Optimal Wireless Equalizers

A Thesis

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To Sree and Chicchu

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7. V. Kavitha and V. Sharma, " Tracking performance of an LMS-Linear Equalizer for fading channels using ODE approach", Technical report no:TR-PME-2005-11, DRDO-IISc programme on mathematical engineering, ECE Dept., IISc, Bangalore, Oct 2005. (downloadable from http://pal.ece.iisc.ernet.in/PAM/tech_rep05.html)

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Abstract

Wireless channels bring in new challenges unforeseen in wireline environments, e.g., ISI becomes time varying. Traditionally the equalizers (which nullify the ISI) were designed to optimize their performance (e.g., MSE) as a standalone component (equalizers were designed using sufficient amount of training that is transmitted prior to the actual data transmission). This approach was well suited for time invariant wireline channels. However with time varying nature of the wireless channels, training sequence needs to be sent frequently and the optimality of the above approach is questionable. Information theoretic measures are better suited to design an equalizer in such an environment. The first goal of the thesis is to obtain the best wireless equalizer (blind/semi-blind or training), using novel information-theoretic arguments (which study the trade-off between the BW lost and accuracy gained) for a given wireless scenario.

Any practical communication system uses training algorithms and they are still optimal, as long as one optimizes their performance as a standalone component (i.e., when the loss in BW due to the training sequence is not considered). A training algorithm commonly uses MMSE criterion to obtain the solutions (e.g., MMSE channel estimate, MMSE equalizer etc). The LMS, an iterative and computationally efficient algorithm, is often used to converge to the MMSE solution. With the advent of (time varying) wireless communications it becomes important to understand the tracking behavior of a wireless component (e.g., channel estimator, equalizer etc). Until now, the theoretical tracking behavior (of a channel estimator) is obtained by modeling the wireless channel either as a first order AR process (e.g., Random Walk model) or as a deterministic periodic process. Block fading model is also used to study a slow fading channel. However, an

AR(2) process models a fading channel better and can model most of the channel dynamics required for the receiver design. By modeling the underlying wireless channel as an AR(2) process, the second part of the thesis studies the tracking behavior of an LMS Linear/Decision Feedback Equalizer in training or decision directed (DD) mode. Along the way, the thesis also attempts to solve some long standing issues in a fixed (also a quasi static) channel environment like, obtaining an MMSE DFE, convergence of an LMS-DFE and convergence of a DD-LMS-LE.

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Chapter 1

Introduction

Communication systems have seen a paradigm shift from predominantly wireline systems to fast emerging wireless technologies. While wireless Cellular, LAN systems have already gained popularity, emerging WiMAX and 4G systems promise to revolutionize the way we communicate. As wireless systems emerge, they bring in new challenges unforeseen in wireline environments, e.g., multipath fading and time varying channel characteristics.

In any communication system, the transmitted signal is distorted through a channel, which can also introduce Inter Symbol Interference (ISI). These undesirable effects are cancelled at the receiver by an equalizer ([21], [32], [43]). The problem is aggravated with a time varying wireless channel, since the ISI also becomes time varying. Then one needs a time varying equalizer in the receiver to undo the channel distortions.

The equalizers are usually linear FIR (finite impulse response) filters, which may also use feedback. For a good performance, one needs to learn/design the optimal values of the equalizer coefficients. There are various ways to do it. One can either directly design an equalizer or can design it using an estimated channel model. For this purpose, often a known training sequence is sent prior to the actual data transmission. Then the MSE (Mean Square Error) optimal filter called Wiener Filter (WF) may be directly calculated, or calculated via an adaptive scheme like the Least Mean Squares (LMS) algorithm ([25]). One can also design an equalizer using a blind method, where no training sequence is used. A blind and a training method may also be combined to obtain a semi-blind equalizer.

Hence a large variety of equalizers are available and one needs to make a good choice for a given wireless scenario, i.e., one needs an *optimal wireless equalizer*.

When an equalizer is treated as a stand-alone component (for example when the BW lost due to training sequence is not considered), the best criterion for optimality would be *MMSE* (Maximum Likelihood/Viterbi equalizer is the optimal equalizer, but often one cannot realize this in practical systems because of its computational complexity). However to obtain a good MSE optimal equalizer one needs sufficient training sequence. Hence the above notion of optimality is good only for a time invariant channel (like wireline systems) where the training sequence is transmitted only once.

