_j)$, others are bounded uniformly [1] in ρ) and recall $\rho = \max\{\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(j)}\}$ and for all $1 \leq j \leq M$:

$$\begin{aligned} |\Gamma_{T-1}| &\leq \zeta_{T-1}(Y_{T-1}, h_{T-1}) \text{ with} \\ \zeta_{T-1}(Y_{T-1}, h_{T-1}) &:= \rho \left| \sum_{j=1}^{M-1-d} c^j_T I^j_{T-2-d} \right| + \rho \left| \sum_{j=1}^{M-1-d} u^j_T I^j_{T-2-d} \right| + \rho \sum_{l=1}^{d} c^l_T h_{T-l} + \rho c_T. \end{aligned} \tag{9.14}$$

The coefficients $\{c^j_T, u^j_T\}_{j\geq 1}$ in the definition of ζ_{T-1} are appropriately defined, converge to finite constants as $\rho \to 0$ and do not depend upon Y_{T-1} or h_{T-1} (see Table 5.1, equation (9.13)). Thus in all, for any Y_{T-1}, and h_{T-1} (from (9.12)) we have:

$$\begin{aligned} E[v_T(Y_T)|Y_{T-1}, h_{T-1}] &= \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \\ &\quad + E_{Y_{T-1}, h_{T-1}} \left[\sum_{l=1}^{d} h_{T-l} \eta^l_T - \sum_{j=1}^{M-d} I^j_{T-1-d} \psi^j_T + \delta_T \right]. \end{aligned} \tag{9.15}$$

Clearly $\zeta_{T-1} = O(\rho)$, policy (5.11) is optimal for all Y_T and so the result is true for $k = 0$. Now we will prove the result for $k = 1$. Using DP equations

$$\begin{aligned} v_{T-1}(Y_{T-1}) &= \inf_{h_{T-1} \in [0, \bar{h}]} J(h_{T-1}, Y_{T-1}) \text{ with} \\ J(h_{T-1}, Y_{T-1}) &:= r_{T-1}(h_{T-1}, Y_{T-1}) + E[v_T(Y_T)|Y_{T-1}, h_{T-1}]. \end{aligned} \tag{9.16}$$

Fix Y_{T-1}. By Lemma 9.3, and equation (9.15) the objective function $J(\cdot)$ can be split and rewritten as:

$$\begin{aligned} J(h_{T-1}, Y_{T-1}) &= \tilde{J}(h_{T-1}, Y_{T-1}) + \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \text{ with} \\ \tilde{J}(h_{T-1}, Y_{T-1}) &:= \sum_{l=0}^{d} h_{T-1-l} \left(\bar{\gamma}_l + \eta^{l+1}_T \mathbb{1}_{l<d} - \mathbb{1}_{l=d} \sum_{j=1}^{M-d} \psi^j_T \gamma^{j-1}_0 \right) \\ &\quad - \sum_{j=1}^{M-1-d} \left(\rho^{(j)} \bar{\varpi}_{d,j+1} + \sum_{r=j}^{M-1-d} \psi^{r+1}_T \rho^{(j)} \varpi^r_{0,j+1} \right) I^j_{T-2-d} \\ &\quad + \alpha \left[1 - \Phi \left(\sum_{l=0}^{d} h_{T-1-l} \tilde{\gamma}^{M-1}_l - \sum_{j=1}^{M-d} I^j_{T-2-d} \tilde{\varpi}^{M-1}_{d,j+1} \right) \right] + \delta_T. \end{aligned} \tag{9.17}$$

[1] In first line of (9.13), the first two and fourth terms inside the expectation are bounded by a finite constant for all realizations and then the probability can be upper bounded by Lemma 9.4. Similarly, the second and third lines of (9.13) are bounded by Lemma 9.4.

The function $\tilde{J}(\cdot)$ is like the objective function of Theorem 9.6 and observe hypothesis (9.30) is satisfied by condition (5.10) with $k = 1$. By this theorem, the optimal policy and value function are respectively:

$$\tilde{v}_{T-1}(Y_{T-1}) := \inf_{0 \le h_{T-1} \le \bar{h}} \tilde{J}(h_{T-1}, Y_{T-1}), \tag{9.18}$$

respectively equal:

$$\tilde{h}^*_{T-1}(Y_{T-1}) = \begin{cases} \bar{h} & \text{on } \bar{\mathcal{A}}_{T-1} \\ 0 & \text{on } \underline{\mathcal{A}}_{T-1} \\ \frac{1}{\tilde{\gamma}_0^{M-1}} \left[-\sum_{l=1}^{d} h_{T-1-l} \tilde{\gamma}_l^{M-1} + \sum_{j=1}^{M-d} I^j_{T-2-d} \tilde{\varpi}^{M-1}_{d,j+1} + a_{T-1} \right], & \text{on } \mathcal{A}^c_{T-1} \end{cases}$$

