On-Demand OFDMA: Control, Fairness and Non-Cooperation

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***Abstract*— Motivated by recent work on improving the efficiency of the IEEE 802.11 protocol at high speeds, we consider an OFDMA system in which the users make reservations requests over a collision channel. The controller schedules from only amongst the successful requests using an alpha-fair scheduler that balances the network throughput and fairness to nodes. The first part of this work analyses the performance of the alpha-fair scheduler when used with an Aloha reservation channel. In the second part we assume that the network prescribes reservation rates to active nodes. However, nodes can attempt reservations more aggressively so as to be scheduled more frequently (and unfairly). A simple game theoretic analysis of this interaction between the Aloha reservation channel and the scheduler shows that in the presence of other cooperative users, a node attempting at a rate higher than that prescribed indeed obtains a larger (unfair) throughput. For such a network we propose a robust alpha-fair scheduler that penalizes aggressive users. This scheduler along with the prescribed reservation rates forms a Nash equilibrium. Extensive numerical results illustrate the scheduling algorithms.**

I. Introduction

Orthogonal frequency division multiple access (OFDMA) is the transmission technique for most of the current and emerging high speed networking technologies including IEEE 802.11 WLANs, LTE and WiMax. In OFDMA, the available spectrum is divided into a number of narrow subbands, or subchannels. The subbands can be dynamically allocated in different combinations to different users. This allows networks to use sophisticated opportunistic allocation mechanisms to provide high spectral efficiency. In addition to spectral efficiency, there is significant interest in developing scheduling mechanism to achieve other objectives. Examples of such objectives are the stabilizing queues (e.g., [1]), minimizing delays (e.g., [1], [2]), and providing fairness (e.g., [3]).

An OFDMA scheduling algorithm essentially allocates different combinations of resources—subchannels, transmission rate (by prescribing the modulation scheme) and transmission power to achieve an objective. A typical scheme works as follows. Time is divided into frames. At the beginning of each frame, the active users provide 'local state' information like channel state, queue length and possibly even battery state. The controller uses this information to perform a resource allocation to achieve a network objective. The schemes in, among others, [1]–[3] have this structure. It is easy to see that such a system can lead to selfish behavior by this nodes and this has led to the design and analysis of these systems assuming non cooperative nodes, e.g., [3], [4].

An interesting recent use of OFDMA has been in easing the PHY inefficiencies of the 802.11 MAC protocol. It is now well known that CSMA/CA protocol of 802.11 becomes increasingly inefficient with increasing PHY data rates because the overheads remain fixed but the data transmission times shrink with increasing PHY rates. This has led to several schemes being devised to improve the efficiency. Some of these reduce the overhead with clever changes to PHY e.g., [5], [6] while some amortize the fixed overheads by exploiting features of OFDMA, e.g., the fine grained channel (FICA) scheme of [7]. FICA works as follows. Time is divided into frames and OFDM subcarriers are divided into D groups called subchannels. A subset of subcarriers of each group, R per subchannel, are allocated for RTS by the nodes to the access point (AP). Depending on the channel gains, each node can choose to transmit in any of the subchannels. For the chosen subchannel(s), the node randomly selects a subcarrier to signal an RTS. If more than one subcarrier is successful in a subchannel the AP performs a collision resolution and allocates the subchannel to one of the nodes using a collision resolution mechanism. To summarize, FICA improves efficiency by stretching the data transmission time (by simultaneously allocating smaller portions of the bandwidth to a larger number of users) while keeping the overheads constant.

The FICA scheme, omitting the protocol details, can be seen to be an OFDMA channel in which the user-to-controller signaling is through a collision channel. In this paper we take a cue from this scheme and develop scheduling algorithms for such a system. In keeping with the terminology of OFDMA literature, the controller or the access point will be referred to as the base station (BS) and the user nodes will be called mobile stations (MS).

The rest of the paper is organized as follows. The notation, system details and some preliminary results are presented in the next section. In Section III we develop the game theoretic description of a system of non cooperative nodes interacting with an alpha-fair scheduler. In Section IV we describe and analyze the robust scheduler that will lead to a Nash equilibrium in which the nodes request reservations at the prescribed rate and the controller performs alpha-fair scheduling. We present extensive numerical results in Section V and conclude with a discussion in which we compare our system with related OFDMA systems and preview some future work.

II. Notation and System Preliminaries

In this section, we first introduce the basic notations used in the paper. This is followed by a detailed description of the Aloha reservation channel. We then describe an alpha-fair scheduler that will be implemented at the base station which will allocate channel to the mobile stations based on some fairness criteria. We will also consider the packet arrival model describing the nature of data packet arrival at the mobiles.

Much like other OFDMA systems in the literature, time is divided into frames and each frame has two phases—reservation and data phases. In the reservation phase, the M mobiles in the system transmit RTS (request to send) to the BS according to a randomized algorithm over the Aloha reservation channel. The BS decodes the successful RTS's and applies its scheduling algorithm to grant channel access to a subset of the successful nodes. This schedule is conveyed via a common CTS (clear to send) signal. The data phase consists of D, $D \geq 1$, parallel data channels and the mobiles that are scheduled will transmit in these channels.

The following notational convention is used. Lower case letters will represent flags or state indicators (some of which are random) while the corresponding bold letters (in lower case) represent the respective vectors. $\mathcal{X}_{\{.\}}$ will be the indicator of the event in the subscript. The positive part of a variable is represented by $(\cdot)^+$, i.e., $(x)^+ = \max\{x, 0\}$. Mobiles are indexed by m and i, data channels by j and frames by t.

The state of channel j for MS m is denoted by $h_{m,j}$ with $\mathbf{h}_m := [h_{m,1}, \cdots, h_{m,D}]$ denoting the channel state vector for MS m. The channel states $\mathbf{h} := [\mathbf{h}_1, \cdots, \mathbf{h}_M]$ are assumed to be independent processes across mobiles and across time frames. Like in other such OFDMA systems, MS m will send $\mathbf{h}_m$ to the BS in the RTS.

The indicator that MS m has transmitted an RTS signal in the frame will be denoted by a_m with $\mathbf{a} := [a_1, a_2, \cdots, a_M]$ being the corresponding vector. Following conventional game theoretic notation, $\mathbf{a}_{-m} := [a_1, \cdots a_{m-1}, a_{m+1}, \cdots, a_M]$, i.e., $\mathbf{a}$ after excluding the m-th component.

A. The Aloha Reservation Channel

We consider two type of reservation channels which use Aloha as the MAC.

1) Aggregated Reservation: Each frame has a single reservation phase and MS m will transmit an RTS with probability p_m. If the total number of nodes transmitting an RTS is less than or equal to R, then all those that transmitted will be deemed successful. If more than R transmit, then the BS cannot decode any RTS and a collision is said to occur. Recall that a_m is the flag indicating MS m transmitted an RTS. Let flag b_m indicate that the RTS from m was successful. Note that $p_m = \mathsf{Pr}\left(a_m = 1\right)$. In the event of a collision in the reservation channel, the corresponding data channel is wasted because it cannot be assigned. The probability of successful RTS is $p_m^{\text{succ}} = \mathsf{Pr}\left(b_m = 1\right)$, is easy to calculate. For example,

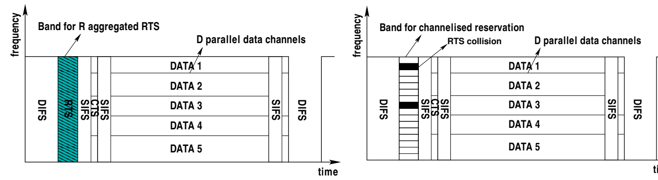

Fig. 1. Frequency v/s time diagram of a frame with aggregated and channelised reservation scheme.

with $p_m = p$ for all m, this will be the same for all m and is

$$p^{\text{succ}} = p \sum_{i=0}^{R-1} \binom{N-1}{i} p^i (1-p)^{N-1-i}.$$

2) Channelized Reservation: In this scheme, each frame has R Aloha channels either via FDM or via TDM. Mobile m will transmit in each of these channels with probability p_m independently of the others. If more than one MS choose the same reservation channel, then there is a collision on the channel and the BS cannot decode any RTS. Assuming $p_m = p$ for all m, the probability that MS m has a successful RTS in a frame is

$$p^{\text{succ}} = 1 - (1 - p(1-p)^{N-1})^R.$$

For both cases, the p^{succ} for the case of asymmetric RTS attempt probability can be obtained albeit with messier expressions. Let $\mathbf{p} := [p_m]$ be called the reservation rate vector. Fig. 1 represent the aggregated and channelized reservation scheme respectively.

