To Participate or Not in a Coalition in Adversarial Games

Ranbir Dhounchak, Veeraruna Kavitha and [!]Yezekael Hayel

IIT Bombay India and [!]University of Avignon France

Abstract. Cooperative game theory aims to study complex systems in which players have an interest to play together instead of selfishly in an interactive context. This interest may not always be true in an adversarial setting. We consider in this paper that several players have a choice to participate or not in a coalition in order to maximize their utility against an adversarial player. We observe that participating in a coalition is not always the best decision; indeed selfishness can lead to better individual utility. However this is true under rare yet interesting scenarios. This result is quite surprising as in standard cooperative games; coalitions are formed if and only if it is profitable for players. We illustrate our results with two important resource-sharing problems: resource allocation in communication networks and visibility maximization in online social networks. We also discuss fair sharing using Shapley values, when cooperation is beneficial.

Keywords: Cooperative Game theory, Shapely value, Social Networks

1 Introduction

Resource allocation problem is a well-known generic problem that involves complex optimal solutions. This type of problem is well-known in networking context. One of the most studied models is the proportional framework [13]. Several users share a common resource and each one gets a part of it proportionally to his/her action. In fact, this mechanism induced a utility function which is linear with respect to user's action. This model has lead to the well-known proportional fair solution concept which has been applied with success in numerous resource allocation problems in networking and in security [9]. In another communication domain, social networking, such linear utility function is useful to model user preferences [15]. Indeed, such type of function has been proved to efficiently model mean number of messages on a timeline that belong to a particular source. This mean number is proportional to the ratio of the sending rate of that message over the total rate of all messages. When one is interested in relative visibility, which is defined as the ratio of expected copies of a message currently alive in a social network to the total number of expected copies of all such competing messages, it results in an exponential cost (e.g., [7]). Such message propagation processes are modeled predominantly using branching processes. We consider linear as well as exponential cost functions.

A problem closely related to our first linear model problem is considered in [8]. However, the utility of the adversary is different from that considered in [8]:

in our case the adversary can be optimizing its own utility like any other player, but this will have adversarial influence on the rest. Assuming a mechanism to identify the adversary by the rest of the other players, this type of game is well adapted to formulate a security game with several defenders as in [10]. Further, this player is not interested in participating in cooperation with any other player. The main theme of this paper is to study the possible improvement obtained by the rest of the players, in presence of such an 'adversarial' player. In regular coalition formation games, like the ones of [11], there is no adversarial context. In a wireless communication context, a coalition formation game against multiple attackers has been studied in [12] and the authors have shown an increase of the average secrecy rate per user up to 25%. We also consider coalition game against (one) adversary, our study is theoretical and also deals with the fairness of the Shapley value mechanism.

Two main features are considered in our study which build our contributions to Resource sharing games:

- *Adversarial context*: We consider that players are in an adversarial environment. Particularly, one specific player's objective is to minimize the total utility of all the other players, the social welfare of other players.
- *Cooperation is possible*: To improve the utility at equilibrium, the players can participate into a coalition and merge their efforts to defeat the adversary.

When, players decide to form a coalition, it leads to a non-cooperative game between a coalition (group of players) and one adversary. We show that, to form a coalition is not always the best choice in an adversarial resource sharing games. But this is the case only for some rare scenarios. Another interesting result is, when coalition is beneficial for players, all but one of them should remain silent (their actions at equilibrium are to be inactive/zeros). It is this silencing which not only helps the coalition, but also the opponents (or non-participants), and eventually leads to situations where grand coalition may not be the best. In such cases, the Shapley values fail to divide the gains fairly. All these results are deeply investigated in two resource sharing games: a linear model which describe Kelly's fairness mechanism and visibility competition; and an exponential model that describes the visibility via propagation of messages in social networks.

The paper is organized as follows. In section 2 we study an adversarial resource sharing game in which individual's utility are linear with its action. We first describe the fully non-cooperative scenario in which all individuals play selfishly. Second, we consider the cooperative setting in which players form coalitions in order to enhance, if possible, their individual utilities. Section 5 is devoted to another scenario inspired by social networks. Players aim is to maximize the visibility of their contents. This visibility is measured in terms of expected number of time-lines reached through the process of re-posting or forwarding. This utility function is exponential in the player's preference parameters.

2 Linear model

Consider a system with many players competing against each other, for shared resources. In such scenarios, it might be beneficial for the players to cooperate with each other. Further, assume there exists an adversary whose aim is to harm the rest of the players. Alternatively the adversary might be aiming for its own benefit, but its actions adversely influence the utilities of the rest of the players. Possibly the adversary is the player that is not interested in participating in any sort of cooperation. We refer the rest of the players, willing to explore cooperation benefits, as C-players. We study the gain of the C-players, with and without participating into cooperation, in an interaction with an adversary. We deal with two possible scenarios: a) Non-cooperative scenario (NCS): This is a complete non-cooperative game between all the players (C-players and the adversary); b) Cooperative scenario (CS): C-players participate into a common coalition and we are faced with a two-player non cooperative game, one of them being the aggregate player formed by the C-players and second player is the adversary.

We also derive a transferable utility (TU) game, that defines worth for each sub-coalition of C-players (in the presence of adversary), to share fairly the benefits derived by cooperation (when beneficial) using well known cooperative solution concepts, e.g., Shapley value.

2.1 Fully Non-Cooperative scenario

The resource sharing game involves C-players and the adversary in a standard non-cooperative game. Let $\mathbf{n}$ (with $\mathbf{n} \geq 2$) represent the number of C-players. The utility of C-players is given by (with $\{\lambda_i\}$ the influence factors, γ the cost factor):

$$U_i(\mathbf{a}) = \frac{\lambda_i a_i}{\sum_{j=0}^{\mathbf{n}} \lambda_j a_j} - \gamma a_i, \mathbf{a} := (a_0, a_1, \cdots, a_{\mathbf{n}}) \text{ for any } 1 \leq i \leq \mathbf{n}, \quad (1)$$

where the action of player i, $a_i \in [0, \bar{a}]$, for some $\bar{a} < \infty$. The utility function of the adversarial player, denoted by U_0, is given by (with $a_0 \in [0, \bar{a}]$):

$$U_0(\mathbf{a}) = -\frac{\sum_{i=1}^{\mathbf{n}} \lambda_i a_i}{\sum_{j=0}^{\mathbf{n}} \lambda_j a_j} - \gamma a_0, \quad \text{or equivalently,} \quad U_0(\mathbf{a}) \ = \ \frac{\lambda_0 a_0}{\sum_{j=0}^{\mathbf{n}} \lambda_j a_j} - \gamma a_0.$$

Let $\mathbf{U} := (U_0, U_1, \cdots, U_{\mathbf{n}})$, represent the utility functions of all players. Thus we have an $(\mathbf{n}+1)$-player non-cooperative strategic form game given by tuple $\langle \{0, 1, \cdots, \mathbf{n}\}, [0, \bar{a}]^{\mathbf{n}+1}, \mathbf{U} \rangle$. We analyze this game using the well known solution concept, the Nash Equilibrium (NE). This game setting is an adversarial extension of the well-known Kelly's problem about optimal resource allocation, particularly studied in communication networks [13]. This utility function represents a compromise between proportional share of a global resource for each user and a cost which depends on the action taken.

We first derive the NE when it lies in the interior of the strategy space, for each player. *Throughout we consider the actions in a bounded domain with* $\bar{a} > \mathbf{n}/\gamma$. A generalization could be of future interest.

Lemma 1. (Positive NE) *Define* $\mathbf{s} := \sum_{j=0}^{\mathbf{n}} 1/\lambda_j$. *Assume* $\mathbf{s} > \mathbf{n}/\lambda_j$ *for all* j. *Then there exists unique NE* $\mathbf{a}^* = (a_0^*, \cdots, a_{\mathbf{n}}^*)$ *which lies in the interior, i.e.,* $\mathbf{a}^* \in (0, \bar{a})^{\mathbf{n}+1}$. *The NE and the corresponding utilities are given by (for any* j*):*

$$a_j^* = \frac{\mathbf{n}\left(\mathbf{s} - \frac{\mathbf{n}}{\lambda_j}\right)}{\gamma \lambda_j \mathbf{s}^2} \textit{ and } U_j^* = \left(\frac{\mathbf{s} - \frac{\mathbf{n}}{\lambda_j}}{\mathbf{s}}\right)^2. \tag{2}$$

Proof. The proof is given in Appendix A. □

We now consider the situation for which the conditions of the above Lemma are not satisfied, i.e., when $\mathbf{s} \leq \mathbf{n}/\lambda_j$ for some j. In this case one can expect that at least one of the players would be silent in the NE, i.e., we meant $a_j^* = 0$ for at least one j. Towards investigating this, we first claim that there exists a unique subset of players $\mathcal{J} \subset \{0, 1, \cdots, \mathbf{n}\}$ with $s_{\mathcal{J}} := \sum_{i \in \mathcal{J}} 1/\lambda_i$ such that

$$\mathbf{s}_{\mathcal{J}} > \frac{|\mathcal{J}| - 1}{\lambda_j} \text{ for all } j \in \mathcal{J} \text{ and } \mathbf{s}_{\mathcal{J}} \leq \frac{|\mathcal{J}| - 1}{\lambda_j} \text{ for all } j \in \mathcal{J}^c. \tag{3}$$

When the above happens, one can expect a NE with non-zero components for only players in $\mathcal{J}$ (as in Lemma 1). We will show that such a $\mathcal{J}$ indeed exists, and then show that the game has unique NE with the said zero/non-zero components. Towards this, *consider the permutation π on the set of players $\{0, 1, \cdots, \mathbf{n}\}$ such that $\lambda_{\pi(0)} \geq \lambda_{\pi(1)} \geq \cdots \geq \lambda_{\pi(\mathbf{n})}$. Let $\lambda'_i := \lambda_{\pi(i)}$ and $\lambda'_{\mathbf{n}+1} =$ any value.*

Theorem 1. *i) There exists a unique $1 \leq k^* \leq \mathbf{n}$ such that ($\mathbf{1}$ is indicator):*

$$\mathbf{1}_{\{k^* < \mathbf{n}\}} \left(\frac{k^* + 1}{\lambda'_{k^*+1}} - \sum_{j=1}^{k^*+1} \frac{1}{\lambda'_j} \right)^{-1} \leq \lambda'_0 < \left(\frac{k^*}{\lambda'_{k^*}} - \sum_{j=1}^{k^*} \frac{1}{\lambda'_j} \right)^{-1}.$$

The unique set satisfying (3) is given by, $\mathcal{J}^* := \{\pi(0), \pi(1), \cdots, \pi(k^*)\}$, and further, $\mathcal{J}^* \subset \{j : \mathbf{s} > \mathbf{n}/\lambda_j\}$.
ii) There exists unique NE, $(a_0^*, \cdots, a_{\mathbf{n}}^*)$, with non-zero components only in J^*:

$$a_j^* = \frac{(|\mathcal{J}^*| - 1)\left(\mathbf{s}_{\mathcal{J}^*} - \frac{|\mathcal{J}^*|-1}{\lambda_j}\right)}{\gamma \lambda_j \mathbf{s}_{\mathcal{J}^*}^2} \mathbf{1}_{\{j \in \mathcal{J}^*\}} \textit{ and } U_j^* = \left(\frac{\mathbf{s}_{\mathcal{J}^*} - \frac{|\mathcal{J}^*|-1}{\lambda_j}}{\mathbf{s}_{\mathcal{J}^*}}\right)^2 \mathbf{1}_{\{j \in \mathcal{J}^*\}},$$

is the optimal utility for any $0 \leq j \leq \mathbf{n}$ at NE, where $\mathbf{s}_{\mathcal{J}} := \sum_{j \in \mathcal{J}} 1/\lambda_j$.

Proof is in Appendix A. □

If hypotheses of Lemma 1 are satisfied, it is clear that $k^* = \mathbf{n}$ and all the players have non-zero utility at NE. It is beneficial for the weaker agents (given by $(\mathcal{J}^*)^c$) to remain silent and the players forced to remain silent is determined by relative values of the inverses $\{1/\lambda_i\}_i$.

2.2 Cooperative scenario (CS)

In cooperative scenario, C-players explore the cooperative opportunities, if any, whereas recall that the adversary does not take part in any coalition. The adversary remains a particular player. Each player among the C-players, seeks to form appropriate coalition with the other C-players, such that they have the best share in presence of the adversary. We *first study the grand coalition of all C-players.*

Towards this consider a two-player non-cooperative game: the adversary is one player and the C-players join together to form one aggregate player. The utility of the aggregate player equals the sum of the utilities of all C- players, i.e., $U_{ag} = \sum_{j \geq 1} U_j$, while that of the adversary equals $U_{ad} = U_0$. The strategy set of the aggregate player equals the product strategy set $[0, \bar{a}]^{\mathbf{n}}$ again with $\bar{a} > \mathbf{n}/\gamma$. We study the NE of this two player game, with the aim to study the maximum improvement possible by 'grand coalition' of C-players. We call the corresponding NE as Cooperative NE (CNE), to distinguish it from the NE of the previous sub-section. Thus the strategic form game to study the cooperative scenario is described as $\left\langle \{a_d, \mathbf{a}_g\}, [0, \bar{a}]^{\mathbf{n}+1}, \{U_{ad}(.), U_{ag}(.)\} \right\rangle$ with $\mathbf{a}_g := (a_1, \cdots, a_{\mathbf{n}})$ and $a_d := a_0$ and the utilities of aggregate and adversary are

$$U_{ag}(\mathbf{a}_g, a_d) = \frac{\sum_{j=1}^{\mathbf{n}} \lambda_j a_j}{\sum_{j=0}^{\mathbf{n}} \lambda_j a_j} - \gamma \sum_{i \geq 1} a_i, \quad U_{ad}(a_d, \mathbf{a}_g) = \frac{\lambda_0 a_0}{\sum_{j=0}^{\mathbf{n}} \lambda_j a_j} - \gamma a_0.$$

Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\mathbf{n}}$, throughout the paper. The CNE is given by (proof is in Appendix A):

Lemma 2. *i) When $\lambda_1 > \max_{j \geq 2} \lambda_j$, the CNE is*

$$a_1^* = a_0^* = \frac{\lambda_1 \lambda_0}{\gamma \big(\lambda_1 + \lambda_0\big)^2}, \text{ and } a_j^* = 0 \text{ for all } j > 1.$$

The utilities are $U_{ag}^* = \left(\frac{\lambda_1}{\lambda_1 + \lambda_0}\right)^2$ *and* $U_{ad}^* = \left(\frac{\lambda_0}{\lambda_1 + \lambda_0}\right)^2$. (4)

ii) When $|\mathcal{J}_m| > 1$, with $\mathcal{J}_m := \arg\max_{j \geq 1} \lambda_j$, we have infinitely many CNE:

$$\Big\{(a_0^*, a_1^*, \cdots, a_{\mathbf{n}}^*) : \sum_{j \in \mathcal{J}_m} a_j^* = a_0^*, \ a_j^* = 0 \ \forall j \notin \mathcal{J}_m\Big\} \text{ and } a_0^* = \frac{\lambda_1 \lambda_0}{\gamma \big(\lambda_1 + \lambda_0\big)^2}.$$

But the utilities at any CNE remains the same and equals that given in (4). □

Remarks: 1) A close look at the CNE reveals that (all) the weaker C-players are silenced, i.e., $a_j^* = 0$ for $j \geq 2$. Thus the benefit of cooperation (if any) is obtained by the weaker players *agreeing to remain silent. Observe that (all) these players may not remain silent in non-cooperative scenario.*
2) The proof (provided in Appendix A) basically shows that the utility (and thus the best response) of the aggregate player, at any action profile and against any a_0, is dominated by the utility (respectively the best response) at an appropriate action profile with non-zero value only for player 1. The rest of the proof is obtained by solving the extremely simplified reduced game (the aggregate player uses only one component action). This result is readily applicable for any sub-coalition (with more than one players), even when it includes adversary (as would be required for completely defining the TU game).
3) When some C-players are of equal influence, say if $\lambda_1 = \lambda_2$, then one can have multiple CNE, but the aggregate utility at any CNE remains the same.