$$\begin{aligned} \text{with } \mathcal{A}_{T-1} &:= \underline{\mathcal{A}}_{T-1} \cup \bar{\mathcal{A}}_{T-1}, \\ \underline{\mathcal{A}}_{T-1} &= \left\{ \sum_{j=1}^{M-d} I^j_{T-2-d} \tilde{\varpi}^{M-1}_{d,j+1} < -a_{T-1} + \sum_{l=1}^{d} h_{T-1-l} \tilde{\gamma}_l^{M-1} \right\}, \\ \bar{\mathcal{A}}_{T-1} &= \left\{ \sum_{j=1}^{M-d} I^j_{T-2-d} \tilde{\varpi}^{M-1}_{d,j+1} > \bar{h} \tilde{\gamma}_0^{M-1} - a_{T-1} + \sum_{l=1}^{d} h_{T-1-l} \tilde{\gamma}_l^{M-1} \right\} \end{aligned}$$

with constants, e.g., a_{T-1}, η^l_{T-1}, ψ^j_{T-1}, as in Table 5.1 and

$$\tilde{v}_{T-1}(Y_{T-1}) = \begin{cases} \tilde{J}\left(\bar{h}, Y_{T-1}\right) & \text{on } \bar{\mathcal{A}}_{T-1} \\ \tilde{J}\left(0, Y_{T-1}\right) & \text{on } \underline{\mathcal{A}}_{T-1} \\ \sum_{l=1}^{d} h_{T-1-l} \eta^l_{T-1} - \sum_{j=1}^{M-d} I^j_{T-2-d} \psi^j_{T-1} + \delta_{T-1}, & \text{on } \bar{\mathcal{A}}^c_{T-1}. \end{cases}$$

Further from (9.16)-(9.18) and (9.14), for any Y_{T-1}

$$\begin{aligned} v_{T-1}(Y_{T-1}) &\le \inf_{0 \le h_{T-1} \le \bar{h}} \tilde{J}(h_{T-1}, Y_{T-1}) + \sup_{0 \le h_{T-1} \le \bar{h}} \left| \Gamma_{T-1}(h_{T-1}, Y_{T-1}) \right|, \\ &\le \tilde{v}_{T-1}(Y_{T-1}) + \varepsilon, \end{aligned} \tag{9.19}$$

with

$$\varepsilon := \sup_{0\leq h_{T-1}\leq \bar{h}} \zeta_{T-1}(h_{T-1}, Y_{T-1}) = \rho \left|\sum_{j=1}^{M-1-d} c_T^j I_{T-2-d}^j\right| + \rho \left|\sum_{j=1}^{M-1-d} u_T^j I_{T-2-d}^j\right|$$
$$+\rho \sum_{l=2}^{d} c_T^l h_{T-l} + \rho c_T^1 \bar{h} + \rho c_T, \tag{9.20}$$

Again using (9.15)-(9.17) we have:

$$\begin{aligned} v_{T-1}(Y_{T-1}) &\leq \left(\tilde{J}\left(\tilde{h}^*_{T-1}(Y_{T-1}), Y_{T-1}\right) + \Gamma_{T-1}\left(\tilde{h}^*_{T-1}(Y_{T-1}), Y_{T-1}\right)\right) + \varepsilon \\ &\quad -\Gamma_{T-1}\left(\tilde{h}^*_{T-1}(Y_{T-1}), Y_{T-1}\right) \\ &\leq J\left(\tilde{h}^*_{T-1}(Y_{T-1}), Y_{T-1}\right) + 2\varepsilon. \end{aligned}$$

Thus $\tilde{h}^*_{T-1}(Y_{T-1})$ is an ε-optimal policy (note $\varepsilon = O(\rho)$) for any Y_{T-1} and thus (5.11) is true for $k = 1$.