B. Scheduler

In each frame, a set of mobiles will have submitted a reservation request via a successful RTS. The scheduler at the BS schedules a subset of these mobiles and sends CTS to the scheduled set. The scheduler could choose an MS with maximum utility, in which case network efficiency is maximized; the allocation may not be fair to the mobiles. It could also choose to allocate such that the minimum utility of the scheduled nodes is maximized, i.e., use a max-min fair schedule. The latter is of course not very efficient. An alpha-fair scheduler helps achieve a desired trade-off between fairness and network efficiency via a suitable choice of the tuning parameter α, $0 \leq \alpha \leq \infty$. Specifically, we define and use the following alpha-fair scheduler described in, among others, [8], [10].

Recall that $b_m := \mathcal{X}_{\{\text{successful RTS by } m\}}$; Define $\hat{h}_{m,j} := h_{m,j} b_m$, i.e., $\hat{h}_{m,j}$ is the effective state of channel j for MS m after accounting for RTS. Define $\hat{\mathbf{h}} := [[\hat{h}_{m,j}]]$. Let $f(h)$ be the utility from a channel with channel state h with $f(h) > 0$ if $h \neq 0$ and $f(0) = 0$. f could, for example, indicate the number of bits per frame per channel that the MS can transmit when the channel state is h. The utility that MS m achieves on being scheduled for transmission in data channel j is $f(\hat{h}_{m,j})$.

A scheduler β maps every channel state $\hat{\mathbf{h}}$ to decisions $[[\beta_{m,j}]]$ where $\beta_{m,j} \in [0,1]$ is the probability with which MS m is allocated data channel j. Let

$$u_{m,j} := \mathsf{E}\left[f(\hat{h}_{m,j})\beta_{m,j}\right]; \quad u_m := \sum_j u_{m,j}, \quad (1)$$

$$\Gamma_\alpha(u) = \frac{u^{1-\alpha}}{1-\alpha}\mathcal{X}_{\{\alpha\neq 1\}} + \log(u)\ \mathcal{X}_{\{\alpha=1\}}. \quad (2)$$

Define $\mathbf{u} = [u_1,\ u_2,\ \cdots, u_M]$. Assume that utilities are additive, i.e., if MS m is allocated more than one data channel in a frame, then the total utility is the sum of the utilities from each data channel. The alpha-fair scheduler will send CTS to a subset of MS selected according to (see [8], [10])

$$\arg\max_\beta \sum_m \Gamma_\alpha(u_m). \quad (3)$$

The efficient scheduler maximizes the sum throughput and is obtained by using $\alpha = 0$ in (3); for this case we have

$$\begin{aligned}\beta^0_{m,j} &= \mathcal{X}_{\{m=\arg\max_i f(\hat{h}_{m,j})\ a_i\}} \quad \text{for all } m,j \\ &= \mathcal{X}_{\{m=\arg\max_i f(h_{i,j})\ a_i\}} b_m. \quad (4)\end{aligned}$$

The max-min fair scheduler maximizes the minimum of all the utilities and is obtained with $\alpha = \infty$ in (3); intermediate levels of fairness-throughput trade-offs are obtained with $\alpha \in (0,\infty)$.

Unlike for $\alpha = 0$, for $\alpha > 0$ we do not have a closed form for the scheduler. However, using the concavity of $\Gamma_\alpha(u)$ and the linear dependence of $\mathbf{u}$ on the scheduler, we obtain a fixed point structure for the alpha-fair scheduler as described in the following theorem, with proof in Appendix.

Theorem 1: Assume the channel states $h_{m,j}$ are independent across mobiles m, data channels j, and time frames.
(1) Consider , $\Lambda(\cdot) := [\Lambda_m(\cdot)]_m$, defined component wise by

$$\Lambda_m(\mathbf{u}) = \sum_j \mathsf{E}\left[\frac{f(h_{m,j})}{\left|\arg\max_i \frac{f(\hat{h}_{i,j})}{u_i^\alpha}\right|}\mathcal{X}_{\left\{m\in\arg\max_i \frac{f(\hat{h}_{i,j})}{u_i^\alpha}\right\}}\right], \quad (5)$$

with $|A|$ denoting the cardinality of set A. If there exists a fixed point $\mathbf{u}^*$ of the above function, then an alpha-fair scheduler maximizing (3) can be defined using this fixed point $\mathbf{u}^* := [u_1^*, \cdots, u_m^*]$ as

$$\beta^\alpha_{m,j} = \frac{\mathcal{X}_{\left\{m\in\arg\max_i \frac{f(\hat{h}_{i,j})}{(u_i^*)^\alpha}\right\}}}{\left|\arg\max_i \frac{f(\hat{h}_{i,j})}{(u_i^*)^\alpha}\right|} \text{ for all } m,j. \quad (6)$$

(2) Assume for each m and j that $h_{m,j}$ is a continuous random variable with continuous density $g_{m,j}(\cdot)$, and that f is integrable, i.e., that $E[f(h_{m,j})] < \infty$. Then there exists a fixed point $\mathbf{u}^*$ of the function (5). $\square$

The theorem says that if a fixed point exists for $\Lambda_m(\mathbf{u})$, then the alpha-fair scheduler is obtained from the fixed point. This structure will be used in further analysis as well as in constructing practical policies.

The case with $D = 1$ and with ideal reservation channel is considered in [8] where an iterative algorithm that asymptotically maximizes the alpha-fair utility (3) is proposed. This algorithm implements the fixed point equation of the preceding theorem (see [10] for more details). We now define the following iterative algorithm which obtains the fixed point of the above theorem for general D and for Aloha reservation channel. For any time frame $t+1$, any MS m and for the j-th data channel, the iteration in the t-th time step is

$$\begin{aligned}u_{m,j,t+1} &= u_{m,j,t} + \mu\left(f(h_{m,j,t+1})\beta^\alpha_{m,j,t} - u_{m,j,t}\right) \quad (7)\\ \beta^\alpha_{m,j,t} &:= \mathcal{X}_{\left\{m=\arg\max_i f(\hat{h}_{i,j,t+1})(u_{i,t}+d_i)^{-\alpha}\right\}} \quad (8)\\ u_{m,t} &= \sum_{j=1}^D u_{m,j,t}.\end{aligned}$$

Here the constant $\mu > 0$ is the step size and $[d_i]$ are small stabilizing constants.

C. Packet Arrival Model

We consider two packet arrival models—saturated and non saturated. In the saturated model, all the M MS will always have data to send. All our theoretical analysis are for this case. Note that this model is widely used in the analysis of a variety of network protocols and is not unreasonable.

A more realistic model is to assume that the packets arrive at each MS according to a random process and that an MS attempts an RTS in the reservation phase only if it has a non empty packet queue. This is the non-saturated condition. The scheduling algorithms developed for the saturated case can be extended to the non saturated case and numerical results to indicate the performance will be presented. The analysis for this case is not presented in this paper.