The above 'optimal' equalizer may not be optimal for a performance measure that pays penalty for the loss in BW. Due to time varying nature of the wireless channels, one will need frequent transmission of the training sequence resulting in significant loss of BW. Hence in a wireless channel, it is better to optimize a performance measure that considers the loss in BW. In such systems, optimizing information theoretic measures (for example *Maximizing information capacity* of a system with the equalizer) would be a better criterion.

However, in general, any practical wireless system uses the first notion of optimality to obtain a training based equalizer and hence is of practical importance. Hence, in this thesis we will be dealing with both the notions of optimality.

1.1 The Problem

An equalizer is an important component of a receiver and its performance critically affects the performance of the overall communication system. Therefore, extensive studies have been made over the years in designing and determining their performance (see [1], [3], [15], [21], [31], [32] [37], [43] and the references therein).

In this thesis we study the performance of various equalizers when used with a wireless channel. Due to multipath fading, the characteristics of a wireless channel change with time. This introduces new complications (compared to a wireline channel) which may make an equalizer well suited for wireline channels to perform quite poorly in a wireless scenario. For example, training based equalizers are extensively used in wireline systems. They are also used in wireless systems, but unlike in wireline systems, the training sequence needs to be sent frequently. Therefore, a significant ($\sim$ 18% in GSM) fraction of the channel capacity is consumed by the training sequence. The training based (MMSE) equalizer as a stand-alone component is still optimal (as here we are not bothered about the loss in BW due to training), but the communication system performance with this training based equalizer may not be the best. The usual blind equalization techniques have also been found to be inadequate [21] due to their slow convergence and/or high computational complexity. Because of this, the blind algorithms may not be able to *track* the time variations effectively in a wireless channel. Hence, one needs to look at semi-blind algorithms, which use some training sequence along with blind techniques (Chapter 7 of [21] and references therein), as they may provide the optimal solution.

However, Semi-blind/blind equalizers are believed to work unsatisfactorily in fading channels as compared to training based methods. But there is no theoretical study till today which confirms the belief. This is because, no systematic comparison of training, blind/semi-blind algorithms seems to be available so far. We aim to fill this gap.

Any practical communication system uses training algorithms and they are still optimal, as long as one optimizes their performance as a stand-alone component. With the advent of (time varying) wireless communications it becomes important to understand the tracking behavior of a wireless component. Thus, we next consider the tracking performance of the training equalizers. As mentioned above, a training based equalizer is most often designed based on Minimum Mean Square Error (MMSE) criterion. The MSE optimal equalizer, also called Wiener filter is given by,

$$\underline{\theta}^* = \arg\min_{\underline{\theta}} E\left[\underline{\theta}^t \underline{X} - s\right]^2,$$

where the vector $\underline{X}$ includes the channel outputs and decisions (decisions are included only in case of a DFE, the Decision Feedback Equalizer) while s represents the channel input and $\underline{\theta}^t$ is the transpose of vector $\underline{\theta}$. The Wiener filter (WF), often involves a matrix inverse computation and hence is computationally expensive. The Least Mean Square

(LMS) algorithm is used as an alternative. LMS is a very popular, easily implementable and a widely used iterative algorithm ([4], [25], [43]) usually designed to converge to the MMSE solution. It is given by,

$$\underline{\theta}_{k+1} = \underline{\theta}_k - \mu_k \underline{X}_k \left(\underline{X}_k^t \underline{\theta}_k - s_k \right),$$

where μ_k is a sequence of constants. However, prior to using the LMS, one needs to know whether it really converges to the MMSE solution, the WF. In addition, the LMS usually has the ability to track a time varying system if we keep $\mu_k \geq \mu > 0$ for all k ([25], [43]). In this thesis we will also address the corresponding issue for tracking (i.e., whether an LMS really tracks the instantaneous WF ?) while designing an equalizer via LMS for a wireless channel.

Equalizers are most commonly designed as linear FIR (finite impulse response) filters. One can easily see that, for an LE (Linear Equalizer) with a fixed channel, the unique WF is given by,

$$\underline{\theta}^* = R_{XX}^{-1} R_{Xs},$$

where $R_{XX} = E\left[\underline{X}\underline{X}^t\right]$ is the auto-correlation matrix of the channel outputs and $R_{Xs} = E\left[\underline{X}s\right]$ is the cross correlation vector. Although the training based LMS-LE has been studied extensively with wireline (fixed) channel (convergence to the WF for a fixed channel is shown in [4], [37]), its performance with a time varying channel is studied only via simulations and approximations. In this thesis we address the tracking problem theoretically.