Further as in the previous step (i.e., step with $k = 0$) one can write (e.g., once again $P(\mathcal{A}_{T-1}) = O(\rho)$, and so on)

$$\begin{aligned} E[\tilde{v}_{T-1}(Y_{T-1})|Y_{T-2}, h_{T-2}] &= E_{Y_{T-2},h_{T-2}}\left[\sum_{l=1}^{d} h_{T-1-l}\eta^l_{T-1} - \sum_{j=1}^{M-d} I^j_{T-2-d}\psi^j_{T-1} + \delta_{T-1}\right] \\ &\quad +\tilde{\Gamma}_{T-2}, \end{aligned}$$

where $\tilde{\Gamma}_{T-2}$ can be upper-bounded like in (9.14). Define

$$\Gamma_{T-2} := E_{Y_{T-2},h_{T-2}}\left[v_{T-1}(Y_{T-1}) - \tilde{v}_{T-1}(Y_{T-1})|Y_{T-2}, h_{T-2}\right] + \tilde{\Gamma}_{T-2}$$

and note that[2]

---

[2] It is clear from (9.17) that

$$\begin{aligned} J(h_{T-1}, Y_{T-1}) &= \tilde{J}(h_{T-1}, Y_{T-1}) + \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \\ &\geq \tilde{J}(h_{T-1}, Y_{T-1}) - |\Gamma_{T-1}(Y_{T-1}, h_{T-1})| \\ &\geq \tilde{J}(h_{T-1}, Y_{T-1}) - \sup_{h_{T-1}\in[0,\bar{h}]} |\Gamma_{T-1}(Y_{T-1}, h_{T-1})|, \\ \text{so, } v_{T-1}(Y_{T-1}) &\geq \tilde{v}_{T-1}(Y_{T-1}) - \sup_{h_{T-1}\in[0,\bar{h}]} |\Gamma_{T-1}(Y_{T-1}, h_{T-1})| \\ &\geq \tilde{v}_{T-1}(Y_{T-1}) - \varepsilon, \end{aligned}$$

and hence from (9.19), $|v_{T-1} - \tilde{v}_{T-1}| \leq \varepsilon$.

$$|\Gamma_{T-2}| \leq \varepsilon + \left|\tilde{\Gamma}_{T-2}\right|.$$

Recall $\tilde{\Gamma}_{T-2}$ can be upper-bounded like in (9.14) and using Lemma 9.4, it is easy to verify the following:

$$\begin{aligned}
|\Gamma_{T-2}| &\leq \zeta_{T-1}(Y_{T-2}, h_{T-2}) \text{ where} \\
\zeta_{T-1}(Y_{T-2}, h_{T-2}) &= \rho\left|\sum_{j=1}^{M-1-d} c^j_{T-1} I^j_{T-3-d}\right| + \rho\left|\sum_{j=1}^{M-1-d} u^j_{T-1} I^j_{T-3-d}\right| + \sum_{l=1}^{d} \rho c^l_{T-1} h_{T-1-l} + \rho c_{T-1},
\end{aligned}$$

for appropriate coefficients that converge to finite constants as $\rho \to 0$ and which do not depend upon Y_{T-2} or h_{T-2}. This completes the proof for $k = 1$.

The rest of the proof is completed using induction, assume the result is true for $k = n$. Now we will prove the result for $k = n + 1$. Using DP equations

$$\begin{aligned}
v_{T-n-1}(Y_{T-n-1}) &= \inf_{h_{T-n-1} \in [0,\bar{h}]} J(h_{T-n-1}, Y_{T-n-1}) \text{ with} \qquad (9.21)\\
J(h_{T-n-1}, Y_{T-n-1}) &:= r_{T-n-1}(h_{T-n-1}, Y_{T-n-1}) + E[v_T(Y_{T-n})|Y_{T-n-1}, h_{T-n-1}].
\end{aligned}$$

Fix Y_{T-n-1}. By Lemma 9.3, the objective function $J(\cdot)$ can be split and rewritten as:

$$\begin{aligned}
J(h_{T-n-1}, Y_{T-n-1}) &= \tilde{J}(h_{T-n-1}, Y_{T-n-1}) + \Gamma_{T-n-1}(Y_{T-n-1}, h_{T-n-1}) \text{ with} \qquad (9.22)\\
\tilde{J}(h_{T-n-1}, Y_{T-n-1}) &:= \sum_{l=0}^{d} h_{T-n-1-l}\left(\bar{\gamma}_l + \eta^{l+1}_{T-n}\mathbb{1}_{l<d} - \mathbb{1}_{l=d}\sum_{j=1}^{M-d}\psi^j_{T-n}\gamma_0^{j-1}\right) \\
&\quad - \sum_{j=1}^{M-1-d}\left(\rho^{(j)}\bar{\varpi}_{d,j+1} + \sum_{r=j}^{M-1-d}\psi^{r+1}_{T-n}\rho^{(j)}\varpi^r_{0,j+1}\right) I^j_{T-n-2-d} \\
&\quad + \alpha\left[1 - \Phi\left(\sum_{l=0}^{d} h_{T-n-1-l}\tilde{\gamma}^{M-1}_l - \sum_{j=1}^{M-d} I^j_{T-n-2-d}\tilde{\varpi}^{M-1}_{d,j+1}\right)\right] + \delta_{T-n}. \qquad (9.23)
\end{aligned}$$