3 Benefit of Cooperation

In the previous sections, we studied the utilities derived by C-players in non cooperative and cooperative frameworks. The natural question then arises, "whether forming a grand coalition among C-players ameliorates the total utility driven by the said players as obtained in non cooperative framework." Towards this, let $U_T^* := \sum_{j=1}^{\mathbf{n}} U_j^*$ be the total/aggregate utility in non cooperative scenario, where $\{U_j^*\}_j$ are the utilities of C-players under the Nash Equilibrium (Theorem 1), and recall U_{ag}^* is the utility of the aggregate player under the CNE (given by Lemma 2). We define an appropriate indicator, which we refer as 'Benefit of Cooperation' (BoC) Ψ for measuring the normalized advantage (if any) obtained by forming coalition:

$$\Psi = \frac{U_{ag}^* - U_T^*}{U_{ag}^* + U_T^*} \times 200.$$

If the C-players are better without cooperation, i.e., if their utilities under the NE are better than those under CNE, Ψ will be negative. Otherwise we have positive BoC, Ψ. We study the influence of adversary player on BoC. In particular, we analyze the variations in Ψ, as λ_0 increases from zero to a large value, much larger than λ_1.

From Theorem 1, the total utility of C-players in non cooperative scenario is

$$U_T^* = \sum_{j\geq 1} U_j^* = |\mathcal{J}^*| - 2(|\mathcal{J}^*| - 1)\frac{\sum_{j\geq 1, j\in\mathcal{J}^*} 1/\lambda_j}{\mathbf{s}_{\mathcal{J}^*}} + (|\mathcal{J}^*| - 1)^2 \frac{\sum_{j\geq 1, j\in\mathcal{J}^*} 1/\lambda_j^2}{\mathbf{s}_{\mathcal{J}^*}^2}.$$

Further using Lemma 2, we have: $$\Psi = 200\frac{\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - U_T^*}{\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 + U_T^*}. \tag{5}$$

3.1 Equal C-players

We begin with the study of the special case, when $\lambda_j = \lambda_1$ for all $j \geq 1$. In this case $\mathbf{s} = \mathbf{n}/\lambda_1 + 1/\lambda_0$ and all the C-players satisfy the hypothesis of Lemma 1 (clearly $\mathbf{s} > \mathbf{n}/\lambda_1$). However, the adversary may or may not satisfy the same and we study the different sub-cases in the following (proof in Appendix A).

Lemma 3. *1)If the adversary is weak, i.e., when $\mathbf{s} \leq \mathbf{n}/\lambda_0$, the BoC Ψ is decreasing in λ_0 as below:*

$$\Psi = 200\left(\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \frac{1}{\mathbf{n}}\right) \Bigg/ \left(\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 + \frac{1}{\mathbf{n}}\right), \text{ with } \mathcal{J}^* = \{1, \cdots, \mathbf{n}\}.$$

Further, we have the smallest BoC at

$$\lambda_0^* = \frac{\mathbf{n}-1}{\mathbf{n}}\lambda_1 \text{ and the minimum BoC equals } \Psi(\lambda_0^*) = 200\frac{\mathbf{n}^3 - (2\mathbf{n}-1)^2}{\mathbf{n}^3 + (2\mathbf{n}-1)^2}.$$

2) If the adversary is strong i.e., when $\mathbf{s} > \mathbf{n}/\lambda_0$ (i.e., if $\lambda_0 > \lambda_1(\mathbf{n}-1)/\mathbf{n}$), the BoC is increasing with λ_0 and reaches $200(\mathbf{n}-1)/(\mathbf{n}+1)$ as $\lambda_0 \to \infty$:

$$\Psi = 200\frac{\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \mathbf{n}\left(\frac{\lambda_1}{\mathbf{n}\lambda_0+\lambda_1}\right)^2}{\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 + \mathbf{n}\left(\frac{\lambda_1}{\mathbf{n}\lambda_0+\lambda_1}\right)^2} \text{ with } \mathcal{J}^* = \{0, 1, \cdots, \mathbf{n}\}. \qquad \square$$

To Participate or Not: Thus the BoC decreases monotonically with λ_0 until $\lambda_0^* = (\mathbf{n}-1)\lambda_1/\mathbf{n}$ and increases monotonically afterwards. The minimum possible BoC $\Psi(\lambda_0^*)$ provided by the above Lemma can easily be computed. It is easy to verify that the minimum BoC is positive for all $\mathbf{n} > 2$. When $\mathbf{n} = 2$, it equals -200/17 approximately and by continuity (which can easily be verified) we have a range of adversary strength λ_0 (around λ_0^*) for which the BoC is negative. Thus it is *always beneficial to participate in cooperation when there are more than two equal strength C-players.* However *when there are only two equal strength C-players it may not always be beneficial to cooperate.* These results are reinforced in the numerical example of Figure 1.

As the number of C-players increases, irrespective of weak or strong adversary, U_T^* always decreases in $\mathbf{n}$ and $\lim_{\mathbf{n}\to\infty} U_T^* = 0$. This is kind of obvious and is mainly because the interference is increasing. However the more interesting fact is that the cooperative scenario is lot more beneficial. From Lemma 3, the BoC

$$\Psi = 200\frac{\left(\frac{\lambda_1}{\lambda_0+\lambda_1}\right)^2 - \frac{1}{\mathbf{n}}}{\left(\frac{\lambda_1}{\lambda_0+\lambda_1}\right)^2 + \frac{1}{\mathbf{n}}} \to 200 \text{ and } \Psi = 200\frac{\left(\frac{1}{\lambda_0+\lambda_1}\right)^2 - \mathbf{n}\left(\frac{\lambda_1}{\mathbf{n}\lambda_0+\lambda_1}\right)^2}{\left(\frac{1}{\lambda_0+\lambda_1}\right)^2 + \mathbf{n}\left(\frac{\lambda_1}{\mathbf{n}\lambda_0+\lambda_1}\right)^2} \to 200,$$

as $\mathbf{n} \to \infty$, respectively with weak and strong adversary.

In all, *irrespective of the strength of the adversary, a large number of equally strong C-players benefit to the maximum extent by participating in coalition.* One may have very different results with unequal C-players, we next study the same.

Unequal C-players

We numerically compute the BoC (5) for some examples in Figures 1-2. We plot Ψ as a function of λ_0, ranging from 0 to 3 and with $\gamma = 0.4$. In Figure 1 with equal players, the BoC Ψ decreases initially with λ_0 till $\lambda_1/2$, takes minimum value $-200/17$ at $\lambda_0 = \lambda_1/2$, and further on increases towards 200/3. This is true for all the examples of the Figure and is exactly as depicted by Lemma 3.

On the other hand, when $\lambda_1 \neq \lambda_2$ as in Figure (2), we have a similar trend for initial values of λ_0. BoC decreases as λ_0 increases, reaches a minimum value and starts raising again. However, *in contrast to the equal C-player case, the BoC eventually decreases to zero.* It is clearly the case for $\lambda_2 \leq .95$ and one can observe a similar trend even for the case with $\lambda_2 = 1.2$. One can compute the strength of the adversary for which BoC is zero, using Theorem 1. For example, when $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$ the BoC is zero at $\lambda_0 = 0.3974$, by using:

$$\Psi/200 = \left(\frac{\lambda_1}{\lambda_0+\lambda_1}\right)^2 - \frac{\lambda_1^2+\lambda_2^2}{(\lambda_2+\lambda_1)^2} = \frac{2.25}{(\lambda_0+1.5)^2} - \frac{2.25+0.25}{4} = 0.$$

We notice again the cases with negative BoC surrounding such zero BoC cases.

When to participate: For equal C-player case, it is beneficial to cooperate once the number of C-players is greater than two. However, this may not be true when the players are of uneven strengths. In Table 1, we tabulate the cases with negative BoC for the case with $\mathbf{n} = 4$. Thus *we have a second contrast with respect to the equal player case: cooperation may not be beneficial even when there*

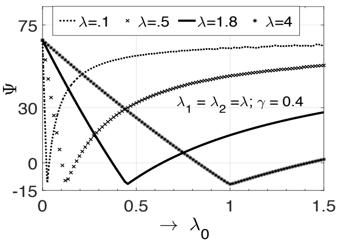

Fig. 1: $\lambda_1 = \lambda_2$, $\mathbf{n} = 2$, BoC increases, rate of convergence faster with small λ_1.

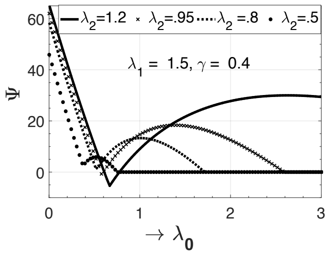

Fig. 2: $\lambda_1 \neq \lambda_2$, $\mathbf{n} = 2$, BoC decreases, increases and finally converges to 0.

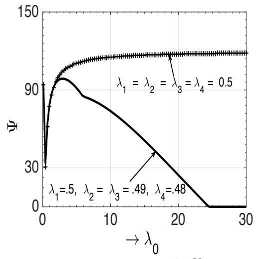

Fig. 3: Drastic difference between equal and unequal agents

are more number of C-players. It rather depends upon the relative strengths of the C-players and that of the adversary. For example, one can have huge number of C-players, however most of them are not sufficiently strong, and in effect we have only two players with non-zero components in NE (can be verified easily):

Lemma 4. *When $\lambda_0 \geq \lambda_1\lambda_2/(\lambda_1 - \lambda_2)$ with $\lambda_1 > \lambda_2$, $\mathcal{J}^* = \{0, 1\}$ irrespective of the number of C-players, and so*

$$U_T^* = \left(\frac{\lambda_1}{\lambda_1 + \lambda_0}\right)^2 \text{ and hence } \Psi = 0 \ \forall \ \mathbf{n} \geq 2. \qquad \square$$

Observe that λ_1, slightly greater than λ_2, is sufficient for this zero BoC case. In Figure 3 we considered two examples with $\mathbf{n} = 4$ to further illustrate this. The BoC for the case with equal agents converges towards 120 (as given by Lemma 3), while that with unequal agents converges to zero, as λ_0 increases. The differences (0.01, 0.02, etc.) in the strengths of the respective agents, in the two examples is negligible, however the outcome is drastically different.

Sr. No.	λ_0	λ_1	λ_2	λ_3	λ_4	$\mathcal{J}$	Ψ	$\{\lambda_0 : \Psi \leq 0\}$
1	2.1	3	1.2	1	0.5	$\{0, 1\}$	0	(2,∞)
2	1.36	2.6	2.4	1.3	1.3	$\{0, 1, 2\}$	-6.17	[1.18, 1.6]
3	1.3	2.7	2.4	1.32	1.1	$\{0, 1, 2, 3\}$	-5.63	[1.2, 1.55]
4	1.34	2.7	2.4	1.32	1.31	$\{0, 1, 2, 3, 4\}$	-5.44	[1.22, 1.62]

Table 1: Example scenarios in which Cooperation Fails

Lemma 4 also illustrates *a third contrast: there would be scenarios when BoC converges to zero (or is zero) as the number of C-players increases.*

4 Fair Sharing

In the previous section, we studied the BoC derived by the grand coalition of C-players. In the scenarios where BoC is positive, it is important to quantify the sharing of the worth among C-players. Towards this we form a relevant Transferable utility (TU) game (e.g., [1]), which has many cooperative solutions (e.g., core, Shapley value etc). We consider only Shapley value in the current paper, while other solution concepts could be of future interest.

4.1 Transferable utilities:

Let $\mathcal{S} := \{C : C \subset \{1, \cdots, \mathbf{n}\}\}$, represent the collection of all subsets of the players, basically the collection of all possible coalitions. The characteristic function

$v : \mathcal{S} \to \mathcal{R}$, maps every coalition to a value on real line $\mathcal{R}$, which represents the worth of the coalition.

We first discuss a commonly used practice, that defines a TU game from any given strategic form game. Consider any coalition $C \subset \{1, \cdots, \mathbf{n}\}$. The worth of this coalition $v(C)$ is defined as the value of the two player zero-sum game, where the first player is the aggregate of all the players from C and the second player is aggregate of the rest of the players $C^c := \{1, \cdots, \mathbf{n}\} - C$. The first aggregate player maximizes the sum of utilities (utilities as given in strategic form game) of all the players from C, while the second aggregate player minimizes the same sum utility. We need some changes to this procedure to suit the problem under hand: a) each player pays some cost for using an action; b) we have an adversary. To incorporate the second factor, we include the adversary as a part of the second aggregate player. To incorporate the first factor we define the reward of the second aggregate player as negative of the reward of the first aggregate player and then include the cost for choosing the particular action(s). Thus we define the worth of the coalition C as the utility obtained by the first aggregate player using the NE of the following two player strategic game:

$$U_{ag,1} = \frac{\sum_{j \in C} \lambda_j a_j}{\sum_{j \in C} \lambda_j a_j + \sum_{j \in C^c} \lambda_j a_j + \lambda_0 a_0} - \gamma \sum_{j \in C} a_j \text{ and}$$

$$U_{ag,2} = 1 - \frac{\sum_{j \in C} \lambda_j a_j}{\sum_{j \in C} \lambda_j a_j + \sum_{j \in C^c} \lambda_j a_j + \lambda_0 a_0} - \gamma \left(\sum_{j \in C^c} a_j + a_0 \right).$$

The analysis of this game is exactly as in Lemma 2. By Lemma 2 and remarks thereafter, this aggregate game may have multiple NE but the utility of the two aggregate players remain the same irrespective of the CNE. The worth of coalition C (for any $C \in \mathcal{S}$) is defined as the utility under a CNE, i.e., from Lemma 2:

$$v(C) := \left[\frac{\lambda_C}{\lambda_C + \lambda_C^c} \right]^2 \text{with } \lambda_C := \max_{j \in C} \lambda_j, \lambda_C^c := \max\{\lambda_0, \lambda_{C^c}\}, \ \lambda^c_{\{1,\cdots,\mathbf{n}\}} = \lambda_0. \quad (6)$$

Thus we have a TU game $\langle \{1, \cdots, \mathbf{n}\}, v(\cdot) \rangle$. There are many fair solutions for TU games, and we consider the well known Shapley Value. The **Shapley value** of player k, ϕ_k, is given by the following (e.g., [1]):

$$\phi_k = \sum_{C \in \mathcal{S}, k \notin C} \frac{|C|!(\mathbf{n} - |C| - 1)!}{\mathbf{n}!} \left[v(C \cup \{k\}) - v(C) \right] \text{ for any player } k. \quad (7)$$