Sometimes, an initial training based equalizer is improved upon using the decisions of the current symbols in place of the training sequence. This mode, called a decision directed (DD) mode, can improve the performance of the equalizer as well as the system using this equalizer (as now one can get better performance without using extra training sequence). Further, one can also track the channel variations via the DD mode itself (i.e., without using an extra training sequence). In this mode, after obtaining a sufficiently 'good' estimate of the equalizer from the training sequence, one uses the previous decisions in place of the training sequence to further improve the equalizer as

$$\underline{\theta}_{k+1} = \underline{\theta}_k - \mu_k \underline{X}_k \left(\underline{X}_k^t \underline{\theta}_k - \hat{s}_k \right),$$

where $\hat{s}_k$ is the decision of the current symbol. However, there can be decision errors, i.e. $\hat{s_k} \neq s_k$, which can affect the probability of error and the tracking performance of the system. This important issue of these equalizers is not well understood. We intend to take a close look at this problem.

Decision feedback equalizers (DFE) are nonlinear equalizers, which can provide significantly better performance than an LE (especially when there are deep nulls in the frequency response of the channel). A DFE feeds back the previous decisions of the transmitted symbols, to nullify the ISI due to them and makes a better decision about the current symbol. Although these equalizers have also been used for quite sometime, due to feedback their behavior is much more complex than LEs. Hence their performance is not well understood. Furthermore, there is no known technique to provide an optimal MMSE DFE even for a fixed channel ([10], [32], [43]). Since the LMS is used to provide MMSE solutions, it could possibly be used here too. It has indeed been used to obtain a DFE solution ([43]). However, it is not known whether an LMS-DFE converges to the WF even for a fixed channel. We will address this issue for fixed as well as time varying channels.

To study the tracking behavior, one needs a theoretical model of the fading channel. Auto Regressive (AR) processes have been shown to model such channels quite satisfactorily ([2], [28], [34], [60]). Further, lower order AR processes can capture most of the channel dynamics required for the receiver design ([28]). However, in literature till today, the tracking behavior of a wireless component (e.g., channel estimator) has been understood only using either a first order AR process, or a deterministic periodic process ([23], [36]) or using a block fading model (e.g., [1], [24]). Recently, it is shown in [28] that a second order AR process many a times can capture almost all the channel dynamics required for a receiver design, while an AR(1) process fails to do the same. Hence the tracking behavior may be better understood, if the underlying channel is modeled as an AR(2) process. However the AR(2) processes depend upon 2 previous values. Hence it would be difficult to theoretically deal with these processes (e.g., ODE approximation, which is extensively used for studying adaptive processes ([4], [6], [29]), is obtained only for processes that depend upon one previous value). In this thesis, we model the wireless channel by an AR(2) process and obtain the theoretical tracking performance.

1.2 Previous Work

In this section, we survey the literature related to the problems mentioned above. Pioneering work towards blind algorithms for channel equalization has been done by Sato in 1975 ([44]). Since then, there has been considerable work on blind algorithms ([15], [21], [22], [46], [57], [59]). Among these, CMA ([22]) has been one of the most studied and used algorithms. Its convergence/ ill-convergence (see [14], [15], [21], [47], [53] and the references therein), bounds on its MSE performance ([45]) and its distance from Wiener receivers ([65]), have been obtained. Also, blind, semi-blind and training algorithms are compared, for a given training length N_t, using the Cramer-Rao bounds by D-Carvalho and Dirk Slock (see Chapter 7 of [21] and the references therein). But these analysis do not discuss the tradeoff between accuracy in channel estimation and the bandwidth lost (for training in training and semi-blind methods), i.e., a larger training sequence provides a better equalizer enabling more data rate, but, reduces the time available to send the data.

An optimum training sequence for the training based methods have been obtained in [1] and [24]. They obtain a lower bound on the channel capacity and find the optimal training sequence length (and also placement in case of [1]). However, they do not provide a comparison with blind or semi-blind methods.

A recent survey on pilot assisted wireless transmissions is available in [58]. This survey gives a exhaustive list of the current literature which provides optimum training length, sequence and placement. Information theoretic methods (usually tight bounds on the capacity) are used to design the optimal training sequence ([58] and the references therein). The authors also mention that a semi-blind method (using CMA) can improve the performance. But this is achieved once again using Cramer-Rao bounds. As mentioned above the Cramer-Rao bounds are not sufficient for proper comparison of a blind algorithm with a training based method.