The function $\tilde{J}(\cdot)$ is like the objective function of Theorem 9.6 and observe hypothesis (9.30) is satisfied by condition (5.10) with $k = n + 1$. By this theorem, the optimal policy and value function are respectively:

$$\tilde{v}_{T-n-1}(Y_{T-n-1}) := \inf_{0 \leq h_{T-n-1} \leq \bar{h}} \tilde{J}(h_{T-n-1}, Y_{T-n-1}), \qquad (9.24)$$

respectively equal:

$$\tilde h^*_{T-n-1}(Y_{T-n-1}) = \begin{cases} \bar h & \text{on } \bar{\mathcal{A}}_{T-n-1} \\ 0 & \text{on } \underline{\mathcal{A}}_{T-n-1} \\ \frac{1}{\tilde\gamma_0^{M-1}}\Big[-\sum_{l=1}^{d} h_{T-n-1-l}\tilde\gamma_l^{M-1} & \\ +\sum_{j=1}^{M-d} I^j_{T-n-2-d}\tilde\varpi^{M-1}_{d,j+1} + a_{T-n-1}\Big], & \text{on } \mathcal{A}^c_{T-n-1} \end{cases}$$

$$\begin{aligned} \text{with } \mathcal{A}_{T-n-1} &:= \underline{\mathcal{A}}_{T-n-1} \cup \bar{\mathcal{A}}_{T-n-1}, \qquad (9.25)\\ \underline{\mathcal{A}}_{T-n-1} &= \left\{ \sum_{j=1}^{M-d} I^j_{T-n-2-d}\tilde\varpi^{M-1}_{d,j+1} < -a_{T-n-1} + \sum_{l=1}^{d} h_{T-n-1-l}\tilde\gamma_l^{M-1} \right\},\\ \bar{\mathcal{A}}_{T-n-1} &= \left\{ \sum_{j=1}^{M-d} I^j_{T-n-2-d}\tilde\varpi^{M-1}_{d,j+1} > \bar h\tilde\gamma_0^{M-1} - a_{T-n-1} + \sum_{l=1}^{d} h_{T-n-1-l}\tilde\gamma_l^{M-1} \right\} \end{aligned}$$

with constants, e.g., a_{T-n-1}, η^l_{T-n-1}, ψ^j_{T-n-1}, as in Table 5.1 and

$$\tilde v_{T-n-1}(Y_{T-n-1}) = \begin{cases} \tilde J(\bar h, Y_{T-n-1}) & \text{on } \bar{\mathcal{A}}_{T-n-1} \\ \tilde J(0, Y_{T-n-1}) & \text{on } \underline{\mathcal{A}}_{T-n-1} \\ \sum_{l=1}^{d} h_{T-n-1-l}\eta^l_{T-n-1} - \sum_{j=1}^{M-d} I^j_{T-n-2-d}\psi^j_{T-n-1} & \\ +\delta_{T-n-1}, & \text{on } \bar{\mathcal{A}}^c_{T-n-1}. \end{cases}$$

Further from (9.21)-(9.24) and (9.14), for any Y_{T-n-1}

$$\begin{aligned} v_{T-n-1}(Y_{T-n-1}) &\le \inf_{0\le h_{T-n-1}\le \bar h} \tilde J(h_{T-n-1}, Y_{T-n-1}) + \sup_{0\le h_{T-n-1}\le \bar h} \left|\Gamma_{T-n-1}(h_{T-n-1}, Y_{T-n-1})\right|,\\ &\le \tilde v_{T-n-1}(Y_{T-n-1}) + \varepsilon \text{ with} \qquad (9.26)\\ \varepsilon &:= \sup_{0\le h_{T-n-1}\le \bar h} \zeta_{T-n-1}(h_{T-n-1}, Y_{T-n-1})\\ &= \rho\left|\sum_{j=1}^{M-1-d} c^j_{T-n} I^j_{T-n-2-d}\right| + \rho\left|\sum_{j=1}^{M-1-d} u^j_{T-n} I^j_{T-n-2-d}\right| + \rho\sum_{l=2}^{d} c^l_{T-n} h_{T-n-l}\\ &\quad +\rho c^1_{T-n}\bar h + \rho c_{T-n}. \end{aligned}$$

$$\begin{aligned} v_{T-n-1}(Y_{T-n-1}) &\le \left(\tilde J\left(\tilde h^*_{T-n-1}(Y_{T-n-1}), Y_{T-n-1}\right) + \Gamma_{T-n-1}\left(\tilde h^*_{T-n-1}(Y_{T-n-1}), Y_{T-n-1}\right)\right) + \varepsilon\\ &\quad -\Gamma_{T-n-1}\left(\tilde h^*_{T-n-1}(Y_{T-n-1}), Y_{T-n-1}\right)\\ &\le J\left(\tilde h^*_{T-n-1}(Y_{T-n-1}), Y_{T-n-1}\right) + 2\varepsilon. \end{aligned}$$