For the non saturated model, packets arrive at MS m according to an arrival process of rate λ_m. Let $x_m(t)$ be the number of bulk arrivals to MS m in frame t. Each arrival demands S units of data transfer and the buffer at each MS is considered to be an infinite buffer. Let $q_{m,t}$ be the queue length at MS m at the beginning of frame t. Let r be the number of units of data transmitted per unit utility. For modeling convenience, we will assume that the MS's attempt reservation in all the frames with probability p_m but the RTS will be deemed failure if $q_{m,t} = 0$, i.e., the flag b_m is

$$b_m = a_m \mathcal{X}_{\{q_m>0\}}\ \mathcal{X}_{\{\text{RTS did not collide}\}} \quad (9)$$

Note that the value of $\mathcal{X}_{\{\text{RTS did not collide}\}}$ depends on the reservation channel. The queue length updates according to

$$q_{m,t+1} = \left(q_{m,t} - r \sum_j f(h_{m,j,t}) \beta^{\alpha}_{m,j,t} + S x_m(t) \right)^+$$

Recall that $\beta^{\alpha}_{m,j,t}$ is the scheduling decision for time frame t for node m on channel j.

In reality, an MS m will attempt RTS in frame t with probability p_m if and only if $q_{m,t} > 0$. The collision probability with this model is lower but this difference reduces with increasing load.

III. Selfish Mobiles: Towards a Game

We begin by showing that an MS has an incentive to be selfish. It can improve its utility by choosing a higher RTS reservation rate. This motivates a game theoretic problem formulation.

The system that we consider has two parts—the Aloha reservation channel and the data channel scheduled by an alpha-fair scheduler. While the literature on Aloha is extensive, a game theoretic understanding is emerging, e.g. [11] for an early survey, and [10], [12] for some recent work. Also, there is significant literature on OFDMA scheduling some of which were pointed out in Section I. Our interest in this paper is to analyze the interaction between the success probability in the Aloha channel and the long term average scheduled throughput obtained on the data channel. More specifically, recall that MS m obtains a utility

$$\begin{aligned} u_m(\mathbf{p}, \beta) &= \sum_j \mathsf{E}\left[f(h_{m,j}) \beta_{m,j} \right] \\ &= \sum_j \sum_{\mathbf{a}} \mathsf{E}\left[f(h_{m,j}) \beta_{m,j} | \mathbf{a} \right] \mathsf{Pr}\left(\mathbf{a} \right). \end{aligned}$$

Note that this utility is a function of $\mathbf{p}$. The MS attempt RTS independent of one another and hence the joint probability of attempting is $\mathsf{Pr}\left(\mathbf{a}\right) = \Pi_m \mathsf{Pr}\left(a_m\right)$. Hence,

$$u_m(\mathbf{p}, \beta) = \sum_j \sum_{\mathbf{a}} \mathsf{E}\left[f(h_{m,j}) \beta_{m,j} | \mathbf{a} \right] \prod_i \mathsf{Pr}\left(a_i \right)$$

$$= p_m \sum_j \sum_{\mathbf{a}_{-m}} \mathsf{E}\left[f(h_{m,j}) \beta_{m,j} | \mathbf{a}_{-m}, a_m = 1 \right] \prod_{i \neq m} \mathsf{Pr}\left(a_i \right). \quad (10)$$

The BS derives the scheduling decision in each frame by maximizing the utility according to (3). Thus following [10], we can define a natural utility function for the BS to be

$$u_{BS}(\mathbf{p}, \beta) = \sum_m \Gamma_{\alpha}(u_m(\mathbf{p})). \quad (11)$$

We call u_{BS} as the *network utility.*

The utility of MS m can be split as $u_m = u^r_m + u^b_m$ where

$$\begin{aligned} u^r_m &:= p_m \Pi_{i \neq m}(1 - p_i) \sum_j \mathsf{E}\left[f(h_{m,j}) \right] \\ u^b_m &:= p_m \sum_{j, \mathbf{a}_{-m} \neq (0, \cdots, 0)} \mathsf{E}\left[f(h_{m,j}) \beta_{m,j} | \mathbf{a}_{-m}, a_m = 1 \right] \mathsf{Pr}\left(a_{-m} \right). \end{aligned}$$

In the above, u^r_m is the utility obtained by MS m, when no other MS attempts RTS. In such a situation, there is only one MS contending and hence the MS is always scheduled. Thus u^r_m is the utility which cannot be changed or controlled by the scheduler and we call this the *private utility* of MS m. u^b_m is a function of the parameter of the scheduler and we call this the *public utility* of mobile m. Note that private utility can also influence total utility of other mobiles, e.g when an MS attempts RTS more aggressively than the prescribed rate.

A. The Game

A game theoretic setting is clear from the preceding discussion—the mobiles can choose p_m to maximize individual utility while the BS can choose a scheduling scheme to desirably trade-off the network utility and the achieved fairness.

Consider a system where the mobiles are assigned reservation attempt rates, say p^{ref}_m for MS m; denote $\mathbf{p}^{\text{ref}} := \left[p^{\text{ref}}_m \right]$. Clearly, $\mathbf{p}^{\text{ref}}$ determines the long term average utility that each of MS can obtain. The data channel is only granted to an MS with a successful RTS. Thus the MS have an incentive to attempt at a rate $p_m > p^{\text{ref}}_m$ to improve their utility. We thus have the following non cooperative game that arises naturally.

$$\begin{array}{lcl} \text{Players} & = & \{1, \quad 2, \quad \cdots, \quad M, \quad BS\}, \\ \text{Utilities} & = & \{u_1, \quad u_2, \quad \cdots, \quad u_M, \quad u_{BS}\}, \\ \text{Actions} & = & \{p_1, \quad p_2, \quad \cdots, \quad p_M, \quad \beta\}. \end{array} \quad (12)$$

Observe that (10) suggests that u_m increases linearly with the reservation rate p_m. Indeed this is true for $\alpha = 0$ i.e., for the efficient scheduler of (4) u_m increases linearly with p_m. One can use continuity arguments to show that this behavior is true even for small values of $\alpha > 0$. Using an example with finite channel states, we will now show that, a mobile can cheat even at high values of α.

B. An Example

We illustrate the preceding discussions, with the help of a simple example. Assume $D = 1$, $M = 3$. Two mobiles are far from the BS and have similar random variations in their channel states. Both can communicate with the BS at one of two rates 5 and 3 units with probability 0.2 and 0.8 respectively. A third MS is close to the BS and can communicate with the BS at one of two rates 12 and 10 units with probability 0.8 and 0.2 respectively. Assume that $p^{\text{ref}}_i = p$ for all i.

When an efficient scheduler (4) is used at BS, MS 3 obtains a higher utility than the other two. However as α increases, higher priority is given to fairness. Maximum fairness is obtained when MS 3 is scheduled only when there are no other MS with successful RTS. If such a fair scheduler exists, then the three mobiles would obtain the following utilities:

$$
\begin{aligned}
u_3 &= p(1-p)^2(0.8*12+0.2*10) = 11.6p(1-p)^2, \\
u_1 = u_2 &= (p(1-p)^2 + p^2(1-p))(0.8*3+0.2*5) \\
&+p^2(1-p)(0.8*0.2*5+0.2*0.2*\frac{5}{2}+0.8*0.8*\frac{3}{2}) \\
&= 3.4p(1-p)^2 + 5.86p^2(1-p).
\end{aligned}
$$

Using part 1 of Theorem 1 (equation (6)), such a scheduling is implemented by an alpha-fair scheduler maximizing (3) for all those values of α which satisfy $12/u_3^\alpha < 1/u_1^\alpha$, i.e., when $12^{1/\alpha}\left(3.4p(1-p)^2 + 5.86p^2(1-p)\right) < 11.6p(1-p)^2$. The above inequality is satisfied for all $\alpha > 2.63$ when the common RTS reservation rate $p = .45$. Fig. 2 plots the utility for the three mobiles as a function of α for different combinations of $\mathbf{p}$. We see that MS 3 indeed has a nearly constant utility, close to minimum, for all $\alpha > 2.6$. Also observe that the utility of the MS 3 when it increases it's $p = 0.75$. Clearly, MS 3 has a higher utility when it uses a higher p. The remaining two have reduced, unfair utilities.