4.2 Shapley value: Strong adversary case

We first consider the case with $\lambda_0 \geq \lambda_j$ for all $j \geq 1$. For this case, from (6) $\lambda_C^c = \lambda_0$ for any coalition C. We first compute the improvement in worth of coalition C when player k joins it:

$$v(C \cup \{k\}) - v(C) = \begin{cases} 0 & \text{if } \lambda_k \leq \lambda_C \text{ and} \\ \left(\frac{\lambda_k}{\lambda_k + \lambda_0} \right)^2 - \left(\frac{\lambda_C}{\lambda_C + \lambda_0} \right)^2 & \text{if } \lambda_k > \lambda_C, \text{ and} \\ \left(\frac{\lambda_k}{\lambda_k + \lambda_0} \right)^2 & \text{if } C = \emptyset, \text{ the empty set.} \end{cases}$$

Fix player k. From above it is clear that (note $\lambda_C = \lambda_{C\cup\{k\}}$)

$$v(C\cup\{k\}) - v(C) = 0 \text{ for any coalition } C \text{ such that, } C\cap\{1,\cdots,k-1\} \neq \emptyset.$$

Thus to compute the Shapley value ϕ_k, we consider the improvement in worth when $\{k\}$ joins sub coalitions, that are subsets of $\{k+1,\cdots,\mathbf{n}\}$:

$$v(C\cup\{k\}) - v(C) = \left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 - \left(\frac{\lambda_C}{\lambda_C+\lambda_0}\right)^2 > 0, \text{ for any } C\subset\{k+1,\cdots,\mathbf{n}\}.$$

Fix any $1\le m\le \mathbf{n}-k$ and consider further sub-coalitions as below:

$$\text{any } C\subset\{k+m,\cdots,\mathbf{n}\} \text{ and } (k+m)\in C.$$

Note that $\lambda_C = \lambda_{k+m}$, $\lambda_{C\cup\{k\}} = \lambda_k$ for all such C and so the improvement:

$$v(C\cup\{k\}) - v(C) = \left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_0}\right)^2.$$

We will have $\binom{\mathbf{n}-k-m}{r}$ such coalitions of size $r+1$, with $r=0,\cdots,\mathbf{n}-k-m$ and hence the contribution to the Shapley value from such coalitions equals:

$$\sum_{C\subset\{k+m,\cdots,\mathbf{n}\} \text{ and } (k+m)\in C} \frac{|C|!(\mathbf{n}-|C|-1)!}{\mathbf{n}!}\left[v(C\cup\{k\}) - v(C)\right]$$

$$= \delta_{k,m}\left[\left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_0}\right)^2\right] \text{ with}$$

$$\delta_{k,m} := \sum_{r=0}^{\mathbf{n}-k-m}\binom{\mathbf{n}-k-m}{r}\frac{(r+1)!(\mathbf{n}-r-2)!}{\mathbf{n}!} = \frac{1}{\mathbf{n}}\frac{\mathbf{n}}{(k+m-1)(k+m)}.$$

Thus in all, from (7) the Shapley value for any $1\le k\le\mathbf{n}$

$$\begin{aligned}\phi_k &= \frac{1}{\mathbf{n}}\left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 + \sum_{m=1}^{\mathbf{n}-k}\frac{1}{(k+m)(k+m-1)}\left[\left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_0}\right)^2\right]\\ &= \frac{1}{k}\left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^2 - \sum_{m=k+1}^{\mathbf{n}}\frac{1}{m(m-1)}\left(\frac{\lambda_m}{\lambda_m+\lambda_0}\right)^2. \end{aligned} \tag{8}$$

4.3 Shapley value for weak adversary

Now consider the case when $\lambda_0\le\lambda_1$. One can compute the Shapley value for this case as in the previous case. The computations are slightly more complex and are carried out in Appendix A. The Shapley value for this case is given by:

$$\phi_1 = \frac{1}{\mathbf{n}}\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 + \sum_{m=2}^{\mathbf{n}-1}\frac{1}{(1+m)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 - \left(\frac{\lambda_{1+m}}{\lambda_{1+m}+\lambda_1}\right)^2\right]$$
$$+\sum_{m=2}^{\mathbf{n}-1}\frac{1}{(m+1)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_{1+m}}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right] + \frac{1}{\mathbf{n}}\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right],$$

and for any $k \geq 2$:

$$\phi_k = \frac{1}{\mathbf{n}}\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 + \sum_{m=1}^{\mathbf{n}-k}\frac{1}{(k+m)(k+m-1)}\left[\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_1}\right)^2\right]$$
$$+\sum_{m=k+1}^{\mathbf{n}}\frac{1}{m(m-1)}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_m}\right)^2 - \left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_k}\right)^2\right] + \frac{1}{\mathbf{n}}\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_k}\right)^2\right].$$

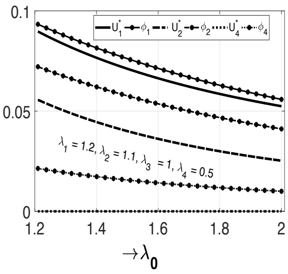

Fig. 4: Shapley values are always fair

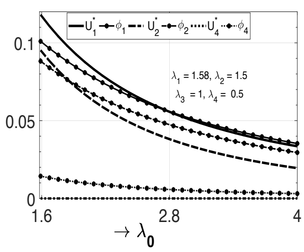

Fig. 5: Shapley values are not always fair

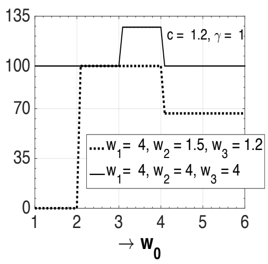

Fig. 6: OSN: Boc for exponential utilities

Numerical observations: Figures 4-5 show the Shapley values and the utilities under NE for two examples with strong adversary ($\lambda_0 \geq \lambda_1$) and $\mathbf{n} = 4$. Both the examples have positive BoC. In Figure 4, the sharing as given by Shapley values (lines with markers) is fair: all players derive more than their respective utilities at NE (non cooperative scenario). In Figure 5 the allocation is not fair: the stronger agents derive lesser utilities than those under NE while the weaker ones improve. Thus the conclusion is that *the Shapley values are not always fair for this system.* The *game is not super-additive (and hence not convex) and this could be the reason. Further the grand coalition may not be the best in this case: one can easily gather examples for which* $v(\{1\}) = v(\{1, \cdots, N\})$.

The above discussion shows that one needs an alternate way of sharing the utilities among C-players. We intend to work towards this in future. We close this discussion by suggesting a heuristic, which could be a part of our future work:

$$\psi_i := U_i^* + \frac{\lambda_i}{\sum_{j\geq 1}\lambda_j}\left(U_{ag}^* - U_T^*\right). \tag{9}$$

Few initial remarks: Basically share the overall gain (above the non-cooperative utility), among the C-players, proportional to their influence factors. Note that (9) need not be individually rational, as $v(\{1\}) = v(\{1, \cdots, N\})$ and $v(\{1\}) \gg$

U_1^* for some examples (e.g., Figure 5). These observations are mainly due to the following: *"when players cooperate they not only aid themselves, they also aid the opponents. And this is because the weaker ones remain silent is the best strategy for cooperative scenario, which reduces the interference to all, including the opponents".*

5 Game with Exponential Utilities

Online Social Networks (OSNs) play an instrumental role in marketing of products or services of the organizations. Due to immense activities of the users, OSNs become predominant vehicles to proliferate contents therein. The marketing/advertising companies make use of these platforms to publicize their "content of interest." We use the model and results of [7] (studies content propagation properties using branching processes), for analysing the importance of cooperation when 'relative visibility' of a content is important to its content provider (CP). We first summarize their model and results.

Each user of an OSN possesses time-line (TL) structure, basically an inverse stack of certain length, where different contents appear on different levels based on their newness. The old content is pushed down by the arrival of new content to the TL. On the other hand, the content gets 'replicated', if a user forwards the post to its friends. After a series of such events a particular content, depending upon the interest it generates, either gets extinct (gets deep down the TLs, to attract any further attention) or gets viral, i.e., the copies of the post grow exponentially fast due to rigorous sharing (see [7]), after considerable time t.

A marketing content is posted to some users initially by a Content Provider (CP). The extinction/virality of a post depends upon network parameters as well as the quality of the post represented by η. It is shown that the number of copies of a content grows in accordance with branching process under certain assumptions. Using the theory of branching processes the growth rate of the content is shown to be proportional to η (see [7]). And growth rate characterizes the visibility of the content: the more the growth rate, the more the visibility.

The authors in [7] obtained the expected number of TLs ($E[X(t)]$) containing the post at time t, after inception with one copy. We reproduce the expression for this expected number, for one interesting example scenario in which the shared post always sits on the top of the recipient TL immediately after sharing (see [7, Theorem 1 and Lemma 1] with $\rho_i = \mathbf{1}_{i=1}$):

$$E[X(t)] \approx \bar{k}e^{\alpha t} = ke^{cw\eta}, \text{ where} \tag{10}$$

$$\alpha := \Big((1-\theta)md_1d_2w\eta - 1 + \theta d_2\Big)(\lambda+\nu), \quad \text{is the virality co-efficient,}$$

$$r_i = d_1 d_2^i, \quad \text{is the probability that user reads the post at level } i \text{ of TL,}$$

$$\lambda, \nu - \text{ rates (of appropriate Poisson processes) at which users visit the OSN,}$$

$$w - \text{ the influence factor of CP,} \qquad m - \text{ the mean number of friends}$$

$$\bar{k} = \frac{1}{1-d_2}\frac{(1-\theta)md_2w\eta}{(1-\theta)md_2w\eta + \theta d_2 - \theta}, \quad k = \bar{k}e^{\Big(-1+\theta d_2\Big)(\lambda+\nu)t},$$

$$c := (1-\theta)md_2(\lambda+\nu)t \qquad \text{and} \quad \theta = \frac{\lambda}{\lambda+\nu}.$$

The above is the expected number of TLs, with the post under consideration, at some level of the TL. Here $w\eta$ represents the probability that a typical user shares the post. Further, we consider the case with d_2 close to 1 (users read post from a good number of levels), and we hence approximate $\bar{k} \approx 1/(1-d_2)$, a constant independent of η. Thus in the expression (10) for the expected number of posts, k and c are constants independent of the action/strategy η.

Further the probability of virality (i.e., the probability that a post gets viral) is positive if and only if the virality coefficient $\alpha > 0$ ([7, Lemma 2]). Thus $E[X(t)] > 0$ if and only if $\alpha > 0$. Hence $E[X(t)] = 0$ for any $\eta \leq \overline{\eta}$ where:

$$\underline{\eta} = \frac{1-\theta d_2}{(1-\theta)md_2} \times \frac{1}{w}, \tag{11}$$

and if $\eta > \underline{\eta}$, then $E[X(t)]$ is given by (10). Thus summarizing, the expected number of TLs with the content of a CP depends on its quality η:

$$y(\eta; w) = \begin{cases} ke^{cw\eta} & \underline{\eta} < \eta \leq \frac{1}{w} \\ 0 & \text{else,} \end{cases} \quad \text{or equivalently} \quad y(x; w) := \begin{cases} ke^{x} & \underline{x} < x \leq c \\ 0 & \text{else,} \end{cases} \tag{12}$$

after the change of variable to $x = cw\eta$ and where $\underline{x} := cw\underline{\eta}$. It is important to note that the constants $c, \underline{x}$ depend only upon the network parameters, and are not altered by w the CP related parameter (see equations (10)-(11)). Thus these boundary points would be the same for any CP using the given OSN.

Content of competing CPs, when propagate through the same OSN and at the same time, create interference to each other by reducing the visibility of each other's post. The visibility of the content of a particular CP is proportional to the number of TLs with its own content and inversely proportional to the number of TLs with content of the competing CPs. We consider $(\mathbf{n}+1)$ number of competing CPs, one among them being an adversary. As before, adversary could be an player that might not be interested in participating in any coalition or the aim of the adversary could be to jam the visibility of the content of other CPs. We study (as before), if it is good to participate in cooperative strategies.

5.1 Non Co-operative N players

Let x_j represent the quality of CP-j content. Its content gets viral only if $x_j > \underline{x}$. If the content gets viral, the expected number of shares is given by (12). The CP incurs a cost proportional to its (actual or non-transformed) quality $\eta_j = x_j/w_j$ and its aim is to get best relative visibility of its content over OSN. Thus the utility of CP j when it creates a content of quality x_j and when others create content with respective qualities $(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{\mathbf{n}})$ equals (see (12)):

$$U_j(x_0, x_1, \cdots, x_{\mathbf{n}}) = \begin{cases} 0 - \frac{\gamma x_j}{w_j c}, & x_j \leq \underline{x} \\ \frac{e^{x_j}}{\sum_i e^{x_i} \mathbf{1}_{x_j > \underline{x}}} - \frac{\gamma x_j}{w_j c}, & \underline{x} < x_j \leq c. \end{cases} \tag{13}$$

This type of utility again induces a $(\mathbf{n}+1)$-player non cooperative strategic form game. We begin with non cooperative scenario where each CP chooses its quality factor to maximize its own utility function.

The objective functions corresponding to this game have discontinuities. Further *it is clear from the above utility that the effective domain of optimization is* $\{0\} \cup (\underline{x}, c]$*, which is not connected.* Thus as expected, the analysis is far more complicated. We observe (through numerical computations) that in many cases pure strategy NE does not even exist. We performed the best response analysis of utility of any typical player (details in Appendix B) and found that the best response is one among $\{0, \underline{x}, c\}$, in most of the cases. Further since $\underline{x}$ is just at the border of virality (the post gets extinct for any $x \le \underline{x}$ and gets viral for any $x > \underline{x}$), it may not be a right choice for practical purposes. Because of all these reasons, we continue further analysis *with binary actions, i.e., the players choose* $a \in \{0, c\}$*. That is, the CPs either prepare the best quality post or do not even participate.*

NE with binary actions We first obtain the NE for this strategic form game.

Lemma 5. *This game can have multiple NE. The set of NE,* $\mathcal{N}^*$*, is given by:*

$$\mathcal{N}^* := \left\{ \mathbf{x} \middle| \sum_{j=0}^{\mathbf{n}} \mathbf{1}_{\{x_j = c\}} = k^*, \ x_j = 0 \text{ if } w_j < k^*\gamma \text{ and } x_j = c \text{ if } w_j \ge (k^*+1)\gamma \right\}$$

$$\text{with } k^* := \max\left\{ 0 \le k \le \mathbf{n} : \sum_{j=0}^{\mathbf{n}} \mathbf{1}_{\{w_j \ge k\gamma\}} \ge k \right\}, \bar{k}^+ := \sum_j \mathbf{1}_{\{w_j \ge (k^*+1)\gamma\}} \text{ and}$$

$$\bar{k}^* := \sum_j \mathbf{1}_{\{w_j \ge k^*\gamma\}}. \text{ Also } |\mathcal{N}^*| = N^* := \binom{\bar{k}^* - \bar{k}^+}{k^* - \bar{k}^+}. \tag{14}$$

In any NE the number of players with non-zero action equals k^**, the action of first* $\bar{k}^+$ *strong players is* c *and there are* N^* *number of NE for this game.* □

Proof. Proof is given in Appendix B.