Thus $\tilde h^*_{T-n-1}(Y_{T-n-1})$ is an ε-optimal policy (note $\varepsilon = O(\rho)$) for any Y_{T-n-1} and thus (5.11) is true for $k = n+1$.

Further as in the previous step (i.e., step with $k = 0$) one can write (e.g., once again $P(\mathcal{A}_{T-n-1}) = O(\rho)$, and so on)

$$
\begin{aligned}
& E[\tilde{v}_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}] \\
= \; & E_{Y_{T-n-2},h_{T-n-2}}\left[\sum_{l=1}^{d} h_{T-n-1-l}\eta^l_{T-n-1} - \sum_{j=1}^{M-d} I^j_{T-n-2-d}\psi^j_{T-n-1} + \delta_{T-n-1}\right] + \tilde{\Gamma}_{T-n-2},
\end{aligned}
$$

where $\tilde{\Gamma}_{T-n-2}$ can be upper-bounded like in (9.14). Define

$$
\Gamma_{T-n-2} := E_{Y_{T-n-2},h_{T-n-2}}\left[v_{T-n-1}(Y_{T-n-1}) - \tilde{v}_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}\right] + \tilde{\Gamma}_{T-n-2}
$$

and note that

$$
|\Gamma_{T-n-2}| \leq \varepsilon + \left|\tilde{\Gamma}_{T-n-2}\right|.
$$

Using the above definition we have:

$$
\begin{aligned}
& E[v_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}] \\
= \; & E_{Y_{T-n-2},h_{T-n-2}}\left[\sum_{l=1}^{d} h_{T-n-1-l}\eta^l_{T-n-1} - \sum_{j=1}^{M-d} I^j_{T-n-2-d}\psi^j_{T-n-1} + \delta_{T-n-1}\right] + \Gamma_{T-n-2}.
\end{aligned}
$$

Recall $\tilde{\Gamma}_{T-n-2}$ can be upper-bounded like in (9.14) and using Lemma 9.4, it is easy to verify the following:

$$
\begin{aligned}
|\Gamma_{T-n-2}| & \leq \zeta_{T-n-1}(Y_{T-n-2}, h_{T-n-2}) \text{ where} \\
\zeta_{T-n-1}(Y_{T-n-2}, h_{T-n-2}) & = \rho\left|\sum_j c^j_{T-n-1} I^j_{T-n-3-d}\right| + \rho\left|\sum_j u^j_{T-n-1} I^j_{T-n-3-d}\right| \\
& \quad + \sum_{l=1}^{d} \rho c^l_{T-n-1} h_{T-n-1-l} + \rho c_{T-n-1},
\end{aligned}
$$

for appropriate coefficients that converge to finite constants as $\rho \to 0$ and which do not depend upon Y_{T-n-2} or h_{T-n-2}. This completes the proof for $k = n + 1$. ■

Lemma 9.3. *For any $k \geq 1$, Y_{T-k} and h_{T-k}*

$$
\begin{aligned}
\tilde{J}(h_{T-k}, Y_{T-k}) & = r_{T-k}(h_{T-k}, Y_{T-k}) \\
& \quad + E_{Y_{T-k},h_{T-k}}\left[\sum_{l=1}^{d} h_{T-k+1-l}\eta^l_{T-k+1} - \sum_{j=1}^{M-d} I^j_{T-k-d}\psi^j_{T-k+1} + \delta_{T-k+1}\right],
\end{aligned}
$$

where $\tilde{J}(h_{T-k}, Y_{T-k})$ is defined (with $k = n + 1$) in (9.23).