We now make the following observations.

- Utility of MS 3 is highest when $\alpha = 0$ and decreases monotonically as $\alpha \to \infty$; see Fig. 2.
- For $\alpha > 2.6$, u_3 does not change with α; this is the best that the scheduler can do towards fairness. For $\alpha > 2.6$, MS 1 or 2 are scheduled whenever their RTS succeeds.
- Private utility of MS 1, 2 is $3.4p(1-p)^2$ and is $11.6p(1-p)^2$ for MS 3. This cannot be changed by a scheduler.
- MS 3 can increase its utility even at high alpha through its private utility by using $p_3 > p$. Also, the cooperating MS receive lower utilities than their fair share. And this is true even for higher α.

We now focus on constructing robust scheduling policies that use additional information to penalize aggressive mobiles. Robustness is shown by arguing that the scheduling policy and $\mathbf{p}^{\text{ref}}$ form a Nash equilibrium of an equivalent game.

IV. A Robust Scheduling Policy

A. Construction of the Robust Policy

The BS can observe $\mathbf{b}_t := \{b_{m,t}\}$, the time sequence of the RTS success flags and estimate their time average. Since time average equals the ensemble average, the BS can estimate the reservation rates $\{\hat{p}_m\}$ using $\hat{p}_m = \hat{b}_m/c_m$ where $\hat{b}_m$ is the time average of the rate of successful RTS from MS m and c_m is obtained below. It can then estimate the extent of any non-cooperation by observing and penalize when MS m when $(\hat{p}_m - p_m^{\text{ref}}) > 0$. This should force the MS to be cooperative.

For the case of saturated mobiles and aggregated reservations, we have $b_m = a_m \mathcal{X}_{\{\sum_{i\neq m} a_i < R\}}$. By independence, $\mathsf{Pr}\,(b_m = 1) = p_m c_m$ where $c_m = \mathsf{Pr}\left(\sum_{i\neq m} a_i < R\right)$. Similarly for R-channelized reservation (for the special case with $p_m^{ref} = p$ for all m):

$$
c_m = \frac{1-(1-p(1-p)^{N-1})^R}{p}.
$$

With the non saturated packet model, the buffer is occasionally empty. However the factorization considered above is still possible. For example, in case of the aggregated reservation channel [1]

$$
c_m := \mathsf{Pr}\left(\sum_{i\neq m} \mathcal{X}_{\{q_i>0\}} a_i < R \,\middle|\, a_m = 1\right). \tag{13}
$$

We would like to conclude that for most cases the factorization $\mathsf{Pr}\,(b_m = 1) = p_m c_m$ is true. The constant c_m however depends on the model and specifically for the non-saturated packet model computing $\mathbf{c} := \{c_m\}$ is difficult. Once c_m is known, one can estimate p_m. The estimates, henceforth, are represented without ˆ to avoid messy equations.

Robust Scheduler: The robust modification of the iterative scheduling algorithms (7) is:

$$
p_{m,t+1} = p_{m,t} + \mu_t\left(b_{m,t}/c_m - p_{m,t}\right) \tag{14}
$$

$$
u_{m,j,t+1} = u_{m,j,t} + \mu_t\left(\frac{f(h_{m,j,t})}{1+\rho_{m,t}}\beta_{m,j,t} - u_{m,j,t}\right) \tag{15}
$$

$$
\begin{aligned}
\omega_{m,t} &= \rho_{m,t} + u_{m,t}, \quad \rho_{m,t} = \Delta\left(p_{m,t} - p_m^{\text{ref}}\right)^+ \\
\beta_{m,j,t} &= \cap_{i\neq m}\mathcal{X}_{\left\{\frac{f(\hat{h}_{m,j,t})}{(1+\rho_{m,t})(\omega_{m,t})^\alpha} \geq \frac{f(\hat{h}_{i,j,t})}{(1+\rho_{m,t})(\omega_{i,t})^\alpha}\right\}}.
\end{aligned} \tag{16}
$$

In the above, the BS estimates the actual reservation rates used by the MS iteratively using (14). It then identifies the selfish MS as the ones with $p_{m,t} > p_m^{\text{ref}}$. It makes robust scheduling decisions $\{\beta_{m,j,t}\}$ by weighing down u_m for the selfish MS using a larger punitive term $(1+\rho_{m,t})(\omega_{m,t})^\alpha$ instead of the original $(u_{m,t})^\alpha$ as in (8). It further reduces the instantaneous utilities $f(\hat{h}_{m,j,t})$ by an extra factor $1+\rho$, to ensure that the private utilities are also punished (whenever the MS is non-cooperative).

B. Analysis

We analyze the proposed robust scheduler, using ODE analysis for the case with saturated packet arrivals. Let $\Theta := [\{p_m\}_m, \{u_{m,j}\}_{m,j}]$ represent a vector of all the components related to the robust algorithm. We will show that the trajectory of the robust scheduler (14)-(16) can be approximated by the

[1] The constant c_m corresponding to the non-saturated case does not have a closed form expression. However it can be estimated for any given reservation rate vector $\mathbf{p}^{\text{ref}}$ using numerical methods (e.g., monte-carlo simulations) once the channel statistics and the arrival rates of all the users are known. For some specific examples one might also get some good approximations. For e.g., in non saturated case with load factor close to 1, one can approximate the non-saturated c_m with the one corresponding to the saturated model.

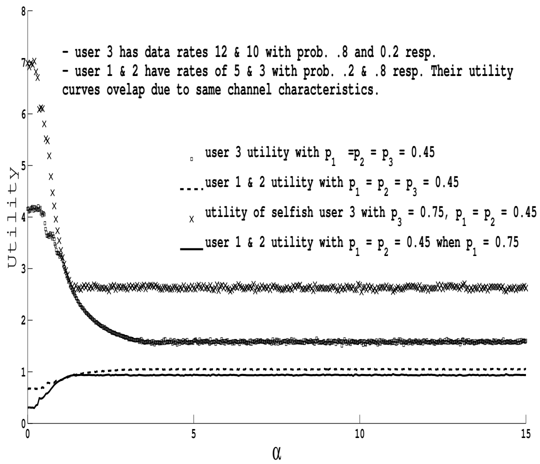

Fig. 2. Discrete channel states: A non-cooperative MS gains at the cost of cooperative ones

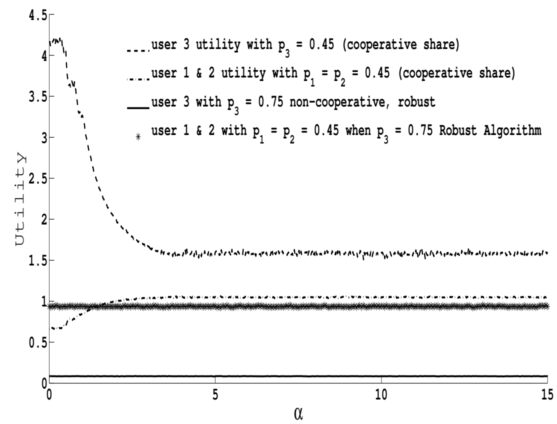

Fig. 3. Discrete channel states: Robust scheduler punishes the non-cooperative MS and aids the cooperative ones

solution $\Theta(t)$ of the following ODE for all $m \leq M$ and j:

$$\begin{aligned} \dot{p}_m(t) &= \frac{\mathsf{Pr}\,(b_m = 1)}{c_m} - p_m(t) && (17) \\ \dot{u}_{m,j}(t) &= E\left[\frac{f(h_{m,j})}{1+\rho_m(t)}\beta_{m,j}(t)\right] - u_{m,j}(t) && (18) \\ \omega_m(t) &:= u_m(t) + \rho_m(t), \quad \rho_m(t) = \Delta(p_m(t) - p_m^{\text{ref}})^+ \\ \beta_{m,j}(t) &= \Pi_{i \neq m}\mathcal{X}_{\left\{\frac{f(\hat{h}_{m,j})}{(1+\rho_{m,t})(\omega_m(t))^\alpha} \geq \frac{f(\hat{h}_{i,j})}{(1+\rho_{m,t})(\omega_i(t))^\alpha}\right\}}. \end{aligned}$$

We consider a slight modification of the non-cooperative game (12) to show the robustness of the proposed algorithm. We replace the utilities defined in the game (12) by the asymptotic time limits of the robust algorithm (14)-(16) and show that the robust algorithm and the assigned reservation rate vector form a Nash equilibrium.