Thus we have multiple NE for this scenario. Our aim is to compute the BoC (benefit of cooperation) and towards this we need $U_T^* = \sum_{j \ge 1} U_j^*$, the total utility of C-players at an 'appropriate' NE. We consider the minimum total utility of C-players among all possible NE as the utility derived in the Non-cooperative scenario.

If the adversary is weak, i.e., if $w_0 < k^*\gamma$, then at any NE $\mathbf{x}^* \in \mathcal{N}^*$

$$U_T^*(\mathbf{x}^*) = 1 - \sum_j \frac{\gamma x_j^*}{w_j c}, \text{ and } \min_{\mathbf{x} \in \mathcal{N}^*} U_T^*(\mathbf{x}) = 1 - \sum_{j=1}^{\bar{k}^+} \frac{\gamma}{w_j} - \sum_{j=\bar{k}^* - k^* + \bar{k}^+ + 1}^{\bar{k}^*} \frac{\gamma}{w_j}. \tag{15}$$

If the adversary is strong, i.e., if $w_0 \ge k^*\gamma$, the minimum total utility equals:

$$\min_{\mathbf{x} \in \mathcal{N}^*} U_T^*(\mathbf{x}) = \frac{k^* - 1}{k^*} - \sum_{j=1}^{\bar{k}^+} \frac{\gamma}{w_j} - \sum_{j=\bar{k}^* - k^* + \bar{k}^+ + 2}^{\bar{k}^*} \frac{\gamma}{w_j}. \tag{16}$$

5.2 Co-operative N players

When the players decide to participate in a coalition, they make a combined post having content of all the participating players. The combined post gets liked by an user of OSN, if user likes the content of any one of them. Thus the probability that a combined post (of grand coalition) is liked by any user equals:

$$\eta_{ag} = 1 - \big(1 - w_1\eta_1\big)\big(1 - w_2\eta_2\big)\cdots\big(1 - w_{\mathbf{n}-1}\eta_{\mathbf{n}-1}\big) = 1 - \prod_{1\le i\le \mathbf{n}}(1 - x_i/c). \tag{17}$$

One can normalize the influence factor of the combined player to 1, i.e., $w_{ag} = 1$. As before define, $x_{ag} := c\eta_{ag}$ and $x_{ad} = cw_{ad}\eta_{ad} = x_0$.

In non-cooperative scenario, content of each CP starts with one copy (at start one user has content stored on its TL). Equivalently for any coalition C we consider that the process starts with $|C|$-copies of the combined post. Thus the aggregated utility of C-players for this exponential-utility game equals (see (12), (13) and with $\mathbf{x} = (x_1, \cdots, x_{\mathbf{n}}) \in [0, c]^{\mathbf{n}}$):

$$U_{ag}(x_0, \mathbf{x}) = \frac{\mathbf{1}_{\{x_{ag}\le \underline{x}\}}\mathbf{n}e^{x_{ag}}}{\mathbf{n}e^{x_{ag}} + e^{x_{ad}}} - \sum_{i=1}^{\mathbf{n}} \frac{\gamma x_i}{w_i c}, \quad \text{with } x_{ag} = c\left[1 - \prod_{1\le i\le \mathbf{n}}\left(1 - \frac{x_i}{c}\right)\right].$$

and that of adversary is:

$$U_{ad}(x_0, \mathbf{x}) = \frac{\mathbf{1}_{\{x_{ad}\le \underline{x}\}}e^{x_{ad}}}{\mathbf{n}e^{x_{ag}} + e^{x_{ad}}} - \gamma\frac{x_{ad}}{w_0 c}, \quad \text{with } x_{ad} = cw_0\eta_{ad} = x_0.$$

As before we consider two player non cooperative game with the above two as the players, and compute the NE, which we again refer as CNE. This two player game is to study the Benefit of cooperation when all C-players form a (grand) coalition. We begin with the following basic result which could also be used while studying the Shapley value. *When more than one players participate in a coalition, optimizing the aggregate utility by only the strongest player's strategy is better than (or as good as) that obtained by optimizing using the actions of all/some of the coalescing players.* Basically it states that the weaker players should remain silent (as in Lemma 2) and this is the best way to cooperate. *Assume throughout* $w_1 \ge w_2 \cdots \ge w_{\mathbf{n}}$.

Lemma 6. *Say* $w_1 = \max_{j\ge 1} w_j$. *For any* x_0, $\mathbf{x}_1^{\mathbf{n}} := (x_1, \cdots, x_{\mathbf{n}}) \in (0 \cup (\underline{x}, c])^{\mathbf{n}}$

$$U_{ag}(x_0, \mathbf{x}_1^{\mathbf{n}}) \le U_{ag}(x_0, x', 0, 0, \cdots, 0) \text{ with } x'(\mathbf{x}_1^n) = \min\left\{c, w_1\sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i}\right\}. \tag{18}$$

Hence for any given x_0*, the best response of the aggregate player is obtained by:*

$$\sup_{(x_1,x_2,x_3\cdots,x_{\mathbf{n}})\in[\underline{x},c]^{\mathbf{n}}} U_{ag}(x_0, x_1, x_2\cdots, x_{n-1}, x_{\mathbf{n}})$$
$$= \sup_{x\in\{0\}\cup(\underline{x},c]} U_{ag}(x_0, x, 0, 0, \cdots, 0) = \sup_{x\in\{0\}\cup(\underline{x},c]}\left(\frac{\mathbf{n}e^x\mathbf{1}_{x>\underline{x}}}{\mathbf{n}e^x + \mathbf{1}_{\{x_0>\bar{x}\}}e^{x_0}} - \frac{\gamma x}{cw_1}\right). \qquad \square$$

Proof. Proof is given in Appendix B.

Remarks: 1) As in (linear case) Lemma 2, the best response is dominated by best response when the weaker C-players remain silent. 2) However the reduced game can't be analysed as easily as in Lemma 2. Nevertheless, as in the linear case, a NE (x_0^*, x^*) of following the reduced game (action profile of aggregate player has only one component) with utilities of the two players as

$$U'_{ag}(x_0, x) = \frac{\mathbf{n}e^x \mathbf{1}_{x > \underline{x}}}{\mathbf{n}e^x + \mathbf{1}_{\{x_0 > \bar{x}\}} e^{x_0}} - \frac{\gamma x}{c w_1}, \; U_{ad}(x_0, x) = \frac{e^{x_0} \mathbf{1}_{x_0 > \underline{x}}}{\mathbf{n}e^x + \mathbf{1}_{\{x_0 > \bar{x}\}} e^{x_0}} - \frac{\gamma x_0}{c w_0}, \quad (19)$$

gives a CNE to the original game, $(x_0^*, x^*, 0, \cdots, 0)$. Once again, the original game can have more CNEs than those derived from the reduced game. For example, when $w_1 = w_2$, if CNE derived from reduced game is $(x_0^*, x^*, 0, \cdots, 0)$ then $(x_0^*, 0, x^*, 0, \cdots, 0)$ is also a CNE (as in Lemma 2.ii). However by virtue of the above Lemma the utility under any CNE of the original game equals that at a corresponding CNE derived using the reduced game. 3) Once again the result is readily true for any sub-coalition, even when it includes adversary, as would be required for defining the TU game (which we might consider in future).

Binary actions We again consider the case with Binary actions, while the case continuous set of actions from $\{0\} \cup (\underline{x}, c]$ is analysed using numerical computations. From (19) the CNE for binary actions equals:

$$(x_0^*, x_1^*) = \begin{cases} (0,0) & \text{if } w_0 < \gamma \text{ and } w_1 < \gamma \\ (0,c) & \text{if } w_0 < \gamma \text{ and } w_1 > \gamma \\ (c,0) & \text{if } w_0 > \gamma \text{ and } w_1 < \gamma \\ (0,c) \text{ or } (c,0) & \text{if } \gamma < w_0 < (\mathbf{n}+1)\gamma \text{ and } \gamma < w_1 < \left(\frac{1}{\mathbf{n}} + 1\right)\gamma \\ (0,c) & \text{if } w_0 < (\mathbf{n}+1)\gamma \text{ and } w_1 > \left(\frac{1}{\mathbf{n}} + 1\right)\gamma \\ (c,0) & \text{if } w_0 > (\mathbf{n}+1)\gamma \text{ and } w_1 < \left(\frac{1}{\mathbf{n}} + 1\right)\gamma \\ (c,c) & \text{if } w_0 > (\mathbf{n}+1)\gamma \text{ and } w_1 > \left(\frac{1}{\mathbf{n}} + 1\right)\gamma \end{cases} \quad (20)$$

Thus there are situations with multiple CNE and unlike in the linear case, we have (drastically) different utilities at different CNE, at $(0,c)$ utility of aggregate player is $1 - \gamma/w_1$ and at $(c, 0)$ it has 0 utility. Thus as in the concept of 'security level', we define the utility in cooperative scenario as the minimum possible utility at any CNE. With this definition

$$\begin{aligned} U_{ag}^{*min} &= \left(\mathbf{1}_{\{w_1 > \gamma\}} \mathbf{1}_{\{w_0 < \gamma\}} + \mathbf{1}_{\left\{w_1 > \left(\frac{1}{\mathbf{n}}+1\right)\gamma\right\}} \mathbf{1}_{\{w_0 < (\mathbf{n}+1)\gamma\}}\right) \left[1 - \frac{\gamma}{w_1}\right] \\ &+ \mathbf{1}_{\left\{w_1 > \left(\frac{1}{\mathbf{n}}+1\right)\gamma\right\}} \mathbf{1}_{\{w_0 > (\mathbf{n}+1)\gamma\}} \left[\frac{\mathbf{n}}{\mathbf{n}+1} - \frac{\gamma}{w_1}\right]. \end{aligned} \quad (21)$$

5.3 Benefit of cooperation

As in linear case, one can define the BoC

$$\Psi = 200 \frac{(U_{ag}^{*min} - U_T^{*min})}{(U_{ag}^{*min} + U_T^{*min})}$$

which can be computed using (15) and (21).

When $w_1 < \gamma$, the utility in cooperative as well as non-cooperative scenario is zero and so is the BoC. When $\gamma \le w_1 < (1/\mathbf{n} + 1)\gamma$ and $w_0 < \gamma$ then $U_{ag}^* = 1 - \gamma/w_1$, while $U_T^{*min} = 1 - \min_{w_j : w_j > \gamma} (\gamma / w_j)$ as $k^* = 1$. Thus the BoC is positive:

$$\Psi = 200 \frac{U_{ag}^{*min} - U_T^{*min}}{U_{ag}^{*min} + U_T^{*min}} = 200 \frac{\min_{w_j : w_j > \gamma} \frac{\gamma}{w_j} - \frac{\gamma}{w_1}}{2 + \min_{w_j : w_j > \gamma} \frac{\gamma}{w_j} + \frac{\gamma}{w_1}}.$$

When $\gamma \leq w_1 < (1/\mathbf{n}+1)\gamma$ and $w_0 > \gamma$ then $U^*_{ag} = 0$, while $k^* = 1$ and $\bar{k}^* \geq 2$. Thus $U_T^{*min} = 0$ as $(c, 0, \cdots 0)$ is one of the NE which gives the minimum total utility for C-players, and hence BoC is again 0.

Thus with $k^* < 2$, we have zero BoC cases, and hence *cooperation may not be beneficial.* But otherwise, it is always beneficial to cooperate (proof in Appendix B):

Lemma 7. *If $k^* \geq 2$, then BoC $\Psi > 0$, i.e., cooperation is always beneficial.* □

When $k^* \geq 2$, which implies $w_1 > 2\gamma > 1/(\mathbf{n}+1)\gamma$, *cooperation is always beneficial.* We consider three further sub-cases to study the extent of the benefit when $w_1 > 1/(\mathbf{n}+1)\gamma$.

Weak Adversary, $w_0 < k^*\gamma$: Thus, $w_0 < (\mathbf{n}+1)\gamma$. By (15) and (21):

$$\Psi = 200\frac{U_{ag}^{*min} - U_T^{*min}}{U_{ag}^{*min} + U_T^{*min}} = 200\frac{1 - \frac{\gamma}{w_1} - 1 + \sum_{j=1}^{\bar{k}^+}\frac{\gamma}{w_j} + \sum_{j=\bar{k}^*-k^*-\bar{k}^++1}^{\bar{k}^*}\frac{\gamma}{w_j}}{1 - \frac{\gamma}{w_1} + 1 - \sum_{j=1}^{\bar{k}^+}\frac{\gamma}{w_j} - \sum_{j=\bar{k}^*-k^*+\bar{k}^++1}^{\bar{k}^*}\frac{\gamma}{w_j}}$$

$$= 200\frac{\sum_{j=1}^{\bar{k}^+}\frac{\gamma}{w_j} + \sum_{j=\bar{k}^*-k^*-\bar{k}^++1}^{\bar{k}^*}\frac{\gamma}{w_j} - \frac{\gamma}{w_1}}{2 - \frac{\gamma}{w_1} - \sum_{j=1}^{\bar{k}^+}\frac{\gamma}{w_j} - \sum_{j=\bar{k}^*-k^*+\bar{k}^++1}^{\bar{k}^*}\frac{\gamma}{w_j}}, \text{ where } \textstyle\sum_{j=1}^{0}\cdot = 0, \sum_{j=1}^{-m}\cdot = 0.$$

With Equal C-Players: Consider that all CPs are of equal strength, i.e., their influence factors are equal $w_i = w_1 \geq \ \forall\ i \geq 1$. Then the above BoC simplifies:

$$\Psi = 200\frac{1 - \frac{\gamma}{w_1} - 1 + k^*\frac{\gamma}{w_1}}{1 - \frac{\gamma}{w_1} + 1 - k^*\frac{\gamma}{w_1}} = \frac{200(k^*-1)\gamma}{2w_1 - (k^*+1)\gamma}, \text{ with } k^* = \min\{\mathbf{n}, \lfloor w_1/\gamma \rfloor\} \quad (22)$$

As, $\mathbf{n} \to \infty$, $k^* \to \lfloor w_1/\gamma \rfloor$ and BoC $\Psi \to \frac{200(\lfloor w_1/\gamma \rfloor \gamma/w_1 - \gamma/w_1)}{2 - \lfloor w_1/\gamma \rfloor \gamma/w_1 - \gamma/w_1}$, and the limit equals 200, if w_1/γ were an integer. On the other hand, if w_1 increases, BoC $\Psi \to 0$.

Strong Adversary, $w_0 > (\mathbf{n}+1)\gamma$: Thus, $w_0 > k^*\gamma$. By (16) and (21)

$$\Psi = 200\frac{\frac{\mathbf{n}}{\mathbf{n}+1} - \frac{k^*-1}{k^*} + \sum_{j=1}^{\bar{k}^+-1}\frac{\gamma}{w_j} + \sum_{j=\bar{k}^*-k^*-\bar{k}^++1}^{\bar{k}^*-1}\frac{\gamma}{w_j} - \frac{\gamma}{w_1}}{\frac{\mathbf{n}}{\mathbf{n}+1} + \frac{k^*-1}{k^*} - \frac{\gamma}{w_1} - \sum_{j=1}^{\bar{k}^+-1}\frac{\gamma}{w_j} - \sum_{j=\bar{k}^*-k^*+\bar{k}^++1}^{\bar{k}^*-1}\frac{\gamma}{w_j}}, \text{ and with equal C-players}$$

$$\Psi = 200\frac{\frac{\mathbf{n}}{\mathbf{n}+1} - \frac{k^*-1}{k^*} + (k^*-2)\frac{\gamma}{w_1}}{\frac{\mathbf{n}}{\mathbf{n}+1} + \frac{k^*-1}{k^*} - k^*\frac{\gamma}{w_1}}, \text{ where } k^* = \min\{\mathbf{n}+1, \lfloor w_1/\gamma \rfloor\}.$$

As $\mathbf{n}$ increases, the BoC, $\Psi \to \frac{200\left(1/\lfloor w_1/\gamma \rfloor + \lfloor w_1/\gamma \rfloor \gamma/w_1 - 2\gamma/w_1\right)}{2 - 1/\lfloor w_1/\gamma \rfloor - \lfloor w_1/\gamma \rfloor \gamma/w_1}$, and the limit equals 200 (if w_1/γ is integer). If w_1 increases, $k^* \to \mathbf{n}+1$ and $\Psi \to 0$.