Proof: The coefficients are defined recursively to satisfy the above, and it can be verified using Table 5.1 and by taking conditional expectation of equation (9.1) of Lemma 9.1. ■

Lemma 9.4. *For any $k \geq 1$, Y_{T-k}, h_{T-k} and the coefficients*

$$P\left(\mathcal{A}_{T-k+2}\middle|Y_{T-k},h_{T-k}\right)\leq 2\exp(-(C/\rho)^2+C/\rho), \tag{9.27}$$

$$E\left[\left|\sum_{j=1}^{M-1-d}c_jI^j_{T-k+1-d}\right|\middle|Y_{T-k},h_{T-k}\right]\leq|\sum_j cc^jI^j_{T-k-d}|+cc^hh_{T-k+1-d}+cc,$$

$$E\left[\sum_j c_jI^j_{T-k+1-d}\middle|Y_{T-k},h_{T-k}\right]\leq|\sum_j cc^jI^j_{T-k-d}|+cc^hh_{T-k+1-d}+cc. \tag{9.28}$$

where the coefficients on the right hand side are appropriately defined. Note that the upper bound in (9.27) is clearly $O(\rho)$.

Proof: For (9.27) (see (9.25)), it suffices to prove the result for the following probability for any $C>0$:

$$p_{\mathcal{A}}:=P\left(\sum_j\bar{e}^jI^j_{T-k-d}>\frac{C}{\rho}\middle|Y_{T-k},h_{T-k}\right)$$

Let

$$\mu:=\sum_j e^jI^j_{T-k-d}+e^hh_{T-k+1-d}=E\left[\sum_j\bar{e}^jI^j_{T-k+1-d}\middle|Y_{T-k},h_{T-k}\right]$$

represent the conditional mean. Thus we need to prove that for a Gaussian random with mean μ the above implication:

$$\begin{aligned}
p_{\mathcal{A}} &= \int_{C/\rho}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\omega^2}\right)\frac{dt}{\sqrt{2\pi\omega^2}}\\
&= \int_{(C/\rho-\mu)/\omega}^{\infty}\exp\left(-\frac{t^2}{2}\right)\frac{dt}{\sqrt{2\pi}}\\
&\leq 2\exp\left(-\frac{(C/\rho-\mu)^2}{\omega^2}\right)\\
&= 2\exp(-(C/\rho)^2)\exp(-\mu^2+2\mu C/\rho)\leq 2\exp(-(C/\rho)^2+C/\rho).
\end{aligned}$$

Here, ω^2 is absorbed by the constant C. For (9.28), first observe that $\sum_j c_jI^j_{T-k+1-d}$ conditioned on (Y_{T-k},h_{T-k}) is a Gaussian random variable and let its conditional expected value be:

$$\tilde{\mu}=E\left[\sum_j c_jI^j_{T-k+1-d}\middle|Y_{T-k},h_{T-k}\right]$$

and note further that $\tilde{\mu}$ is linear in (Y_{T-k},h_{T-k}). Let ω be the corresponding variance. Using the above definitions (without loss of generality when $\tilde{\mu}>0$),

$$\begin{aligned}
E\left[\left|\sum_j c_jI^j_{T-k+1-d}\right|\middle|Y_{T-k},h_{T-k}\right] &= \int_{-\infty}^{\infty}|t|\exp\left(-\frac{(t-\tilde{\mu})^2}{2\omega^2}\right)\frac{dt}{\sqrt{2\pi\omega^2}}\\
&= \int_{-\infty}^{\infty}|t+\tilde{\mu}|\exp\left(-\frac{t^2}{2\omega^2}\right)\frac{dt}{\sqrt{2\pi\omega^2}}\\
&\leq 2\int_0^{\infty}t\exp\left(-\frac{t^2}{2\omega^2}\right)\frac{dt}{\sqrt{2\pi\omega^2}}+\tilde{\mu}. \qquad \blacksquare
\end{aligned}$$

Lemma 9.5. *Assume $\alpha>(1+\tilde{\phi})\beta M\sqrt{2\pi}/2$ for some $\tilde{\phi}>0$ and let $\rho:=\max\{\rho^{(1)},\rho^{(2)},\ldots,\rho^{(j)}\}$ for all $1\leq j\leq M$. Then there exists an upper bound, $1\geq\bar{\rho}>0$, such that condition (5.10) is satisfied (for all k) and for all load factors $\rho\leq\bar{\rho}$.*

Proof: Define the following function,

$$f_\eta : (0,1] \to \mathbb{R}^{T+1}$$

i.e., $\rho \xrightarrow{\mathrm{f}_\eta} \left(\bar{\gamma}_0 + \{\eta_t^1\}_{t\geq 0}\right)$. For every $\rho > 0$, $f_\eta(\rho)$ is well defined and continuous (see Table 5.1). Now we define the continuous extension of f_η at $\rho = 0$, i.e.,

$$f_\eta(0) \triangleq \lim_{\rho\to 0} f_\eta(p).$$

Note here $f_\eta(0)$ is not corresponding to any system and is only defined to obtain a continuous function: f_η is a continuous function on $[0,1]$ by construction.