ODE Approximation: We begin by discussing the existence of solutions of the above ODEs. The ODE (17) and (18) have a unique solution as will be established below:

***Lemma** 1:* The ODE (18) has unique bounded solution for any initial condition in $\mathcal{A}$ with $\mathcal{A} := [0,1]^M \times \mathcal{R}_+^{MD}$ and it is bounded as below where $\eta := \sup_{m,j} E[|f(h_{m,j})|]$.

$$|u_{m,j}(t)| \leq \eta - (\eta - |u_{m,j,0}|)e^{-t} \text{ for all } t.$$

Proof: One can obtain the following upper bound easily (with $< .,. >$ representing the inner product):

$$\left\langle E\left[\frac{f(h_{m,j})}{1+\rho_m}\beta_{m,j}\right] - u_{m,j}(t), u_{m,j}\right\rangle \leq Bu_{m,j} - u_{m,j}^2.$$

Using this and using the comparison methods of [14, pp. 169-170] the proof is obtained as is done in Lemma 1 in [9]. □

We now define the limit set of the ODEs (17)-(18) and its δ-neighborhood:

$$\mathbb{L}^{ODE} := \lim_{t\to\infty} \cup_{\Theta\in\mathcal{A}}\{\Theta(s) : s \geq t \text{ and } \Theta(0) = \Theta\}.$$

$$\mathbb{B}_\delta(\mathbb{L}^{ODE}) := \left\{\Theta : |\Theta - \tilde{\Theta}| \leq \delta \text{ for some } \tilde{\Theta} \in \mathbb{L}^{ODE}\right\}.$$

The following theorem now establishes that the trajectory $\{\Theta_t; t\}$ with $\Theta_t := \{\{p_{m,t}\}_m, \ \{u_{m,j,t}\}_{m,j}\}$, ultimately spends time in the limit set defined above (Proof in Appendix).

***Theorem** 2:* Assume the following:

- There exists a sequence $\epsilon_k \to \infty$ with $\lim_k \sup_{0\leq l\leq\epsilon_k} \mu_{k+l}/\mu_k = 0$.
- The channel state $\{\mathbf{h}_k\}$ is an independent and identically distributed (IID) sequence with finite mean and variance.
- The average rates are bounded by the same constant, i.e., $|f(h_{m,j})| \leq B$ for all m, j.
- $\{\mathbf{h}_k\}$ has continuous and bounded density.

Then for every $\delta > 0$, the fraction of time the tail of the algorithm $\{\Theta_\tau\}_{\tau\geq t}$ for any initial condition $\Theta_0 \in \mathcal{A}$ spends in the δ-neighborhood of the limit set $\mathbb{B}_\delta(\mathbb{L}^{ODE})$ tends to one as $t \to \infty$. □

Further analysis is obtained by studying the limit set of the above ODEs. The first ODE (17) has a unique solution and a unique attractor:

$$p_m(t) = p_m^* - (p_m^* - p_{m,0})\,e^{-t}, \qquad p_m^* = \frac{\mathsf{E}\,[b_m]}{c_m}. \quad (19)$$

That means, the BS via iteration (14) estimates the RTS reservation rates $[p_m]$ used by all the MS's. And as we will see below for any MS m, it uses only the excess (w.r.t. the assigned reservation rate) given by $(p_m^* - p_m^{ref})^+$ to punish it. When one MS becomes non cooperative, the c_m of the other MS's actually should decrease, however the BS uses the larger one corresponding to all cooperative case. As seen from equation

(19) this results in a reduced estimate for the reservation rates of the other MS's. Since one uses only the positive part of the excess this would not alter the analysis.

We now study the limit set of the second ODE (18). In particular we study its equilibrium points. Any equilibrium point of the ODE (18) satisfies the following fixed point equation:

$$u_m^* = \sum_j E\left[\frac{f(h_{m,j})}{1+\rho_m^*}\beta_{m,j}\right], \tag{20}$$

$$\omega_m := u_m^* + \rho_m^*, \quad \rho_m^* = \Delta(p_m^* - p_m^{\text{ref}})^+,$$
$$\beta_{m,j} = \cap_{i\neq m}\mathcal{X}_{\left\{\frac{f(\hat{h}_{m,j})}{(1+\rho_m^*)(\omega_m)^\alpha} \geq \frac{f(\hat{h}_{i,j})}{(1+\rho_i^*)(\omega_i)^\alpha}\right\}}. \tag{21}$$

Existence of a fixed point for the above equation can be established as in Theorem 1. We further have the following:

Lemma 2: The equation (21) has a unique fixed point i.e., the ODE (18) has an unique equilibrium point.

Proof: The proof is in Appendix. □

One still needs to show that the above unique equilibrium is indeed a limit point. This is given by [8, Theorem 2.2] for the case with $\alpha \leq 1$ and without aloha reservation. The proof of [8, Theorem 2.2] can easily be imitated for the cooperative case, i.e., when $p_{m,0} = p_m^{\text{ref}}$ for all m. Note that, with $p_{m,0} = p_m^{\text{ref}}$ for all m, from (19) that $p_m(t) = p_m^* = p_m^{\text{ref}}$ for all t and m and hence one can treat ω, ρ like constants and then [8, Theorem 2.2] can be extended. One need to show that the unique equilibrium point of Lemma 2 is a limit point for general case and this is work in progress.

Nash Equilibrium: By Theorem 2, the robust algorithm converges weakly to the limit set of the ODEs (17)–(18) as discussed earlier. The analysis of the robust algorithm (14)–(16) can be obtained by further studying these equilibrium points and it is easy to verify the following result:

Lemma 3: At any reservation rate vector $\mathbf{p} = [p_1, \cdots, p_M]$, the unique equilibrium point is upper bounded for all m by:

$$u_m^* \leq \frac{\sum_j E[f(h_{m,j})}{1+\Delta(p_m^* - p_m^{\text{ref}})^+}.$$

Proof: The proof is immediate from the fixed point equation (20). □

So, one can chose Δ large enough such that

$$u_m^* < u_m^{\text{ref}} \text{ for all } m \text{ with } p_m^* > p_m^{\text{ref}},$$

where u_m^{ref} is the limit when all the MS are cooperative, i.e., the fixed point of Lemma 2 with $\mathbf{p}^{\text{ref}}$. With this Δ, the algorithm (14)-(16) converges weakly (for any MS m with w representing weak convergence) to:

$$\lim_{t\to\infty} u_{m,t} \stackrel{w}{=} u_m^* < u_m^{\text{ref}} \text{ when } p_m^* > p_m^{\text{ref}}.$$

Thus any MS (say MS m) obtains a smaller u_m when it deviates unilaterally from its designated reservation rate $p_m^* > p_m^{\text{ref}}$. In other words, any MS obtains a smaller utility if it attempts more aggressively. Hence the reservation rates $\mathbf{p}^{\text{ref}}$ and the robust policy β given by (14)-(16) form a Nash Equilibrium for a game with utilities defined by the weak time limits:

Utilities

$$= \left\{\lim_{t\to\infty} u_{1,t}, \cdots, \lim_{t\to\infty} u_{M,t}, \sum_m \Gamma_\alpha\left(\lim_{t\to\infty} u_{m,t}\right)\right\}.$$

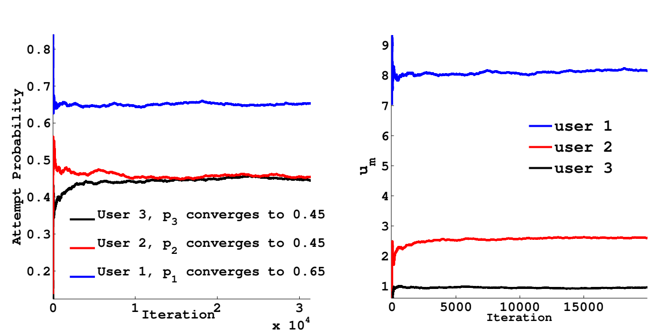

Fig. 4. Estimated transmission probabilities p_m and estimated utility u_m

V. Numerical Examples

We consider few numerical examples to illustrate main characteristics of different scheduling algorithms, considered in this work.