Intermediate Adversary, $\boldsymbol{k^*\gamma \leq w_0 \leq (n+1)\gamma}$: By (16) and (21):

$$\Psi = 200 \frac{1 - \frac{k^*-1}{k^*} + \sum_{j=1}^{\bar{k}^+-1} \frac{\gamma}{w_j} + \sum_{j=\bar{k}^*-k^*-\bar{k}^++1}^{\bar{k}^*-1} \frac{\gamma}{w_j} - \frac{\gamma}{w_1}}{1 + \frac{k^*-1}{k^*} - \frac{\gamma}{w_1} - \sum_{j=1}^{\bar{k}^+-1} \frac{\gamma}{w_j} - \sum_{j=\bar{k}^*-k^*+\bar{k}^++1}^{\bar{k}^*-1} \frac{\gamma}{w_j}}.$$

With equal C-players, $\Psi = 200 \dfrac{1 - \frac{k^*-1}{k^*} + (k^*-2)\frac{\gamma}{w_1}}{1 + \frac{k^*-1}{k^*} - k^* \frac{\gamma}{w_1}}, \; k^* = \min\{\mathbf{n}, \lfloor w_1/\gamma \rfloor\}.$

As $\mathbf{n}$ increases, the BoC, $\Psi \to \frac{200\left(1/\lfloor w_1/\gamma \rfloor + \lfloor w_1/\gamma \rfloor \gamma/w_1 - 2\gamma/w_1\right)}{2 - 1/\lfloor w_1/\gamma \rfloor - \lfloor w_1/\gamma \rfloor \gamma/w_1}$, and the limit equals 200, if w_1/γ were an integer. On the other hand, if w_1 increases $k^* \to \mathbf{n}+1$ and the BoC converges to $1/(2n+1)$.

References

1. Narahari, Yadati. "Game theory and mechanism design." World Scientific, 2014.
2. `http://www.its.caltech.edu/mshum/ec105/matt8.pdf`
3. Gast, Nicolas, and Bruno Gaujal. "Computing absorbing times via fluid approximations." Advances in Applied Probability 49.3 (2017): 768-790.
4. Rosen, J. Ben. "Existence and uniqueness of equilibrium points for concave n-person games." Econometrica: Journal of the Econometric Society (1965): 520-534.
5. Cheng, Shih-Fen, et al. "Notes on equilibria in symmetric games." (2004): 71.
6. Xu, Yuedong, et al. "Efficiency of Adversarial Timeline Competition in Online Social Networks." arXiv preprint arXiv:1602.01930 (2016). APA
7. Dhounchak, Ranbir, Veeraruna Kavitha, and Eitan Altman. "A Viral Timeline Branching Process to Study a Social Network." Teletraffic Congress (ITC 29), 2017 29th International. Vol. 3. IEEE, 2017.
8. Xu, Yuedong, et al. "Efficiency of Adversarial Timeline Competition in Online Social Networks." arXiv preprint arXiv:1602.01930 (2016).
9. A. Vulimiri, G. Agha, P. Godfrey, and K. Lakshminarayanan, *How well can congestion pricing neutralize denial of service attacks?* in Proc. of ACM Sigmetrics 2012.
10. J. Lou, A. Smith, Y. Vorobeychik, *Multidefender Security Games*, in IEEE Intelligent Systems, vol. 32, no. 1, pp. 1541-1672, 2017.
11. W. Saad, Z. Han, M. Debbah, A. Hjorungnes, T. Basar, *Coalitional game theory for communication networks*, in IEEE Signal Processing Magazine, vol. 26, no. 5, 2009.
12. W. Saad, Z. Han, T. Basar, M. Debbah, A. Hjorungnes, *Distributed Coalition Formation Games for Secure Wireless Transmission*, in Mobile Networks and Applications, vol. 16, no. 2, 2011.
13. F. Kelly, A. Maulloo, D. Tan, *Rate control for communication networks: shadow prices, proportional fairness and stability*, The Journal of the Operational Research Society, Vol. 49, No. 3, 1998.
14. E. Altman, P. Kumar, and A. Kumar, *Competition over timeline in social networks*, in Proc. of Social Network Analysis in Applications Workshop (in conjunction with COMSNETS), 2013.
15. A. Reiffers, Y. Hayel, E. Altman , *Game theory approach for modeling competition over visibility on social networks*, in proceedings of IEEE COMSNETS, workshop on Social Networking, 2014.

6 Appendix A: Linear Utilities

Proof of Lemma 1 Recall the utility of a player i is:

$$U_i = \frac{\lambda_i a_i}{\sum_j \lambda_j a_j} - \gamma a_i; \quad i = 0, 1, \cdots, \mathbf{n}.$$

We compute the Nash Equilibrium using best response method. The best response of a player i against fixed a_{-i} (game theoretic notation) is given by:

$$BR_i(a_{-i}) := \arg\max_{a_i} U_i(a_i, a_{-i}).$$

This set can be computed using the method of differentiation[1] as follows:

$$\frac{\partial U_i}{\partial a_i} = \frac{\lambda_i \sum_j \lambda_j a_j - \lambda_i^2 a_i}{\left(\sum_j \lambda_j a_j\right)^2} - \gamma = \lambda_i \frac{\sum_j \lambda_j a_j - \lambda_i a_i}{\left(\sum_j \lambda_j a_j\right)^2} - \gamma \tag{23}$$

$$\frac{\partial^2 U_i}{\partial a_i^2} = -2\lambda_i \frac{\lambda_i \sum_{j:j\neq i} \lambda_j a_j}{\left(\sum_j \lambda_j a_j\right)^3}.$$

We first compute the NE for unconstrained action set/domain and later on show that the computed NE is in fact in the interior of the bounded domain. In view of this, the best response of a player is/are among the stationary points of the corresponding utility function and is given by equating the corresponding/respective first derivative to zero. That is, by $\partial U_i/\partial a_i = 0$, we get the following system of equation

$$\sum_j \lambda_j a_j - \lambda_i a_i = \frac{\gamma}{\lambda_i}\left(\sum_j \lambda_j a_j\right)^2 \quad \forall\ i. \tag{24}$$

On summing, $$\mathbf{n}\sum_j \lambda_j a_j = \gamma \sum_j \frac{1}{\lambda_j}\left(\sum_j \lambda_j a_j\right)^2 \implies \frac{\mathbf{n}}{\mathbf{s}\gamma} = \sum_j \lambda_j a_j. \tag{25}$$

Moreover for the stationary points obtained using (24) and (25), the second derivative at the same is **negative**. Thus these stationary points are maximizers and hence are the best response set. Further, the Nash Equilibrium is the simultaneous solution to the above system of equations [2]. Using (25) equation (24) becomes

$$-\lambda_i^2 a_i = \frac{\mathbf{n}^2}{\mathbf{s}^2\gamma} - \frac{\mathbf{n}\lambda_i}{\mathbf{s}\gamma} \text{ or } a_i = \frac{\mathbf{n}\lambda_i}{\mathbf{s}^2\gamma\lambda_i}\left(\mathbf{s} - \frac{\mathbf{n}}{\lambda_i}\right) \quad \forall\ i = 0, 1, \cdots, \mathbf{n}. \tag{26}$$

One can substitute the above into equation (24) and show that this indeed is a solution. Note that the above solution is in the interior of the domain because

[1] Boundary conditions are considered towards the end.

of the hypotheses of the Lemma. Further one can argue that this is the unique solution of the set of equations given by (24) as follows: a) when $\sum_i \lambda_i a_i$ is considered as a constant (say κ) in (24), then clearly the set of equations have unique solution for each κ; b) the final solution is derived by obtaining correct κ, which is the solution of (25) and it is clear there exists an unique solution to (25).
We now rule out the possibility of a_j^* being 0 or $\bar{a}$ for some player j as follows. Under the hypotheses of Lemma, $\bar{a}$ can not be best response for any player as the utility driven is strictly less than zero ($\bar{a} > \frac{\mathbf{n}}{\gamma}$). Now the only possibility left is the action '0'. Let us say the best response for j player $a_j^* = 0$ for some $j \in \mathcal{J}^c$ in some NE (if possible) and non-zero for the rest of the players.

It is easy to see that the best responses of players in $\mathcal{J}$, against zeros of players from set $\mathcal{J}^c$, are obtained by replacing $\mathbf{n}$ by $|\mathcal{J} - 1|$ in (26) and which further satisfy

$$\frac{|\mathcal{J}-1|}{\gamma s_{\mathcal{J}}} = \sum_{i \in \mathcal{J}} \lambda_j a_j^*; \quad \text{with } s_{\mathcal{J}} := \sum_{i \in \mathcal{J}} 1/\lambda_i. \tag{27}$$

We show that a player from set $\mathcal{J}^c$ derives positive utility at non zero action against the given $(a_i^*, i \in \mathcal{J})$; proving action '0' can not be in Nash Equilibrium. We compute the best response of the least influential player from set $\mathcal{J}^c$; say it to be player j i.e. $\lambda_j \le \lambda_k \ \forall \ k \in \mathcal{J}^c$. Its best response, using (27), is given as

$$\max_{a \in [0,\bar{a}]} \frac{\lambda_j a}{\sum_{i \in \mathcal{J}} \lambda_i a_i^* + \lambda_j a} - \gamma a = \max_{a \in [0,\bar{a}]} \left\{ \frac{\lambda_j a}{\frac{|\mathcal{J}-1|}{\gamma s_{\mathcal{J}}} + \lambda_j a} - \gamma a \right\}.$$

Note that at $a = 0$ player j's utility is 0. Now observe the following

$$s_{\mathcal{J}} = \mathbf{s} - \sum_{k \in \mathcal{J}^c} \frac{1}{\lambda_i} > \frac{\mathbf{n}}{\lambda_i} - \sum_{k \in \mathcal{J}^c} \frac{1}{\lambda_k} = \frac{\mathbf{n} - |\mathcal{J}^c|}{\lambda_i} + \frac{|\mathcal{J}^c|}{\lambda_i} - \sum_{k \in \mathcal{J}^c} \frac{1}{\lambda_k} \ \forall \ i \in \mathcal{J}^c \cup \mathcal{J}.$$

And in particular, we have the following inequality

$$s_{\mathcal{J}} > \frac{\mathbf{n} - |\mathcal{J}^c|}{\lambda_j} + \frac{|\mathcal{J}^c|}{\lambda_j} - \sum_{k \in \mathcal{J}^c} \frac{1}{\lambda_k} \tag{28}$$

$$\text{hence } \ s_{\mathcal{J}} > \frac{\mathbf{n} - |\mathcal{J}^c|}{\lambda_j} = \frac{|\mathcal{J}-1|}{\lambda_j}; \quad \text{as } \sum_{k \in \mathcal{J}^c} \lambda_k \ge |\mathcal{J}^c| \lambda_j. \tag{29}$$

$$\text{Now define } \ G(a) := \frac{\lambda_j}{\frac{|\mathcal{J}-1|}{\gamma s_{\mathcal{J}}} + \lambda_j a} - \gamma; \quad \text{and then } \ G(0) = \frac{\lambda_j}{\frac{|\mathcal{J}-1|}{\gamma s_{\mathcal{J}}}} - \gamma.$$

Appealing to (29), we get $G(0) > 0$. Consider the following neighborhood of zero $(0, \epsilon)$ where ϵ is a positive number arbitrary close to 0, using the continuity of $G(.)$ we can write

$$|G(b) - G(0)| < G(0)/2 \implies G(0)/2 < G(a) < 3G(0)/2 \ \forall \ b \in (0, \epsilon).$$

We thus have $G(b) > 0 \ \forall \ b \in (0, \epsilon)$ which gives $U_j(b, a^*_{-j}) > 0$ as $U_j(a, a^*_{-j}) = G(a)a$ where a^*_{-j} standard game theoretic notation. Thus action '0' can not best response for player j ruling out the possibility of zero action (by j player) in NE. Which contradicts the fact that Nash strategy of player j is zero. This completes the proof. $\square$

Proof of Theorem 1: For ease of notations, represent λ'_i by λ_i for each i. This notation is used only in this proof. Given that

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{\mathbf{n}}. \tag{30}$$

We claim there exists exactly one n^* such that with

$$\mathcal{J}^* := \{1, \cdots, n^*\}, \text{ we have } \mathbf{s}_{\mathcal{J}} > (n^* - 1)/\lambda_j \text{ for all } j \leq n^* \text{ and } \\ \mathbf{s}_{\mathcal{J}} \leq (n^* - 1)/\lambda_j \text{ for all } j > n^*. \tag{31}$$

It is clear that such an n^* and $\mathcal{J}^*$ would satisfy (3). We first provide the proof of the above claim. For any $k \leq \mathbf{n}$, define $\mathbf{s}_k := \sum_{j=0}^{k-1} 1/\lambda_j$ and start with Step 0 given below.
Step 0: Start with $k = 2$. It is easy to verify that

$$\frac{\mathbf{s}_k}{k-1} \geq \frac{1}{\lambda_i} \text{ for all } i \leq k. \tag{32}$$

Step 1: For the given k we have:

$$\frac{\mathbf{s}_k}{k-1} \geq \frac{1}{\lambda_i} \text{ for all } i \leq k. \ \text{ If further, } \frac{\mathbf{s}_{k+1}}{k} \leq \frac{1}{\lambda_{k+1}} \tag{33}$$

then condition (31) is satisfied with $n^* = k$ and <u>stop</u>. If the above is negated, then by Lemma 8

$$\frac{\mathbf{s}_{k+1}}{k} > \frac{1}{\lambda_{k+1}} \geq \frac{1}{\lambda_i} \text{ for all } i \leq k+1.$$

Now set $k \leftarrow k+1$ and loop back to **Step 1** till $k \leq \mathbf{n}$. Else <u>stop</u> with $n^* = \mathbf{n}$. Remark: Using (32), one can easily see that $\{0, 1\} \in \mathcal{J}^*$ always. Next we prove the uniqueness of n^*, we begin with two case as follows

- When $n^* = \mathbf{n}$, uniqueness is obvious.
- When it stops with $n^* < \mathbf{n}$, we have

$$\frac{\mathbf{s}_{n^*+1}}{n^*} \leq \frac{1}{\lambda_{n^*+1}}, \ \text{ and then using Lemma 8 recursively } \frac{\mathbf{s}_{k+1}}{k} \leq \frac{1}{\lambda_{k+1}} \ \forall \ k \geq n^*.$$

This gives uniqueness of $|\mathcal{J}|$ satisfying (31), which is same as the condition (3).
From equation (33) as applied to n^*:

$$\frac{\mathbf{s}_{n^*}}{n^*-1} > \frac{1}{\lambda_{n^*}},$$

then one can easily verify that

$$\frac{\mathbf{s}}{\mathbf{n}} = \frac{\mathbf{s}_{n^*}}{n^*-1}\frac{n^*-1}{\mathbf{n}} + \frac{\sum_{j\geq n^*} 1/\lambda_j}{\mathbf{n}} > \frac{1}{\lambda_{n^*}}\frac{n^*-1}{\mathbf{n}} + \frac{\mathbf{n}+1-n^*}{\mathbf{n}}\frac{1}{\lambda_{n^*}} = \frac{1}{\lambda_{n^*}}.$$

Thus by monotonicity of (see (30)) ,

$$\frac{\mathbf{s}}{\mathbf{n}} > \frac{1}{\lambda_{n^*}} \geq \frac{1}{\lambda_i} \text{ for all } i \leq n^*,$$

and thus we have:

$$\{0, 1, \cdots, n^*\} = \{j : \mathbf{s} > \mathbf{n}/\lambda_j\}.$$

This proves $\mathcal{J}^* \subset \{j : \mathbf{s} > \mathbf{n}/\lambda_j\}$.