Before we proceed further one can easily verify the following (see Table 5.1):

$$\begin{aligned} \frac{\tilde{\gamma}_l^{M-1}}{\tilde{\gamma}_0^{M-1}} \to 0 \text{ for all } l \geq 1, &\quad \bar{\gamma}_i \to 0, \text{ for all } i > 0, \\ \bar{\gamma}_0 \to \frac{M}{2}, \text{ and}, &\quad \psi_t^j \to 0 \text{ for all } j, t > 0 \text{ as } \rho \to 0, \end{aligned} \tag{9.29}$$

which would be instrumental in proving this lemma.

If one proves $f_\eta(0) = \lim_{\rho\to 0} f_\eta(\rho) > 0$ (i.e., all components are strictly positive), then by continuity there exists a $\bar{\rho} > 0$ such that $f_\eta(\rho) > 0$ for all $\rho \leq \bar{\rho}$. We prove this by proving each η_t^l converges to zero, but the limit of the overall co-efficient is non zero because of non-zero-limit of $\tilde{\gamma}_0^{M-1}$. Towards this, we begin with $\{\eta_T^l\}_l$ and from Table 5.1 and (9.29) for any $l \geq 1$:

$$\lim_{\rho\to 0} \eta_T^l = \lim_{\rho\to 0} \frac{1}{\tilde{\gamma}_0^{M-1}} \left(\left(\bar{\gamma}_l + \mathbb{1}_{l<d}\eta_{T+1}^{l+1} - \mathbb{1}_{l=d} \sum_{j=1}^{M-d} \psi_{T+1}^j \gamma_0^{j-1} \right) \tilde{\gamma}_0^{M-1} - \left(\bar{\gamma}_0 + \eta_{T+1}^1\right) \tilde{\gamma}_l^{M-1} \right) = 0.$$

Using the above, and using backward recursion it is clear that

$$\lim_{\rho\to 0} \eta_{T-k}^l = 0 \text{ for all } l > 0, k \geq 0.$$

Thus the RHS of condition (5.10) has the following non-zero-limit:

$$\lim_{\rho\to 0} \frac{\left(\bar{\gamma}_0 + \eta_{T-j}^1\right)\sqrt{2\pi}\omega}{\tilde{\gamma}_0^{M-1}} = \beta\frac{M\sqrt{2\pi}}{2},$$

and now there exists a $\bar{\rho} > 0$ (by continuity of f and others) such that

$$\frac{\left(\bar{\gamma}_0 + \eta_{T-j}^1\right)\sqrt{2\pi}\omega}{\tilde{\gamma}_0^{M-1}} \leq (1+\tilde{\phi})\beta\frac{M\sqrt{2\pi}}{2} \text{ for all } \rho \leq \bar{\rho}$$

By hypothesis $\alpha > (1+\tilde{\phi})\beta M\sqrt{2\pi}/2$, and hence the condition (5.10) is satisfied for all $\rho \leq \bar{\rho}$. ∎

Theorem 9.6. *Let g be function of the following type: Φ is the CDF of normal random variable with mean 0 and variance ω^2*

$$\begin{aligned} g(h, \mathbf{Y}) &= \sum_{l=0}^{d} h_{-l}\bar{z}_l - \sum_{j=1}^{M-1-d} I^j \bar{z}^j + \alpha\left(1 - \Phi\left(\sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j\right)\right) + \delta \\ &\quad \textit{with } \mathbf{Y} = [\{h_{-l}\}_{l>0}, I^1, I^2, \ldots, I^M], \end{aligned}$$

with $\bar{z}_0 > 0$, $\alpha > 0$, $\tilde{z}_0 > 0$ and for notational simplicity let $h_0 = h$. Consider the following optimization problem:

$$g^*(\mathbf{Y}) := \min_{h\in[0,\bar{h}]} g(h, \mathbf{Y}) \textit{ and } h^* := \arg\min_h g(h, \mathbf{Y}).$$

Assume

$$\alpha > \frac{\bar{z}_0\sqrt{2\pi}\omega}{\tilde{z}_0} \text{ and define } a := \omega\sqrt{-2\log\left(\frac{\bar{z}_0\sqrt{2\pi}\omega}{\tilde{z}_0\alpha}\right)}. \tag{9.30}$$

Then there exists an unique optimizer to this problem and the optimizer is given by:

$$h^* = \begin{cases} 0 & \text{if } -\sum_{l=1}^{d} h_{-l}\tilde{z}_l + \sum_{j=1}^{M-d} I^j \tilde{z}^j < -a \\ \bar{h} & \text{if } \bar{h}\tilde{z}_0 + \sum_{l=1}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} I^j \tilde{z}^j < a \\ \frac{1}{\tilde{z}_0}\left[-\sum_{l=1}^{d} h_{-l}\tilde{z}_l + \sum_{j=1}^{M-d} I^j \tilde{z}^j + a\right], & \text{else.} \end{cases} \tag{9.31}$$