Example 1 We begin with the discrete channel example of Section III-B when the common assigned reservation rate equals 0.45. We consider the case when MS 3, is selfish and uses an increased RTS reservation rate $p_3 = 0.75$. We plot the asymptotic utilities $\lim_k u_{m,k}$ as a function of α. These limits are plotted for the non robust (equation (7)) as well as the robust scheduler (equation (16)) respectively in figures 2 and 3. Both the figures also show the limits corresponding to the cooperative case, referred to as cooperative shares.

While the utility of MS 3 increases with increased reservation rate, it however reduces the utility of the other two MS's resulting in unfair allocations (see Fig. 2). When the scheduler is replaced with the proposed robust algorithm, MS 3 is penalized while the utilities of other MS's are improved. Note that the utility curves for MS 1 and 2 overlap (due to identical channel statistics) while MS 3 has almost 0 utility in Fig. 3 with robust scheduler.

Example 2: We now consider continuous channel states. Fig. 4 shows the convergence of the scheduling algorithm for the case with $\alpha = 0$, $M = 3$ and with 2 data channels. The MS's have asymmetric channel states and the $h_{m,j}$'s are truncated Rayleigh distributed with parameter σ_m (as in [9], [10]). For this example $\sigma_1 = 60, \sigma_2 = 30$ and $\sigma_3 = 10$. The MS's are again assigned equal reservation rates, i.e., $p_m^{\text{ref}} = 0.45$ for all m and $\mu = 4 \times 10^{-5}$. MS 1 is a non-cooperative node and uses an increased reservation rate $p_1 = 0.65 > p_1^{ref}$.

The first plot in Fig. 4 shows the convergence of estimates $p_{m,k}$ for the 3 users; $p_{1,k}$ converges to 0.65 while $p_{2,k}$ and $p_{3,k}$ converge to 0.45. The second part of Fig. 4 shows the convergence of estimated utilities $u_{m,k}$. In summary, the trajectories converge to their corresponding limits, even when started from an unknown initial point different from the limit, reaffirming the stochastic approximation Theorem 2.

Example 3 : We continue with Example 2. As in Example 1, we compare the two scheduling algorithms (7) with (16)). The asymptotic limits ($\{\lim_k u_{m,k}\}_m$) are plotted for the non robust (blue curves) as well as the robust scheduler (black curves) in Fig. 5. Orange curves represent the cooperative shares. From Fig. 5 near $\alpha = 0$, the selfish behavior of MS 1 increases $\lim_k u_{1,k}$ corresponding to non robust scheduler (continuous blue curve), which is at the cost of the same for other users (other two blue curves). There is a significant improvement in MS 1 utility for all the values of α. Such a behavior is not seen in the corresponding black curves, wherein the robust algorithm actually penalizes the selfish user. Note actually that the utility of MS 1 is almost equal to 0. Further, it improves the utility of the cooperative users.

In all the above examples, we observe that the robust algorithm not only punishes non-cooperative MS but also improves the utilities of the cooperative MS (see Fig. 3 and 5). Thus it exhibits anti jamming property, whenever the noncooperation is within manageable limits.

Non saturated case

We now switch to a non saturated case in Fig. 6. In this case data arrives at random instants of time, in contrast to the saturated case, wherein the mobiles always have data to transmit. We again consider the previous example, but for the packets to be transmitted. We assume that the packets arrive according to a Poisson process (with rates λ_1, λ_2 and λ_3) and each packet demands transfer of S units of information. When allocated in a slot and in a subchannel, at maximum $f(h_{m,j})$ units of information is transmitted. A mobile attempts RTS only if it has data to be transmitted. When it has data, it attempts RTS as in the saturated case, with a reservation rate. As anticipated, we notice from Fig. 6, that the cooperative shares of this system are close to that of the saturated case. Note that the vector of cooperative shares of all the mobiles in saturated case, can represent the maximum load factor each mobile can accept. For example at $\alpha = 0$, the saturated cooperative share of $MS1$ is approximately 5.5 which further reduces as $\alpha \to \infty$. However it demands $\lambda_1 * S = 1.65 * 4 = 6.6$ and so its demand is never satisfied. In other words, MS 1 always operates in saturated condition[2] and one can read this even from the figure (both the cooperative shares of MS 1 are almost equal). Similar is the case with MS 2 which again demands 6.6 units of data transfer per time slot. However, the case with MS 3 is different. With $\alpha = 0$, the saturated cooperative share is close to 1.5 and increases to a value slightly bigger than 3 as $\alpha \to \infty$ (dashed blue curve with rectangles). MS 3 demands $.725*4 = 2.9$ units of data transfer per time slot, which is not satisfied for small values of α, i.e., MS 3 also remains in saturated condition (to be more precise in time asymptotic saturated condition). But as $\alpha \to \infty$, the system can support the demand of $MS3$ and it operates in non saturated condition, for $\alpha > 6$.

The non saturated system or the system with packet arrivals at random instances of time performs close to that of a saturated or the system which always has data to transmit. This is true when the demands/load factors are close to the saturated cooperative shares (as these represent the average units of information that can be transmitted in one time slot). Because of this, we expect that this kind of a non saturated system behaves in a similar way in presence of noncooperative users. That is, it would fail if the existing schedulers are used when an MS attempts more rigorously while it would be robust when the proposed robust scheduler is used. We observe that this indeed is the case in Fig. 7. When MS 1 becomes noncooperative and attempts RTS at aggressive rate 0.75 and when the BS uses the existing α-fair scheduler (7) we notice that MS 1 is able to gain while the other two loose resulting in unfairness. However the proposed robust scheduler (equation (16)) not only punishes MS 1, the utilities of the other two improve.

[2]For the case with demand load factor ($\lambda * S$) greater than the cooperative share in saturated condition, the buffer length increases with time and eventually the MS always has data to transmit with probability close to 1.

VI. Discussion

We have considered and analyzed a combined system with collision based reservation requests and a scheduler that trades-off the system efficiency with desired level of fairness. That the system is prone to noncooperation is established. We have proposed a scheduling mechanism in which users violating a prescribed reservation rates are penalized. However we have not addressed the choice of reservation rates. Two possible criteria to determine this choice are immediate – (a) choosing a reservation rate vector that optimizes a network utility and (b) a reservation rate vector to simultaneously achieve the assigned utility for each MS. This is part of the future work. While the initial numerical analysis of the non-saturated packet arrivals is established, an extensive analysis is being carried out.