The condition to to determine n^* If the below condition is true

$$\mathbf{s} < \frac{\mathbf{n}}{\lambda_{\mathbf{n}}} \text{ or equivalently if } \lambda_0 \geq \left(\frac{\mathbf{n}}{\lambda_{\mathbf{n}}} - \sum_{j=1}^{\mathbf{n}} \frac{1}{\lambda_j}\right)^{-1}$$

then $\mathbf{n} \notin \mathcal{J}^*$, by previous steps of the proof. In a similar way we also have $(\mathbf{n}-1) \notin \mathcal{J}^*$ if additionally one of the following two conditions is satisfied, a) if $\mathbf{s} < \mathbf{n}/\lambda_{\mathbf{n}-1}$ or equivalently if the following true

$$\lambda_0 \geq \left(\frac{\mathbf{n}}{\lambda_{\mathbf{n}-1}} - \sum_{j\geq 1}^{\mathbf{n}} \frac{1}{\lambda_j}\right)^{-1},$$

we have $(\mathbf{n}-1) \notin \mathcal{J}^*$; b) if the above the condition is not true, but if $\mathbf{s}_{\mathbf{n}-1} < (\mathbf{n}-1)/\lambda_{\mathbf{n}-1}$ is true or equivalently if

$$\lambda_0 \geq \left(\frac{\mathbf{n}-1}{\lambda_{\mathbf{n}-1}} - \sum_{j=1}^{\mathbf{n}-1} \frac{1}{\lambda_j}\right)^{-1},$$

then also $(\mathbf{n}-1) \notin \mathcal{J}^*$, and this is obtained by concentrating on the reduced game with players only from $\{0, 1, \cdots, \mathbf{n}-1\}$. This reduced game has to be considered/ results because, when $\mathbf{n} \notin \mathcal{J}^*$ (which means when $a^*_{\mathbf{n}} = 0$) then in all the best responses we have zero influence from player $\mathbf{n}$.

By monotonicity if $(\mathbf{n}-1)$-th player is not eliminated, i.e., if $\mathbf{n} \in \mathcal{J}^*$, then so are the remaining players. In other words NE has all non-zero components only if

$$\lambda_0 < \left(\frac{\mathbf{n}-1}{\lambda_{\mathbf{n}-1}} - \sum_{j=1}^{\mathbf{n}-1} \frac{1}{\lambda_j}\right)^{-1},$$

and $J^* = \{1, \cdots, \mathbf{n}-1\}$ if

$$\left(\frac{\mathbf{n}}{\lambda_{\mathbf{n}}} - \sum_{j=1}^{\mathbf{n}} \frac{1}{\lambda_j} \right)^{-1} \le \lambda_0 < \left(\frac{\mathbf{n}-1}{\lambda_{\mathbf{n}-1}} - \sum_{j=1}^{\mathbf{n}-1} \frac{1}{\lambda_j} \right)^{-1}.$$

Proceeding in the same manner, by induction, we have that $J^* = \{1, \cdots, k\}$ for any k, if the following is true:

$$\left(\frac{k+1}{\lambda_{k+1}} - \sum_{j=1}^{k+1} \frac{1}{\lambda_j} \right)^{-1} \le \lambda_0 < \left(\frac{k}{\lambda_k} - \sum_{j=1}^{k} \frac{1}{\lambda_j} \right)^{-1}.$$

This completes the first part, i.e., part (i) of the Theorem.
Uniqueness: Due to the uniqueness of the set $\mathcal{J}^*$ and appealing to Lemma 1 for the reduced game with players only from $\mathcal{J}^*$, we have unique NE with the following characteristics: a) there exists an unique $\mathcal{J}^*$, and $a_i^* = 0$ for all $i \notin \mathcal{J}^*$; and b) $(a_0^*, \cdots, a_{n^*}^*)$ equals the unique NE in the interior of the (reduced) domain, given by Lemma 1 for the reduced game. □

Proof of Lemma 2 Using $\lambda_1 > \max_{j \ge 2} \lambda_j$. it is easy to see the following

$$1 - \frac{\lambda_0 a_0}{\sum_j \lambda_j a_j} < 1 - \frac{\lambda_0 a_0}{\lambda_0 a_0 + \lambda_1 \sum_{j \ge 1} a_j} = 1 - \frac{\lambda_0 a_0}{\lambda_0 a_0 + \lambda_1 \hat{a}}; \quad \hat{a} := \sum_{j \ge 1} a_j$$

$$\frac{\sum_{j \ge 1} \lambda_j a_j}{\sum_j \lambda_j a_j} - \gamma \sum_{j \ge 1} a_j < 1 - \frac{\lambda_0 a_0}{\lambda_0 a_0 + \lambda_1 \hat{a}} - \gamma \hat{a}$$

Further in the light of unbounded actions, we can as well take $a_1 = \hat{a}$ keeping the inequality intact. And thus for any a_d the unconstrained (more precisely optimization over the non-negative axis) optimization

$$\max_{\mathbf{a}_g \in [0,\infty)^{\mathbf{n}}} U_{ag}(\mathbf{a}_g, a_d) \le \max_{\mathbf{a}_g \in [0,\infty) \times \{0\}^{\mathbf{n}-1}} U_{ag}(\mathbf{a}_g, a_d). \tag{34}$$

For computing the NE, one can consider unconstrained optimization as above, to begin with. Once the NE is computed and if it is within $[0, \bar{a}]^{\mathbf{n}+1}$, then we are done. We are following the same approach here. In view of (34), it suffices to consider action profile of the aggregate player from $[0, \infty) \times \{0\}^{\mathbf{n}-1}$, i.e., set $a_i = 0$ for all $i \ge 2$, and the resulting simplified utility function is

$$U_{ag}(\mathbf{a}_g, a_d) = U_{ag}((a_g, 0, \cdots, 0), a_d) = \frac{\lambda_1 a_g}{\lambda_1 a_g + \lambda_0 a_d} - \gamma a_g.$$

We first check if a pair of interior points becomes an NE. Towards this we compute the respective gradients and attempt to obtain the best responses using their zeros:

$$\frac{\partial U_{ag}}{\partial a_g} = \frac{\lambda_1 \lambda_0 a_0}{\Big(\lambda_1 a_g + \lambda_0 a_0\Big)^2} - \gamma = 0 \tag{35}$$

$$\frac{\partial U_{ad}}{\partial a_0} = \lambda_0 \frac{\lambda_1 a_g}{\Big(\lambda_1 a_g + \lambda_0 a_0\Big)^2} - \gamma = 0 \tag{36}$$

On solving the above set of equations we have:

$$\lambda_0 \lambda_1 a_g = \lambda_1 \lambda_0 a_0 \text{ and } \gamma\Big(1 + \frac{\lambda_0}{\lambda_1}\Big)\lambda_1 a_g = \lambda_0.$$

Thus we get a unique simultaneous critical point (as there is an unique zero of the above equations)

$$\boxed{a_1^* = a_g^* = \frac{\lambda_1 \lambda_0}{\gamma\big(\lambda_1 + \lambda_0\big)^2} = a_0^* \text{ and } \quad a_i^* \ = 0 \text{ for all } i \geq 2.} \tag{37a}$$

It is clear that the a_1^*, a_0^* components of the above solution is in the interior of the domain. To see the behaviour of these critical points, we compute the second derivative at $\{a_0^*, a_g^*\}$

$$\frac{\partial^2 U_{ag}}{\partial a_g^2} = \frac{-2\lambda_0 \lambda_1^2 a_0^*}{\Big(\lambda_1 a_g^* + \lambda_0 a_0^*\Big)^3} \tag{38}$$

which is negative. In similar way, one can show the same for adversary player. Hence $(a_0^*, a_g^*, 0 \cdots, 0)$ is unique NE, because one can further show by direct computations that any combination of boundary and interior points can't be CNE, as well as $(\bar{a}, \bar{a}, 0 \cdots, 0)$ is not a CNE.
ii) Follows trivially using the above part, by first noticing that arguments of part (i) go through even when $\lambda_1 \geq \max_{j\geq 2} \lambda_j$. Further we can have one such CNE with each of the maximizers of $max_{j\geq 1}\lambda_j$. One can also have one CNE with every convex combination of the maximizers (using exactly similar logic), and hence part (ii). □

Proof of Lemma 3 We are given $\lambda_1 = \lambda_2 \cdots = \lambda_{\mathbf{n}}$. We begin with:
1) *Weak Adversary* i.e., $\mathbf{s} \leq \mathbf{n}/\lambda_0$, using Theorem 1 we get $a_0^* = 0$ and $\sum_{j\geq 1} U_j^* = 1/\mathbf{n}$. Further, using Lemma 2 the performance improvement equals:

$$\Psi = 200 \frac{\Big(\frac{\lambda_1}{\lambda_1+\lambda_0}\Big)^2 - \frac{1}{\mathbf{n}}}{\Big(\frac{\lambda_1}{\lambda_1+\lambda_0}\Big)^2 + \frac{1}{\mathbf{n}}} = 200 \frac{\mathbf{n}\lambda_1^2 - (\lambda_1+\lambda_0)^2}{\mathbf{n}\lambda_1^2 + (\lambda_1+\lambda_0)^2}.$$

On differentiating Ψ w.r.t λ_0

$$\frac{\partial \Psi}{\partial \lambda_0} = \frac{-2(\lambda_1+\lambda_0)\Big(\mathbf{n}\lambda_1^2 + (\lambda_1+\lambda_0)^2 + \mathbf{n}\lambda_1^2 - (\lambda_1+\lambda_0)^2\Big)}{\Big(\mathbf{n}\lambda_1^2 + (\lambda_1+\lambda_0)^2\Big)^2} = \frac{-4(\lambda_1+\lambda_0)\mathbf{n}\lambda_1^2}{\Big(\mathbf{n}\lambda_1^2 + (\lambda_1+\lambda_0)^2\Big)^2}.$$

Thus the derivative is negative and hence Ψ is decreasing in λ_0. And the the minimum improvement (while the adversary is being weak) occurs at maximum allowable value of λ_0. As $\mathbf{s} \leq \mathbf{n}/\lambda_0$ we have $\lambda_0 \leq (\mathbf{n}-1)\lambda_0/\mathbf{n}$. Hence

$$\lambda_0^* = \frac{\mathbf{n}-1}{\mathbf{n}}\lambda_1 \text{ and } \Psi(\lambda_0^*) = \frac{\mathbf{n}^3 - (2\mathbf{n}-1)^2}{\mathbf{n}^3 + (2\mathbf{n}-1)^2}200.$$

This completes the first part.
1) *Strong Adversary:* In this case we have

$$\lambda_0 > \frac{(\mathbf{n}-1)\lambda_1}{\mathbf{n}}; \text{ and } \Psi = 200\frac{(\mathbf{n}\lambda_0 + \lambda_1)^2 - \mathbf{n}(\lambda_1+\lambda_0)^2}{(\mathbf{n}\lambda_0 + \lambda_1)^2 + \mathbf{n}(\lambda_1+\lambda_0)^2}.$$

Similarly on differentiating Ψ w.r.t, we get

$$\frac{\partial \Psi}{\partial \lambda_0} = 4(\lambda_0+\lambda_1)(\mathbf{n}\lambda_0+\lambda_1)\mathbf{n}\frac{(\mathbf{n}-1)\lambda_1}{\left((\mathbf{n}\lambda_0+\lambda_1)^2 + \mathbf{n}(\lambda_1+\lambda_0)^2\right)^2} > 0$$

which is always (strictly) positive. Thus, in this case, Ψ is increasing in λ_0 as long as the adversary is strong, i.e., beyond λ_0^*. Thus in all, minimum value of BoC occurs at λ_0^*. Further notice that

$$\lim_{\lambda_0\to\infty} \frac{(\mathbf{n}\lambda_0 + \lambda_1)^2 - \mathbf{n}(\lambda_1+\lambda_0)^2}{(\mathbf{n}\lambda_0 + \lambda_1)^2 + \mathbf{n}(\lambda_1+\lambda_0)^2} = \lim_{\lambda_0\to\infty} \frac{(\mathbf{n}^2-\mathbf{n})\lambda_0^2 + (1-\mathbf{n})\lambda_1^2}{(\mathbf{n}^2+\mathbf{n})\lambda_0^2 + (1+\mathbf{n})\lambda_1^2 + 2\mathbf{n}\lambda_0\lambda_1}$$

Thus $\lim_{\lambda_0\to\infty}\Psi = 200\frac{\mathbf{n}^2-\mathbf{n}}{\mathbf{n}^2+\mathbf{n}} = 200\frac{\mathbf{n}-1}{\mathbf{n}+1}$. □

Shapley values: When $\boldsymbol{\lambda_0 < \lambda_1}$

Similar to the case of strong adversary to compute the Shapley value for any k, we group together coalitions of similar nature that provide same improvement. We begin with $k \geq 2$ and first consider all coalitions in $C \subset \{2, \cdots, \mathbf{n}\}$. The analysis for such sub-coalitions is exactly the same as in previous subsection, in that λ_C^c is the same for all such coalitions. The only difference being that, $\lambda_C^c = \lambda_1 = \lambda_{C\cup\{k\}}^c$. Thus (see (8)):

$$\sum_{C:1,k\notin C} \frac{|C|!(\mathbf{n}-|C|-1)!}{\mathbf{n}!}\left[v(C\cup\{k\}) - v(C)\right]$$
$$= \frac{1}{\mathbf{n}}\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 + \sum_{m=1}^{\mathbf{n}-k}\frac{1}{(k+m)(k+m-1)}\left[\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_1}\right)^2\right].$$

Now we consider the rest of the coalitions, i.e., the ones with $1 \in C$. We derive further analysis, by considering further (appropriate) sub-class of coalitions.

For the coalitions of the type, such that

$$C^c \cap \{2, \cdots, k-1\} \neq \emptyset, \text{ and } 1 \in C, \text{ clearly, } v(C\cup\{k\}) - v(C) = 0.$$

Note that in the above $\lambda_C = \lambda_{C\cup\{k\}} = \lambda_1$.