The convergence of the LMS-LE algorithm to the Wiener filter for a fixed channel has been studied in ([4], [37]). Hence it is known that for a fixed channel (e.g., wired channels), the performance of the limiting value of an LMS is not degraded much with respect to the Wiener filter as the length of the training sequence increases.

However, the above answer is not sufficient for a wireless channel because of its time varying nature. The tracking of a channel estimator has been extensively studied (see e.g., the convergence analysis in [4], [29], [35], [62], measures of tracking performance in [23], [36], [63] and variable step size algorithms in [30], [61] and the references therein). The channel estimates can be used at any time to obtain an equalizer.

Tracking behavior of an LMS equalizer is obtained only via simulations, approximations and upper bounds on probability of error (see, e.g., [25], [28], [34], [54]). We are not aware of any theoretical analysis except [36] which discusses the degree of non-stationarity and optimum step-size. However, the assumptions made in this analysis are unrealistic. They assume deterministic bounded or random IID zero mean time variations. Also they assume that the noise sequence after the equalizer, resulting from residual ISI and the noise passing through the equalizer, is IID.

In [37], it has been shown that the DD-LMS-LE for a fixed channel converges to the WF almost surely, if the initializer is sufficiently close to the WF and if the noise is zero. The authors also observe that the DD attractors are away from the WFs when the noise is non-zero and when the equalizer length is one. However, one needs to understand the DD algorithm under more practical scenarios. Benveniste et al. ([4]) have obtained an ODE approximation for a fixed channel DD-LMS-LE. However they have not studied its attractors. The existence of undesirable local minima are established in [38], [41]. In [21] (chapter 11 and the references therein) the convergence properties (noiseless) and initialization strategies (to 'open' eye) are discussed.

Performance analysis of a DFE is more complicated because of the feedback loop. Existence of a hard decoder inside the feedback loop, makes the study all the more difficult. A DFE mainly exploits the finite alphabet structure of the hard decoder output ([19], [32]) and hence the hard decoder cannot be ignored.

For a DFE, statistics of the previous decisions are not known. Hence the first hurdle: there is no known technique to compute a DFE Wiener filter. One gets around this problem by assuming perfect decisions (see, e.g., [13], [32], [56]) and design a Wiener filter. For convenience, for the rest of the thesis, we will call it IDFE (Ideal DFE). The IDFE often outperforms the linear Wiener filter significantly ([3], [43], [54]). But it is generally believed that the true optimal DFE Wiener filter (considering the decision errors) can significantly outperform even this.

Another way to obtain an optimal solution is to replace the feedback filter at the receiver by a precoder at the transmitter ([10], [43]). This way one can indeed obtain the optimal filter (one no longer has the problem due to the decision errors). But this requires the knowledge of the channel at the transmitter. However for wireless channels, which are time varying, this is often not an attractive solution ([32] [43]).

Some research has been carried out to deal with the decision errors. Either the distribution of the decision errors was approximated in designing an MSE optimal filter (IDFE being one such example) or some other appropriate criterion was used to get the optimal filter considering the errors in decisions. For example, in [50], the authors approximated the errors in decisions with a WGN (White Gaussian Noise) uncorrelated with the input data and obtained the DFE Wiener filter. But as is stated in the paper this approximation is not realistic. In [19], the authors obtain an H^∞ optimal DFE considering decision errors. However it is not compared to the DFE Wiener filter.

LMS-DFE could possibly be used to obtain an optimal DFE. However, its performance is not understood theoretically. For example, the convergence behavior of an LMS-DFE even for a fixed channel is not understood. Trajectory of the LMS-DFE algorithm without a hard decoder in the feedback loop has been approximated by an ODE in [29]. However this ODE does not approximate the DFE with a hard decoder. Beneveniste et al. ([4])

have shown the ODE approximation of an LMS-DFE with a hard decoder. But the ODE obtained by them is not explicit enough and hence does not provide clear insight into the behavior of the LMS-DFE algorithm. Furthermore, they do not relate the attractors of this ODE to the DFE Wiener filter.