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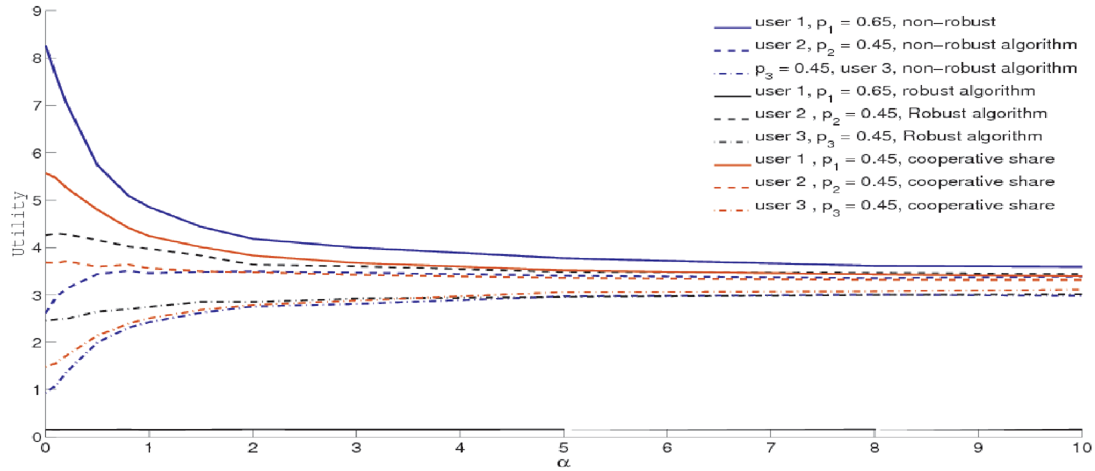

Fig. 5. Continuous channel states: comparison of non robust and robust schedulers

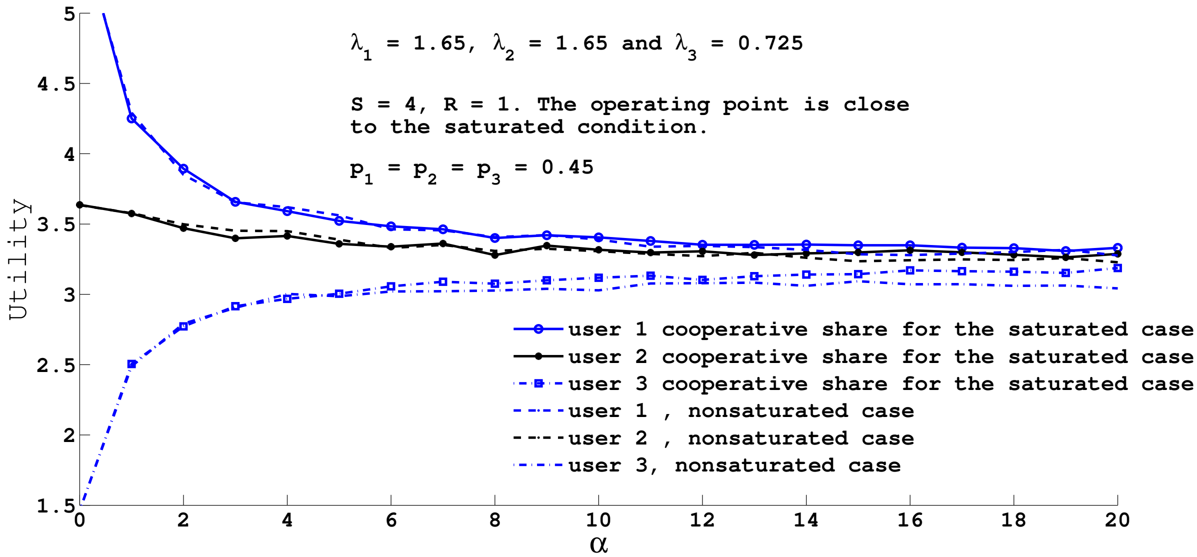

Fig. 6. Non Saturated case: Cooperative shares are close to that of the saturated case at high load factor.

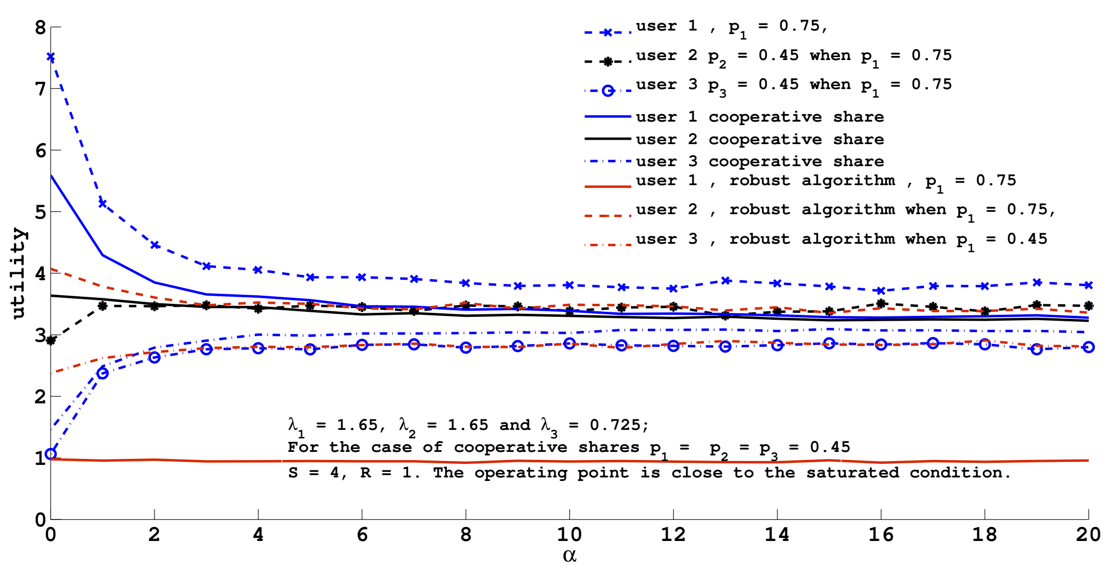

Fig. 7. Non Saturated case: Existing schedulers are non robust while our proposed policy is robust.

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Appendix

Proof of Theorem 1: We first prove the second part. We fix the reservation rates $\mathbf{p}$ and neglect them in the rest of the proof. Consider the function defined component wise by:

$$\Theta_m(\mathbf{u}) := \sum_j E\left[f(h_{m,j}) \Pi_{i\neq m} \mathcal{X}_{\left\{ \frac{f(\hat{h}_{m,j})}{u_m^\alpha} > \frac{f(\hat{h}_{i,j})}{u_i^\alpha} \right\}} \right] \quad (22)$$

Under the theorem hypothesis, Θ_m is continuous in $\mathbf{u}$ because:

a) for any sequence $\mathbf{u}^{(n)} \to \mathbf{u}$, the term inside the expectation:

$$\Pi_{i\neq m} \mathcal{X}_{\left\{ \frac{f(\hat{h}_{m,j})}{\left(u_m^{(n)}\right)^\alpha} > \frac{f(\hat{h}_{i,j})}{\left(u_i^{(n)}\right)^\alpha} \right\}} \to \Pi_{i\neq m} \mathcal{X}_{\left\{ \frac{f(\hat{h}_{m,j})}{(u_m)^\alpha} > \frac{f(\hat{h}_{i,j})}{(u_i)^\alpha} \right\}}$$

at all $\mathbf{h}$ which are the points of continuity and hence for almost all $\mathbf{h}$ and this true for all j.

b) It is easy to see that,

$$|\Theta_m(\mathbf{u})| \leq \sum_j E[f(h_{m,j})] < \infty \text{ for any } \mathbf{u}. \quad (23)$$

Thus the expected value $\Theta_m(\mathbf{u}^{(n)}) \to \Theta_m(\mathbf{u})$ by Dominated Convergence theorem.

c) Sequential continuity implies continuity in a finite dimension space, hence $\Theta := [\Theta_1, \cdots, \Theta_M]$ is continuous in $\mathbf{u}$.

From (23) Θ is bounded and hence by Brouwer's fixed point theorem, there exist a fixed point $\mathbf{u}^* = [u_1^*, \cdots, u_M^*]$ for the map Θ and hence for Λ proving the second part of the theorem.