Now consider sub-coalitions such that (with $2 \le m \le \mathbf{n} - k$)

$$1, 2, \cdots, (k-1) \in C, \quad (k+1), \cdots, (k+m-1) \in C \text{ and } (k+m) \notin C.$$

For all such coalitions we have, with $\widehat{\lambda}_i := \lambda_i \vee \lambda_0$:

$$v(C \cup \{k\}) - v(C) = \left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_{k+m}}\right)^2 - \left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_k}\right)^2.$$

And the contribution towards the Shapley value by such coalitions, as before can be calculated as below (with $r_a := (r + a - 2)$):

$$\begin{aligned}
&\sum_{m=2}^{\mathbf{n}-k} \sum_{r=0}^{\mathbf{n}-k-m} \binom{\mathbf{n}-k-m}{r} \frac{r_{k+m}!(\mathbf{n} - r_{k+m} - 1)!}{\mathbf{n}!} \left[\left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_{k+m}}\right)^2 - \left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_k}\right)^2\right] \\
&\quad = \sum_{m=k+2}^{\mathbf{n}} \sum_{r=0}^{\mathbf{n}-m} \binom{\mathbf{n}-m}{r} \frac{r_m!(\mathbf{n} - r_m - 1)!}{\mathbf{n}!} \left[\left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_m}\right)^2 - \left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_k}\right)^2\right] \\
&\quad = \sum_{m=k+2}^{\mathbf{n}} \frac{1}{m(m-1)} \left[\left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_m}\right)^2 - \left(\frac{\lambda_1}{\lambda_1 + \widehat{\lambda}_k}\right)^2\right]
\end{aligned}$$

The last equality follows because: (using ([sum binom(n-m, k-m) / binom (n-1, k-2), k = m to n] in wolfram))

$$\sum_{r=0}^{\mathbf{n}-m} \binom{\mathbf{n}-m}{r} \frac{r_m!(\mathbf{n} - r_m - 1)!}{\mathbf{n}!} = \sum_{r=m}^{\mathbf{n}} \binom{\mathbf{n}-m}{r-m} \frac{(r-2)!(\mathbf{n} - r + 1)!}{\mathbf{n}!} = \frac{1}{\mathbf{n}} \frac{\mathbf{n}}{m(m-1)}.$$

When $C = \{1, 2, \cdots, (k-1)\}$, the improvement by this coalition:

$$\frac{(k-1)!(\mathbf{n} - k)}{\mathbf{n}!} \left(v(C \cup \{k\}) - v(C)\right).$$

Further one needs to consider all sub-coalitions in which

$$1, 2, \cdots, (k-1) \in C, \text{ and } (k+1) \notin C.$$

The total contribution of all such coalitions towards Shapley value, computing as before equals:

$$\frac{1}{k(k+1)} \left[\left(\frac{\lambda_k}{\lambda_k + \lambda_1}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m} + \lambda_1}\right)^2\right].$$

Thus in all for any $k \geq 2$:

$$\phi_k = \frac{1}{\mathbf{n}}\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 + \sum_{m=1}^{\mathbf{n}-k}\frac{1}{(k+m)(k+m-1)}\left[\left(\frac{\lambda_k}{\lambda_k+\lambda_1}\right)^2 - \left(\frac{\lambda_{k+m}}{\lambda_{k+m}+\lambda_1}\right)^2\right]$$
$$+\sum_{m=k+1}^{\mathbf{n}}\frac{1}{m(m-1)}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_m}\right)^2 - \left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_k}\right)^2\right]$$
$$+\frac{1}{\mathbf{n}}\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_k}\right)^2\right].$$

The last term is due to improvement when $\{k\}$ is added to $C = \{1, \cdots, \mathbf{n}\} - \{k\}$.

When $k = 1$, consider sub-coalitions $C \in \{1+m, \cdots, \mathbf{n}\}$ and such that $1 + m \in C$ for any $m > 1$. Following similar logic as before the contribution by such terms towards Shapley value is given by:

$$\sum_{C:1,2\notin C}\frac{|C|!(\mathbf{n}-|C|-1)!}{\mathbf{n}!}\left[v(C\cup\{1\}) - v(C)\right]$$
$$= \frac{1}{\mathbf{n}}\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 + \sum_{m=2}^{\mathbf{n}-1}\frac{1}{(1+m)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 - \left(\frac{\lambda_{1+m}}{\lambda_{1+m}+\lambda_1}\right)^2\right].$$

Now consider the coalitions of the type with $2 \leq m \leq \mathbf{n}-1$:

$$2, 3, \cdots, m \in C \text{ and } 1, (m+1) \notin C.$$

For these coalitions,

$$\lambda_C = \lambda_2,\ \lambda_{C\cup\{1\}} = \lambda_1,\ \lambda_C^c = \lambda_1 \text{ and } \lambda_{C\cup\{1\}}^c = \widehat{\lambda}_{1+m}.$$

The contribution towards ϕ_1 by such coalitions:

$$\sum_{m=2}^{\mathbf{n}-1}\sum_{r=0}^{\mathbf{n}-m-1}\binom{\mathbf{n}-m-1}{r}\frac{r_{m+1}!(\mathbf{n}-r_{m+1}-1)!}{\mathbf{n}!}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_{1+m}}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right]$$
$$= \sum_{m=2}^{\mathbf{n}-1}\frac{1}{(m+2)(m+1)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_{1+m}}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right]$$

The last equality is because of:

$$\sum_{r=0}^{\mathbf{n}-m-1}\binom{\mathbf{n}-m-1}{r}\frac{r_{m+1}!(\mathbf{n}-r_{m+1}-1)!}{\mathbf{n}!} = \sum_{r=m+1}^{\mathbf{n}}\binom{\mathbf{n}-m-1}{r-m-1}\frac{(r-2)!(\mathbf{n}-(r-2)-1)!}{\mathbf{n}!}$$
$$= \frac{1}{(m+1)m}$$

Thus in all we have:

$$\phi_1 = \frac{1}{\mathbf{n}}\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 + \sum_{m=2}^{\mathbf{n}-1}\frac{1}{(1+m)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_2}\right)^2 - \left(\frac{\lambda_{1+m}}{\lambda_{1+m}+\lambda_1}\right)^2\right]$$
$$+\sum_{m=2}^{\mathbf{n}-1}\frac{1}{(m+1)m}\left[\left(\frac{\lambda_1}{\lambda_1+\widehat{\lambda}_{1+m}}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right] + \frac{1}{\mathbf{n}}\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^2 - \left(\frac{\lambda_2}{\lambda_2+\lambda_1}\right)^2\right]$$

The last term is because of $\{1\}$ being added to the set $\{2, \cdots, \mathbf{n}\}$. □

Appendix B: Exponential Utilities

Proof of Lemma 5: Consider a permutation π on the set of players such that, $w'_0 \geq w'_1 \geq w'_2 \geq \cdots \geq w'_{\mathbf{n}}$ where $w'_i := w_{\pi(i)}$ for each i.

To keep things simple, for this proof, we refer player numbered $\pi(i)$ as player i and note influence factor of player i now equals w'_i. Let $\mathcal{X}\{\cdot\}$ represent the indicator function. Define the function

$$\hbar(\mathbf{x}) := \sum_i \mathcal{X}\{x_i = c\},$$

to count the number of non-zeros action choices in the action profile, $\mathbf{x}$. Let $\mathbf{x}^a_{-i} := (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{\mathbf{n}})$ represent the action profile of all (C-players + adversary and after permutation), other than player i. Now the best response (represented by $B(\cdot)$) of any player i against the action profile of others $\mathbf{x}^a_{-i}$ can be represented as below (see (13)):

$$B_i(\mathbf{x}) = c\mathcal{X}\{\ (\hbar(\mathbf{x}^a_{-i}) + 1)\gamma < w'_i\}.$$

When k^* given by (14) exists, the best response of the player j with $w'_j \geq k^*\gamma$, satisfies the following:

$$B_j(\mathbf{x}_{-j}) = c, \text{ for any action profile, } \mathbf{x} \text{ with } \hbar(\mathbf{x}_{-j}) \leq k^* - 1.$$

Consider any $k < k^*$. Then any action profile $\mathbf{x}$ with k number of c, i.e., when $\hbar(\mathbf{x}) = k$, can not be a NE: best response of any player j with $w'_j > k^*\gamma$ against $(k-1)e^c$ or ke^c (i.e., against action profile with $k-1$ or k number of players choosing c) is c; and there are $\bar{k}^* > k$ number of such players.

Any action profile with k (where $k > k^*$) number of c can not be a NE: this is possible only if the best response of at least k number of players against $(k-1)e^c$ is c, and by definition of k^*, such a thing is not possible for any $k > k^*$.

One can readily prove the remaining results using similar best response based arguments. □

Proof of Lemma 6: We first show that $x_{ag} \leq x'$ for all $(x_1, \cdots, x_{\mathbf{n}}) \in [0, c]^{\mathbf{n}}$ and then using this result, we will prove both the parts of the Theorem.

We begin with the claim that $x_{ag} \leq x'$ i.e.,

$$x_{ag} = c\left[1 - \prod_{i=1}^{n}\left(1 - \frac{x_i}{c}\right)\right] \leq x' = \min\left\{c, w_1 \sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i}\right\} \quad \forall (x_1, \cdots, x_{\mathbf{n}}) \in [0, c]^{\mathbf{n}}.$$

By definition of x_{ag}, it is clear that $x_{ag} \leq c$, thus it suffices to prove the following, to hold the claim

$$c\left[1 - \prod_{i=1}^{n}\left(1 - \frac{x_i}{c}\right)\right] \leq w_1 \sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i} \quad \forall\ (x_1, \cdots, x_{\mathbf{n}}) \in [0, c]^{\mathbf{n}}. \tag{39}$$

We prove (39) via the method of induction as follows.

We begin with $\mathbf{n} = 2$,

$$x_{ag} = c\left(\frac{x_2}{c} + \frac{x_1}{c}\frac{c - x_2}{c}\right).$$

Using $w_1 \geq w_2 \geq \cdots \geq w_{\mathbf{n}}$ and the fact that $\frac{x_1}{c}\frac{c-x_2}{c} \leq \min\left\{\frac{x_1}{c}, \frac{c-x_2}{c}\right\}$, for all $(x_1, x_2) \in [0, c]^2$

$$\begin{aligned} x_{ag} &= c\Big(\frac{x_2}{c} + \frac{x_1}{c}\frac{c - x_2}{c}\Big) \leq c \min\Big\{\frac{x_2}{c} + \frac{x_1}{c}, \frac{x_2}{c} + \frac{c - x_2}{c}\Big\} \\ &= \min\Big\{x_1 + x_2, c\Big\} \leq \min\left\{w_1 \sum_{i=1}^{2} \frac{x_i}{w_i}, c\right\} = x', \end{aligned}$$

thus (39) holds for $\mathbf{n} = 2$.

Induction step: Assume (39) holds true for $\mathbf{n} = k$ i.e

$$c\Big(1 - \prod_{i=1}^{k}(1 - \frac{x_i}{c})\Big) \leq w_1 \sum_{i=1}^{k} \frac{x_i}{w_i} \quad \forall\ (x_1, \cdots, x_k) \in [0, c]^k \tag{40}$$

and then prove (39) for $\mathbf{n} = k + 1$.

It is easy to verify that:

$$1 - \prod_{i=1}^{k+1}\left(1 - \frac{x_i}{c}\right) = \left(1 - \frac{x_{k+1}}{c}\right)\left(1 - \prod_{i=1}^{k}\left(1 - \frac{x_i}{c}\right)\right) + \frac{x_{k+1}}{c}$$

Multiplying with c and using (40), we get for any $(x_1, \cdots, x_k) \in [0, c]^{\mathbf{n}}$

$$\begin{aligned} \left(1 - \frac{x_{k+1}}{c}\right)c\left(1 - \prod_{i=1}^{k}\left(1 - \frac{x_i}{c}\right)\right) + x_{k+1} &\leq \left(1 - \frac{x_{k+1}}{c}\right)\left[w_1 \sum_{i=1}^{k} \frac{x_i}{w_i} + x_{k+1}\right] \\ &\leq \left(1 - \frac{x_{k+1}}{c}\right)\left[w_1 \sum_{i=1}^{k+1} \frac{x_i}{w_i}\right], \end{aligned}$$

as $w_1/w_{k+1} \le 1$, thereby proving (40) for any k by induction. Thus summarizing

$x_{ag} \le x' \le c$ for all $(x_1, \cdots, x_{\mathbf{n}}) \in [0, c]^{\mathbf{n}}$ and so also for $(x_1, \cdots, x_{\mathbf{n}}) \in (\{0\} \cup [\underline{x}^+, c])^{\mathbf{n}}$.

We now prove the first part of the Theorem. Writing the utilities of the aggregated player

$$U_{ag}(x_0, x_1, \cdots, x_n) = \begin{cases} 0 & x_{ag} \le \underline{x} \\ \frac{\mathbf{n}e^{x_{ag}}}{\mathbf{n}e^{x_{ag}} + e^{x_0}} - \gamma \sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i c} & \underline{x} < x_{ag} \le c \end{cases}$$

$$U_{ag}(x_0, x', 0 \cdots, 0) = \begin{cases} 0 & x' \le \underline{x} \\ \frac{\mathbf{n}e^{x'}}{\mathbf{n}e^{x'} + e^{x_0}} - \frac{\gamma x'}{c w_1} & \underline{x} < x' \le c. \end{cases}$$

For the first part, we have to show the following in the specified domain i.e. $(0 \cup [x^+, c])$

$$U_{ag}(x_0, x_1, x_2 \cdots, x_{\mathbf{n}}) \le U_{ag}(x_0, x', 0, \cdots, 0, x_{\mathbf{n}})^{\mathbf{n}}. \tag{41}$$

Trivial case: when $x_{ag} = 0$, $U_{ag}(x_0, x_1, \cdots, x_n) = 0$, and (41) holds trivially as in that case $x' = 0$ also and $U_{ag}(x_0, x', 0 \cdots, 0) = 0$.
We now consider the **non trivial case** i.e., when $\underline{x} < \underline{x}^+ x_{ag} \le x' \le c$. In this case, proving (41) is equal to showing

$$\frac{\mathbf{n}e^{x_{ag}}}{\mathbf{n}e^{x_{ag}} + e^{x_0}} - \gamma \sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i c} \le \frac{\mathbf{n}e^{x'}}{\mathbf{n}e^{x'} + e^{x_0}} - \frac{\gamma x'}{c w_1}. \tag{42}$$

Recall $w_j \ge 1 \ \ \forall\ j = 0, 1, \cdots, \mathbf{n}$, and observe the following

$$\sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i} \ge \frac{x'}{w_1}; \quad x' = \min\left\{c, w_1 \sum_{i=1}^{\mathbf{n}} \frac{x_i}{w_i}\right\}.$$

As $x_{ag} \le x'$ then for any non decreasing function say $f(.)$, $f(x_{ag}) \le f(x')$. In what follows, for any $x_0 \in (0 \cup [\underline{x}, c])$

$$\frac{\mathbf{n}e^{x_{ag}}}{\mathbf{n}e^{x_{ag}} + e^{x_0}} \le \frac{\mathbf{n}e^{x'}}{\mathbf{n}e^{x'} + e^{x_0}} \text{ and further } \sum_{i=1}^{\mathbf{n}} \frac{\gamma x_i}{c w_i} \ge \frac{\gamma x'}{c w_1}$$

And hence $\dfrac{\mathbf{n}e^{x_{ag}}}{\mathbf{n}e^{x_{ag}} + e^{x_0}} - \displaystyle\sum_{i=1}^{\mathbf{n}} \frac{\gamma x_i}{c w_i} \le \dfrac{\mathbf{n}e^{x'}}{\mathbf{n}e^{x'} + e^{x_0}} - \dfrac{\gamma x'}{c w_1}$.