While understanding the tracking behavior of a channel estimator, a time varying channel has been modeled either as a first order AR process (Random Walk model) or as a deterministic periodic process ([23], [36]). Block fading model is also used to study a slow fading channel ([1], [24]). However, an AR(2) process models a fading channel better ([28]). It has been used to improve channel tracking algorithms in [28], [34] and [60]. But it has not been used to obtain better performance analysis of a wireless component.

1.3 Contributions of the thesis

We attempt to obtain an optimal wireless equalizer (either optimal in MSE sense, like DFE-WF, LE-WF etc, or optimal in capacity sense) by theoretically analyzing various wireless equalizers described above. In this connection, we obtain answers to the following questions under certain conditions:

- What is the optimal length of the training sequence ?
- How does a blind/semi-blind equalizer compares itself with respect to a training equalizer ? What is the optimal equalizer ?
- Does LMS really track the instantaneous Wiener filter ?
- When can we use the "decisions of the current symbol" in place of the training sequence ?

Prior to answering these questions, one needs an appropriate model for the wireless channel. A wireless channel can be fast/slow fading and frequency selective/nonselective ([43]). We will be dealing with a slow fading, frequency selective channel in our thesis. We model these channels either as block fading channels or as AR processes. Our AR processes can also model some fast fading channels. In particular we will use AR(2) processes. We obtain the following:

- The optimum training sequence length for the training based and semi-blind algorithms.
- Among the algorithms compared, semi-blind algorithms provide the best performance.
- A training based LMS linear equalizer tracks the instantaneous WF for a stable/unstable channel.
- A decision directed LMS linear equalizer stays close to the instantaneous WF whenever the SNR is high and when it is properly initialized. Hence, we conclude that a DD-LMS-LE can be used to track/obtain the WF under high SNR conditions (without the use of training sequence). However, at low SNRs, the DD attractors are away from the WF.
- A training based LMS decision feedback equalizer stays close to the instantaneous WF at high SNRs. Thus, we conclude that at least under high SNR, LMS can be used to obtain the optimal DFE. We also show that the 'Optimal' DFE obtained by ignoring the decision errors can perform much worse than the LMS DFE even under high SNR (even after designing the former with perfect channel estimate).

Our general tool for analysis is the ODE (Ordinary Differential Equation) approximation often used in stochastic approximation methods ([4], [6], [29] etc).

In Chapter 2, we provide an ODE approximation for an AR(2) process. This appears to be the first time when an ODE approximation is obtained for a recursive equation where the current value depends upon two previous values (a second order difference equation). Unlike previous ODE approximations available in literature ([4], [6], [29]), now depending upon the parameters, the approximating ODE may be a first order or a second order ODE.

We then obtain an ODE approximation for a general system, whose components may depends on two previous values (like in an AR(2) process) in Appendix I, provided at the end of the thesis. Using this, we obtain the ODE approximation for an LMS-LE, DD-LMS-LE or LMS-DFE while tracking an AR(2) process. This ODE approximation is instrumental in obtaining the last three results.

We obtain the comparison of the training based equalizer with (semi) blind equalizers in Chapter 3. Here, we use novel information theoretic arguments to study the trade-off between the BW lost in sending a training sequence versus the BW gained due to *better* channel estimate obtained via a longer training sequence. We use the most popular blind algorithm, CMA, for our comparison purposes. We define a 'composite' channel for each equalizer, and use its capacity as a measure to obtain the optimum training sequence length as well as to compare the three equalizers. The CMA ODE approximation of [47] has made it possible to obtain the capacity.

An LMS-LE is studied in Chapter 4. For an LMS-LE, the error between the instantaneous WF and the LMS trajectory is shown to reduce polynomially/exponentially to zero with time, using the ODEs approximating the stable/unstable channel (AR(2) process) and the LMS-LE trajectories. This error remains bounded for a marginally stable channel (stable, unstable and marginally stable channels are explained in Chapter 2). For stable channels we also show that the MSE of an LMS-LE converges to the instantaneous MMSE exponentially.

For a fixed channel, using implicit function theorem, we obtain the existence of DD-attractors close to the WF at high SNRs. We use similar techniques to compare the LMS-DFE attractors with that of the DFE Wiener filters.

1.4 Organization of the thesis

Chapter 2 studies the AR modeling of a Wireless channel. It also provides the convergence of AR(2) processes to ODEs. The blind/semi-blind equalizers are compared with training equalizers in Chapter 3. It also provides the optimal training sequence length. Tracking behavior of LMS Linear equalizers is studied in Chapter 4. DD-LMS-LE is studied in Chapter 5. LMS DFE for a fixed channel is dealt with in Chapter 6 while its tracking behavior is obtained in Chapter 7. Chapter 8 concludes the thesis. All major proofs are available in appendices included at the end of the corresponding chapters. ODE approximation of a general system (with some of its components depending on two previous values) is obtained in Appendix I, included at the end of the thesis. References are included at the end of the thesis.