Fair scheduler using the fixed point: If there exists a fixed point of Λ, then define β^α by (6) and define,

$$G^\alpha(\beta) = \sum_m \Gamma_\alpha(u_m(\beta)).$$

Since Γ_α is a concave function of u and hence for any β,

$$G^\alpha(\beta) - G^\alpha(\beta^\alpha) \leq \sum_m \left[u_m(\beta) - u_m(\beta^\alpha)\right] d\Gamma_\alpha(u_m(\beta^\alpha)). \quad (24)$$

Consider the following function:

$$\begin{aligned} \beta \quad &\mapsto \quad \sum_m u_m(\beta) d\Gamma_\alpha(u_m(\beta^\alpha)) \\ &= \quad \mathbb{E}_{\mathbf{h}}\left[\sum_m \sum_j f(\hat{h}_m, j) d\Gamma_\alpha(u_m(\beta^\alpha))\beta(m,j) \right]. \quad (25) \end{aligned}$$

From (6) for every m, j and channel state $\mathbf{h}$ and for any scheduler β with $\beta(m,j) \in [0,1]$ and $d\Gamma_\alpha(u) = 1/u^\alpha$,

$$f(\hat{h}_m, j) d\Gamma_\alpha(u_m(\beta^\alpha))\beta(m,j) \leq f(\hat{h}_m, j) d\Gamma_\alpha(u_m(\beta^\alpha))\beta^\alpha(m,j).$$

Hence β^α maximizes the function (25) and so

$$\sum_m \left[u_m(\beta) - u_m(\beta^\alpha)\right] d\Gamma_\alpha(u_m(\beta^\alpha)). \leq 0$$

Hence from (24) $G^\alpha(\beta) - G^\alpha(\beta^\alpha) \leq 0$. Hence β^α maximizes G^α among all the schedulers and hence is an alpha-fair scheduler. $\square$

For the ease of explanation the proof is provided using the above independence assumptions. But the stochastic approximation result can easily be extended to the stationary case as in [8].

Proof of Theorem 2: As a first step, we rewrite the algorithm (14)-(16) as in [13] :

$$\begin{aligned} Y_{m,k}^p \quad &:= \quad b_{m,k}/c_m - p_{m,k}, \quad p_{m,k+1} = p_{m,k} + \mu_k Y_{m,k}^p, \\ Y_{m,j,k}^u \quad &:= \quad \frac{f(h_{m,j,k})}{1+\rho_{m,k}} \beta_{m,j,k}^\Delta - u_{m,j,k} \\ u_{m,j,k+1} \quad &= \quad u_{m,j,k} + \mu_k Y_{m,j,k}^u. \end{aligned}$$

Define

$$\mathcal{F}_k := \sigma\left(\Theta_\tau, \{\{Y_{m,\tau-1}^p\}_m, Y_{m,j,\tau-1}^u\}_{m,j}, \text{ for all } \tau \leq k\right)$$

and let $\mathbb{E}_k$ represent the expectation w.r.t. $\mathcal{F}_k$, the filtration. Under the assumptions **A**.2 and **A**.3 clearly the condition expectation equals (for all m, j, k):

$$\begin{aligned} \mathbb{E}_k[Y_{m,k}^p] \quad &= \quad g_m^p(\Theta_k) := p_m^* - p_{m,k}, \\ \mathbb{E}_k[Y_{m,j,k}^u] \quad &= \quad g_{m,j}^u(\Theta_k) := \sum_j R_{m,j}(\Theta_k) - u_{m,j,k} \\ R_{m,j}(\Theta) \quad &= \quad E\left[\frac{f(h_{m,j})}{1+\rho_m} \right. \\ &\qquad \left. \cap_{i\neq m} \mathcal{X}\left\{ \frac{f(\hat{h}_{m,j})}{(1+\rho_m)(\omega_m)^\alpha} \geq \frac{f(\hat{h}_{i,j})}{(1+\rho_i)(\omega_i)^\alpha} \right\} \right]. \end{aligned}$$

By assumption **A**.3, and with $\mu_k \to 0$ monotonically,

$$u_{m,j,k} \leq \Pi_{l\leq k}(1-\mu_l)u_{m,j,0} + \left(\sum_{l\leq k}\Pi_{n\leq l}\mu_n\right)B$$
$$\leq \Pi_{l\leq k}(1-\mu_l)u_{m,j,0} + \sum_{l\leq k}\mu_0^l B < \infty \text{ for all } m,j,k.$$

Hence the entire trajectory resides inside a bounded set for any given initial condition. This gives tightness also. The first trajectory always lies in [0,1], i.e., $0 \leq p_{m,k} \leq 1$ for all m,k. Hence we have bounded trajectories and hence $\{Y_k^\Theta; k\}$, with $Y_k^\Theta := \{\{Y_{m,k}^p\}\{Y_{m,j,k}^u\}\}$, is uniformly integrable, satisfies assumption A.2.1, pp. 258 [13]. By Assumption **A**.4 $R_{m,j}$ are continuous in Θ as shown in the proof of the first part of Theorem 1 now by bounded convergence theorem. So, the assumptions A.2.2 to A.2.7 of pages 258, 259 [13] are satisfied with $g_k^\Theta = \bar{g}^\Theta = g^\Theta$ and $\beta_k = 0$ $\xi_k = 0$ for all time k.

Then by Theorem 2.3, pp. 259, [13] the trajectory of robust policy Θ_k converges weekly to the trajectory of the solution of the ODE (17)-(18) (in the sense as explained in [13]). Further by the same theorem of [13], for any $\delta > 0$, the fraction of time that the tail sequence $\{\Theta_\tau\}_{\tau\geq k}$, with initializations in $\mathcal{A}$ spends in the δ-neighborhood of the limit set of the ODEs (17)-(18), $\mathbb{B}_\delta(\mathbb{L}^{ODE})$, goes to one (in probability) as $k \to \infty$. ■

Proof of Lemma 2: We first consider the cooperative case, i.e., when $p_m^* = p_m^{ref}$ for all m. If say there exist two distinct fixed points $\bar{\mathbf{u}}^1 = \{\bar{u}_m^1\}$ and $\bar{\mathbf{u}}^2 = \{\bar{u}_m^2\}$ and without loss of generality let

$$\bar{u}_1^1 > \bar{u}_1^2. \tag{26}$$

From the fixed point equation it is clear that a component of a fixed point can increase only when some extra pairwise decisions are in its favor. The scheduler $\beta_{1,j}^\Delta$ is inversely proportional to $\bar{u}_1$ and any pair wise decision depends upon the ratio of two MS's utilities, as in the example given below:

$$\mathcal{X}\left\{f(h_{1,j}) > \left(\frac{\bar{u}_1^1}{\bar{u}_m^1}\right)^\alpha f(h_{m,j})\right\}.$$

Hence (26) is possible only if there exists at least one m, say without loss of generality $m = 2$ such that

$$\frac{\bar{u}_1^1}{\bar{u}_2^1} < \frac{\bar{u}_1^2}{\bar{u}_2^2} \tag{27}$$

because only then we can have more decisions in favor of user 1, with fixed point $\bar{\mathbf{u}}^1$. Inequality in (27) implies (note $\bar{u}_1^1 - \bar{u}_1^2 > 0$) $\bar{u}_2^1 > \bar{u}_2^2$, which in turn using similar logic implies there exists third MS, say $m = 3$ which satisfies:

$$\frac{\bar{u}_2^1}{\bar{u}_3^1} < \frac{\bar{u}_2^2}{\bar{u}_3^2}.$$

Note that this implies $\dfrac{\bar{u}_1^1}{\bar{u}_3^1} < \dfrac{\bar{u}_1^2}{\bar{u}_3^2}$.

Hence further because $u_3^1 > u_3^2$ we would need a fourth MS (different from the first and the second), say $m = 4$ such that:

$$\frac{\bar{u}_3^1}{\bar{u}_4^1} < \frac{\bar{u}_3^2}{\bar{u}_4^2},$$

and this repeats. However when $m = M$, there are no more MS's which can ensure MS M has higher utility with the first fixed point, (i.e., such that $\bar{u}_M^1 > \bar{u}_M^2$) and hence this leads to a contradiction. Thus there exists a unique fixed point. It is easy to see that the proof can be extended to any general $\mathbf{p}$. □