Further,

$$g^*(\mathbf{Y}) = \begin{cases} g(0,\mathbf{Y}) & \text{if } -\sum_{l=1}^{d} h_{-l}\tilde{z}_l + \sum_{j=1}^{M-d} I^j \tilde{z}^j < -a \\ g(\bar{h},\mathbf{Y}) & \text{if } \bar{h}\tilde{z}_0 + \sum_{l=1}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} I^j \tilde{z}^j < a \\ \sum_{l=1}^{d} h_{-l}\eta^l - \sum_{j=1}^{M-d} I^j \psi^j + \delta^* & \text{else, with} \end{cases}$$

$$\eta^l = \frac{\bar{z}_l\tilde{z}_0 - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}, \qquad \psi^j = \bar{z}^j \mathbb{1}_{j<M-d} - \frac{\bar{z}_0\tilde{z}^j}{\tilde{z}_0}, \qquad \delta^* = \alpha\left[1-\Phi(a)\right] + \frac{\bar{z}_0 a}{\tilde{z}_0} + \delta.$$

Proof: Consider the objective function,

$$g^*(\mathbf{Y}) = \min_{h\in[0,\bar{h}]}\left\{\sum_{l=0}^{d} h_{-l}\bar{z}_l - \sum_{j=1}^{M-1-d} I^j \bar{z}^j + \alpha\left(1-\Phi\left(\sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j\right)\right) + \delta\right\}. \tag{9.32}$$

Let h^* be the optimal policy for problem (9.32), i.e.,

$$h^* \in \arg\min(g^*(\mathbf{Y})),$$

either h^* is on the boundary or h^* in interior when $\frac{d}{dh}(g(\mathbf{Y}))\Big|_{h=h^*} = 0$.

The first derivative is given by:

$$\frac{d}{dh}(g(\mathbf{Y})) = \bar{z}_0 - \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{\left(\sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j\right)^2}{2}\right)\tilde{z}_0.$$

When a is well defined (as given by hypothesis (9.30)), there exists a zero of the derivative which satisfies the following:

$$\left(\sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j\right)^2 = a^2, \text{ i.e., } \Rightarrow \sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j = a.$$

If

$$\sum_{l=0}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j < a,$$

then the derivative is negative for all $h \in [0,\bar{h}]$, and hence $h^* = \bar{h}$. For the rest of the cases (i.e., when $a - \bar{h}\tilde{z}_0 < \sum_{l=1}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-d} \tilde{z}^j I^j < a$) we have the interior optimizer if $\alpha > \frac{\bar{z}_0\sqrt{2\pi}}{\tilde{z}_0}$,

$$h^* = \frac{1}{\tilde{z}_0}\left(a - \sum_{l=1}^{d} h_{-l}\tilde{z}_l + \sum_{j=1}^{M-d} \tilde{z}^j I^j\right),$$

as the second derivative (at such h^*) is positive.

Substituting the above, the optimal objective function (when interior point is optimal) equals:

$$\begin{aligned} g^*(\mathbf{Y}) &= h^*\bar{z}_0 + \sum_{l=1}^{d} h_{-l}\bar{z}_l - \sum_{j=1}^{M-1-d} I^j \bar{z}^j + \alpha\left(1 - \Phi\left(h^*\tilde{z}_0 + \sum_{l=1}^{d} h_{-l}\tilde{z}_l - \sum_{j=1}^{M-1-d} I^j \tilde{z}^j\right)\right) + \delta \\ &= \sum_{l=1}^{d} h_{-l}\left(\frac{\bar{z}_l\tilde{z}_0 - \bar{z}_0\tilde{z}_l}{\tilde{z}_0}\right) - \sum_{j=0}^{M-d} I^j\left(\bar{z}^j \mathbb{1}_{j<M-d} - \frac{\bar{z}_0\tilde{z}^j}{\tilde{z}_0}\right) + \frac{\bar{z}_0 a}{\tilde{z}_0} + \alpha\left(1 - \Phi\left(a\right)\right) + \delta \\ &= \sum_{l=1}^{d} h_{-l}\eta^l - \sum_{j=0}^{M-1} I^j\psi^j + \delta^*, \text{ where,} \end{aligned}$$

$\{\eta^l\}$, $\{\psi^j\}$, and δ^* are as defined in hypothesis. ■

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