Chapter 2

Channel models for a Wireless Channel

A wireless channel can be fast/slow fading and frequency selective/non-selective ([43]). We will consider a slow fading, frequency selective channel in this thesis. This model includes a fixed frequency selective channel (i.e., a time invariant channel with ISI) as a special case. One can model a slow fading wireless channel either as a block fading channel or as a continuously varying channel with a slow drift. We will model the continuously varying channel by an AR process. Our AR processes can also handle some fast fading channels.

Auto Regressive (AR) processes have been shown to model wireless channels quite satisfactorily ([2], [28], [34], [60]). We study these processes in detail. We will show that the trajectory of an AR(2) process can be approximated by a system of ODEs. Using these ODEs we show that the trajectory of an AR(2) process can be approximated by an exponential, polynomial, cosine, hyperbolic-cosine, exponential cosine or an exponential hyperbolic-cosine waveform, if the trajectory is suitably scaled in time and space. These ODEs will be used to study the performance of the wireless equalizers in Chapters 4, 5 and 7.

The chapter is organized as follows. Section 2.1 describes the Block-fading model, which will be used in Chapter 3. Section 2.2 provides the AR process model and its approximation with an ODE. In Section 2.3 we demonstrate via some examples that the ODEs can indeed provide a good approximation. The appendices provide the proofs.

2.1 Block-fading model

In a block-fading model, the channel is approximated with a constant value over a frame/block and is assumed to change over to an independent value in the next frame. This is a very commonly used model for a slow fading channel ([1], [24]). For example, this model is suitable in a TDMA wireless system like GSM. In GSM, a packet of a voice call is transmitted in one slot in each frame (consisting of 8 slots). The fading during a slot may be approximately constant. However, a frame later, the fading might have changed significantly.

For this model, the training sequence is transmitted once for every frame, and an equalizer is computed for every frame. We use this model for comparing the blind/semi-blind equalizers with training based equalizers in Chapter 3. The channel co-efficients across frames are assumed to be independent and Gaussian distributed.

2.2 AR process model

A slowly varying channel can sometimes be better modeled (than by block fading model) by a continuously varying process with small variations at a time. To model such a channel, we use an Auto Regressive processes of order p (AR(p) processes),

$$\underline{Z}_k = \sum_{l=1}^{p} d_l \underline{Z}_{k-l} + \mu \underline{W}_k, \quad k \geq 1$$

where $\{W_k\}$ is an IID sequence independent of initial conditions Z_0, Z_{-1}, $\cdots$, Z_{p+1} and μ and d_l, $l = 1, \cdots, p$ are some constants. These processes have been shown to model the slowly varying channels quite satisfactorily ([2], [28], [34], [60]).

Exact modeling of the wireless channel time variations using an AR process is not possible (the autocorrelation function of an AR process is rational, while the wireless channel autocorrelation functions are usually not rational). Nevertheless, AR processes especially for a large p, are shown to model a wireless channel quite accurately in [2]. However, first few correlation terms of the channel are more important for the design of receiver algorithms ([28]) and hence low order AR models may be sufficient.

One can estimate the $p+1$ parameters of the AR(p) process (the p coefficients and the variance of the noise, μ) from the autocorrelation function of the WSSUS model of the fading channel (given by a zero order Bessel function of the first kind, $\mathcal{J}_0(2\pi f_d T k)$, with k the lag, f_d the maximum Doppler frequency and $1/T$ the data rate) using linear Yule-Walker equations ([2], [28], [34]).

In [28], it is shown that a second order AR process correlation matches the true Bessel correlations accurately almost up to 20 lags while the first order AR process may match satisfactorily only for very small lags. Hence an AR(2) process can be a much better model than an AR(1) process. It includes the frequently used models in literature, e.g., the Filtered Random Walk model ([34]) obtained with $d_1 + d_2 = 1$ and $|d_2| \leq 1$ and the Autoregressive Second Order model ([34], [28]) obtained with $d_1 = 2\rho cos w_0$, $d_2 = -\rho^2$, where ρ and w_0 represent the degree of damping and the dominating frequency respectively. Also, an AR(2) process sometimes can capture almost all of the channel dynamics that are required for the receiver design ([28]). Hence we *model the wireless channel by an AR(2) process while studying the tracking behavior of the LMS algorithm* in Chapters 4, 5 and 7.