Hence first part of the Theorem is proved.
Now the second part of the Theorem comes direct via taking supremum of over the closed domain.

$$\begin{aligned} \sup_{(x_0, x_1, \cdots, x_{\mathbf{n}}) \in [\underline{x}, c]^{\mathbf{n}}} U_{ag}(x_0, x_1 \cdots, x_{n-1}, x_{\mathbf{n}}) &= \sup_{x \in [\underline{x}, c]} U_{ag}(x_0, x, 0, 0, \cdots, 0) \\ &= \sup_{x \in [0, c]} \left(\frac{\mathbf{n}e^x}{\mathbf{n}e^x + \mathbf{1}_{\{x_0 > \bar{x}\}} e^{x_0}} - \frac{\gamma x}{c w_1} \right) \mathbf{1}_{x > \underline{x}}. \end{aligned}$$

This completes the proof. □

Proof of Lemma 7: From Lemma 5, $k^* > 1$ and $w_1 \geq w_j$ for all $j \geq 1$ implies,

$$w_1 \geq 2\gamma > \left(\frac{1}{\mathbf{n}} + 1\right)\gamma \text{ for any } \mathbf{n} \geq 2.$$

Case 1: If further $w_0 < (\mathbf{n}+1)\gamma$, then by (21), irrespective of other system parameters:

$$U_{ag}^{*min} = 1 - \frac{\gamma}{w_1}.$$

As $w_1 \geq w_j$ for any j and $k^* \geq 2$, we immediately have the following

$$U_{ag}^{*min} = 1 - \frac{\gamma}{w_1} > 1 - \sum_{j \in \mathcal{J}} \frac{\gamma}{w_j} \text{ for any subset } \mathcal{J} \subset \{1, 2, \cdots, \mathbf{n}\}, \text{ such that } \mathcal{J} = |k^*|$$

and

$$U_{ag}^{*min} = 1 - \frac{\gamma}{w_1} > \frac{k^*-1}{k^*} - \sum_{j \in \mathcal{J}} \frac{\gamma}{w_j} \text{ for any subset } \mathcal{J}, \text{ such that } \mathcal{J} = |k^* - 1|.$$

Thus by (15)-(16), $U_{ag}^{*min} > U_T^{*min}$, and hence BoC $\Psi > 0$.
Case 2: On the other hand, if $w_0 \geq (\mathbf{n}+1)\gamma$ then from (21):

$$U_{ag}^{*min} = \frac{\mathbf{n}}{\mathbf{n}+1} - \frac{\gamma}{w_1} > \frac{k^*-1}{k^*} - \frac{\gamma}{w_1},$$

when $k^* \leq \mathbf{n}$. Using similar logic as above we again have $U_{ag}^{*min} > U_T^{*min}$, and hence BoC $\Psi > 0$, when $k^* \leq \mathbf{n}$. When $k^* = \mathbf{n}+1$, because $\mathbf{n} \geq 2$ we have

$$U_{ag}^{*min} = \frac{\mathbf{n}}{\mathbf{n}+1} - \frac{\gamma}{w_1} > \frac{\mathbf{n}}{\mathbf{n}+1} - \sum_{j=1}^{\mathbf{n}} \frac{\gamma}{w_j} = U_T^{*min}.$$

Thus in all, $\Psi > 0$ once $k^* > 1$. □

Lemma 8. *For any $k \geq 2$, then*

$$\frac{\mathbf{s}_k}{k-1} > \frac{1}{\lambda_{k+1}} \textit{ if and only if } \frac{\mathbf{s}_{k+1}}{k} > \frac{1}{\lambda_{k+1}}.$$

Proof: It is easy to verify that:

$$\frac{\mathbf{s}_{k+1}}{k} - \frac{\mathbf{s}_k}{k-1} = \mathbf{s}_k\left(\frac{1}{k} - \frac{1}{k-1}\right) + \frac{1}{k\lambda_{k+1}}$$

Thus we have:

$$\begin{aligned} \frac{\mathbf{s}_{k+1}}{k} &= \frac{\mathbf{s}_k}{k-1}\left(1 - \frac{1}{k}\right) + \frac{1}{k\lambda_{k+1}} \\ &> \frac{1}{\lambda_{k+1}}\left(\frac{k-1}{k} + \frac{1}{k}\right) = \frac{1}{\lambda_{k+1}}. \end{aligned}$$

Similarly negation of the condition will provide negative result: if

$$\frac{\mathbf{s}_k}{k-1} \leq \frac{1}{\lambda_{k+1}} \text{ then } \frac{\mathbf{s}_{k+1}}{k} \leq \frac{1}{\lambda_{k+1}}.$$

□

6.1 Reference System for BR analysis

We provide the best response analysis for the exponential utility game in both cooperative and non cooperative scenario considering the continuous action set. Consider the following system with $d \geq 0$

$$U(x) = \frac{e^x}{d+e^x} - \frac{\gamma x}{cw}; \ x \ \in (\underline{x}, c] \text{ and } U(x) = 0 \text{ for all } x \leq \underline{x}.$$

Trivial cases: 1) If $\gamma = 0$, the utility function is always increasing in x. Thus the maximizer is c, further $U(c) > 0$ for any w and thus the best response, x^*, is c.
2)When $d = 0$, the best response is given as

$$x^* = \begin{cases} 0 & \text{if } \ w \leq \frac{\gamma \underline{x}}{c} \\ \underline{x}^+ & \text{else}, \end{cases} \tag{43}$$

where $\underline{x}^+ := \lim_{x \downarrow \underline{x}} x$. When we say $\underline{x}^+$ is the optimizer, it means we don't have a precise optimizer but any value bigger than and close to $\underline{x}$ provide a value close to the optimal value.
3) If $cw < 4\gamma$: On differentiating $U(x)$ w.r.t. x, we get

$$\frac{\partial U}{\partial x} = \frac{de^x}{\left(d+e^x\right)^2} - \frac{\gamma}{cw} = \frac{de^x(cw-4\gamma) - \gamma(e^x-d)^2}{\left(d+e^x\right)^2 cw}. \tag{44}$$

The derivative is always negative and hence maxima is at 0 or $\underline{x}^+$. Further directly checking the values at 0 and $\underline{x}^+$ we get the following:

$$x^* = \begin{cases} 0 \quad \text{if} \ \ w \leq \min\left\{ \frac{\gamma \underline{x}(d+e^{\underline{x}})}{ce^{\underline{x}}}, \frac{4\gamma}{c} \right\} \\ \\ \underline{x}^+ \ \text{if} \ \ \min\left\{ \frac{\gamma \underline{x}(d+e^{\underline{x}})}{ce^{\underline{x}}}, \frac{4\gamma}{c} \right\} < w \leq \frac{4\gamma}{c}. \end{cases} \tag{45}$$

4) When $d \geq e^c$, computing the second derivative is

$$\frac{\partial^2 U}{\partial x^2} = \frac{de^x\left(d+e^x\right)^2 - 2de^{2x}\left(d+e^x\right)}{\left(d+e^x\right)^4} = \frac{de^x\left(d-e^x\right)}{\left(d+e^x\right)^3}.$$

which is always positive in this case and hence there does not exist interior maxima. Thus the best response is one among $\{0, \underline{x}^+, c\}$ depending upon the functional values thereof.

Thus in all the above cases the best response is in one of the three boundary points. We now consider the left out case, which is more complicated than the previous ones.

Non trivial case, when $0 < d < e^c$**,** $cw > 4\gamma$**:** We first compute all the stationary points and then characterize each of them (maxima, minima, saddle point etc). The stationary points are the solution of $\partial U/\partial x = 0$ or equivalently the solution of (see (44)),

$$\gamma y^2 + (2\gamma d - dcw)y + \gamma d^2 = 0 \text{ with } y := e^x.$$

And the solutions to this quadratic equation are

$$y_1 = \frac{d(cw - 2\gamma) - d\sqrt{cw(cw - 4\gamma)}}{2\gamma}; \; y_2 = \frac{d(cw - 2\gamma) + d\sqrt{cw(cw - 4\gamma)}}{2\gamma}.$$

We show that y_1 is minimizer. Referring to the second derivative

$$d - y_1 = \frac{d}{2\gamma}\Big(- (cw - 4\gamma) + \sqrt{cw(cw - 4\gamma)}\Big).$$

thus

$$d - y_1 > 0 \implies \frac{\partial^2 U(x)}{\partial x^2}|_{x=y_1} > 0$$

Thus y_1 is minimizer. And by the way of similar expressions one can prove that y_2 is maximizer.

Further if $y_2 < c$ (note $y_2 > \underline{x}$), the best response x^* is one among $\{0, \underline{x}^+, c, y_2\}$ depending on the function values at these points. Thus one can have *interior maximizer only if* y_2 *is the maximizer and* $y_2 < c$. The domain of optimization is $\{0\} \cup (\underline{x}, c]$ and thus we are not interested in the case with $y_2 > c$. Thus we further study to check if $y_2 < c$ is a maximizer. Towards this we further study the first derivative $\partial U/\partial x$.

Recall $\partial U/\partial x = de^x/(d + e^x)^2 - \gamma/(cw)$, note that the first term is varying while the second is constant in above expression. If the minimum of the first term is greater than the second term, *the first derivative is always positive implying maxima of* U *is at boundary.* Towards utilizing this we consider the derivatives of the first term,

$$g(x) := \frac{de^x}{\Big(d + e^x\Big)^2}.$$

The first and second derivatives are:

$$\begin{aligned}
\frac{\partial g}{\partial x} = \frac{\partial^2 U}{\partial x^2} &= \frac{de^x\big(d + e^x\big)^2 - 2de^{2x}\big(d + e^x\big)}{\big(d + e^x\big)^4} \\
&= \frac{de^x(d + e^x) - 2de^{2x}}{\Big(d + e^x\Big)^3} = \frac{de^x(d - e^x)}{\Big(d + e^x\Big)^3} \\
\frac{\partial^2 g}{\partial x^2} &= d\frac{(de^x - 2e^{2x})(d + e^x)^3 - 3(d + e^x)^2 e^x(de^x - e^{2x})}{(d + e^x)^6} \\
&= de^x \frac{(d - e^x)(d - 2e^x) - e^{2x}(d + e^x)}{(d + e^x)^4}.
\end{aligned}$$

The only possible stationary point of g is at $x^* = \log(d)$ (i.e., when $e^{x^*} = d$) and the second derivative at this point

$$\frac{\partial^2 g}{\partial x^2}\Big|_{x^*=\log(d)} < 0$$

Thus the only stationary point is a maxima. Thus the minimizers of the function g are at one of the boundary points $\underline{x}$ or c. In all, if

$$\min\{g(\underline{x}), g(c)\} \geq \frac{\gamma}{cw}$$

then the derivative is always positive (see (44)) and then

$$\boxed{\max_{x\in\{0\}\cup[\underline{x},c]} = \max\{0, U(c)\} \text{ if } \min\{g(\underline{x}), g(c)\} \geq \frac{\gamma}{cw}.}$$

Thus we are left with the following case:

$$d < e^c, \; w > \frac{4\gamma}{c} \text{ and } \min\{g(\underline{x}), g(c)\} < \frac{\gamma}{cw}, \tag{46}$$

for which there is a possibility of interior point maximizer (or best response) for U.

Lemma 9. *Assume $w > 4\gamma/c$ and $d < e^c$. i) When $g(\underline{x}) < \gamma/cw$ then there exists at least one point $y \in \Big(\underline{x}, max\{\underline{x}, \log d\}\Big)$ such that $g(y) - \frac{\gamma}{cw} = 0$, which is minimizer of U.*

ii) When $g(c) < \gamma/cw$ then there exists at least one point $y \in \Big(max\{\underline{x}, \log d\}, c\Big)$ such that $g(y) - \frac{\gamma}{cw} = 0$, which is an interior (local) maximizer of U.

iii) Thus if $\gamma/cw < g(c)$, there exists no interior maximizer of U.

Proof: Let $h(x) := \partial U(x)/\partial x = g(x) - \gamma/(cw)$ and let $x^* = \log(d)$. (i) In this case $h(\underline{x}) < 0$ while (note $e^{x^*} = d$)

$$h(x^*) = \frac{d^2}{4d^2} - \frac{\gamma}{cw} > 0, \text{ as } w > \frac{4\gamma}{c}.$$

Thus by intermediate function theorem there exists an intermediate y at which $h(y) = 0$. Further the second derivative of U for this point is greater than zero, since $(d - e^y) > 0$. Thus it is a minimizer of U. Proof of (ii) is similar.
Towards (iii) observe that the derivative $\partial g/\partial x$ is negative for any $x > \log(d)$, thus $g(x) > \gamma/cw$ for all $x > \log(d)$ if $g(c) > \gamma/cw$. Thus there exists no zero of $h(x)$ for any $x > \log(d)$. It is clear that for any $x < \log(d)$ the second derivative $\partial^2 h/\partial^2 x = \partial g/\partial x > 0$ (thus these points can never be maximum of $h(.)$) and hence (iii) follows. □

In the case (iii) of the above lemma either c or 0 is the maximizer and hence the following:

Non Cooperative: Thus the summary with $0 < d < e^c$

$$x^* = \begin{cases} 0 & \text{if } \ w \leq \min\left\{\frac{\gamma \underline{x}(d+e^{\underline{x}})}{ce^{\underline{x}}}, \frac{4\gamma}{c}\right\} \\ \underline{x}^+ & \text{if } \ \min\left\{\frac{\gamma \underline{x}(d+e^{\underline{x}})}{ce^{\underline{x}}}, \frac{4\gamma}{c}\right\} < w \leq \frac{4\gamma}{c} \\ \textit{interior} & \text{if } \ \frac{4\gamma}{c} < w < \frac{\gamma}{c}\frac{(d+e^c)^2}{de^c} = \frac{\gamma}{cg(c)} \text{ and } U(x^*) \geq \max\{U(\underline{x}^+), 0\}, \\ c & \text{if } w \geq \max\left\{\frac{\gamma}{c}\frac{(d+e^c)^2}{de^c}, \frac{\gamma(d+e^c)}{e^c}\right\} \\ 0 & \text{if } w \geq \frac{\gamma}{c}\frac{(d+e^c)^2}{de^c}, \text{ but } w < \frac{\gamma(d+e^c)}{e^c} \\ \text{Boundary point} & \text{else.} \end{cases} \tag{47}$$

Thus we may have interior points as global maximizers only when $d < e^c$ and when

$$\frac{4\gamma}{c} < w < \frac{\gamma}{c}\frac{(d+e^c)^2}{de^c}.$$

If we exclude w in the following range

$$\boxed{\frac{4\gamma}{c} < w < \max_{e^{\underline{x}} \leq d < e^c} \frac{\gamma}{c}\frac{(d+e^c)^2}{de^c} = \frac{\gamma}{c}\frac{(e^{\underline{x}}+e^c)^2}{e^{\underline{x}}e^c},}$$

we would have BR only on boundaries.