An ODE approximation is often used to study an adaptive process ([4], [29]). An AR(2) process depends on two previous values and an ODE approximation for such a process does not seem to be available in literature. In the next section, we will approximate the trajectory of an AR(2) process by an ODE. We will show in Section 2.3 that the ODE approximation is quite accurate for $f_d T$ upto 0.04, which can for example correspond to a mobile speed of 150 Km/h at 2.4-GHz transmission with symbol time equal to 100μs. We then suggest an ODE whose solution can approximate a more general AR(p) process.

The values of d_1, d_2 determine the stability of the channel. The channel is stable, unstable or marginally stable respectively if its poles are inside, outside or on the unit circle ([25]).

We first study a general system in Appendix A (whose current value depends on two previous values) and use this to obtain an ODE approximation for the AR(2) process. We obtain a first order ODE whose solution approximates the general system (and hence the AR(2) process) with $d_2 \in (-1, 1]$. The AR(2) processes corresponding to this case includes the stable channels and a class of unstable channels. However for d_2 close to -1, we show that a second order ODE better approximates the above system. Later on, we obtain another second order ODE which approximates the process with $d_2 = -1$. The AR(2) process in this case will include a marginally stable channel.

2.2.1 ODE approximation of an AR(2) process

To begin with, we study a general system given by,

$$Z_{k+1} = (1 - d_2)Z_k + d_2 Z_{k-1} + \mu H(W_k, Z_k).$$

In Appendix A, we obtain a system of ODEs (2.5), (2.7), (2.9) for $d_2 \in (-1, 1]$, $d_2 = -1$ and d_2 close to -1 respectively, which approximate the above system under the assumptions:

A.1 W_k is an IID sequence.

A.2 $h(Z) = E\left[H\left(W_k, Z(t)\right)\right]$ is a C^1 function.

A.3 For any compact set Q, there exists a constant $C(Q)$, such that

$$E|H(Z, W)|^2 \leq C(Q) \quad \text{for all } Z \in Q,$$

where the expectation is taken wrt W.

Using the results of Appendix A, we obtain an ODE approximation for the AR(2) process. One can rewrite the AR(2) process as,

$$Z_{k+1} = (1 - d_2)Z_k + d_2 Z_{k-1} + \mu \left(W_k + \frac{d_1 + d_2 - 1}{\mu} Z_k \right).$$

Hence the AR(2) process is an example of the above general system. One can easily see that the assumptions **A**.1–**A**.3 are satisfied by the process if $E\left[|W_k|^2\right] < \infty$. We will assume this in the thesis. Hence using the three theorems of Appendix A, an AR(2) process $\{Z_k\}$ with a small μ can be approximated by the solution of the following system of ODEs (the approximation would be good as long as $d_1 + d_2$ is close to 1),

$$\begin{aligned} (1+d_2)\, \dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \in (-1, 1], \\ \frac{d^2 Z(t)}{dt^2} &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 = -1, \\ \frac{d^2 Z(t)}{dt^2} + \eta_1\, \dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \text{ is close to } -1, \end{aligned} \tag{2.1}$$

where $\eta \stackrel{\triangle}{=} \frac{d_1+d_2-1}{\mu}$ and $\eta_1 \stackrel{\triangle}{=} \frac{1+d_2}{\sqrt{\mu}}$. Note that with d_2 close to -1, two ODEs approximate the AR(2) process (we will discuss this in detail later).

The channel ODE (2.1) has a unique solution (which is bounded for any finite time) given by (with $a \stackrel{\triangle}{=} \frac{1+d_2}{2\sqrt{\mu}} = \frac{\eta_1}{2}$),

$$Z(t) := \begin{cases} C_1 e^{\frac{\eta}{1+d_2} t} - \frac{E(W)}{\eta}, & \eta \neq 0, \quad d_2 \in (-1, 1], \\ \frac{E(W)}{1+d_2} t + C_1, & \eta = 0, \quad d_2 \in (-1, 1], \\ C_1 cosh(\sqrt{\eta}\ \ t) - \frac{E(W)}{\eta}, & \eta > 0, \quad d_2 = -1, \\ C_1 cos(\sqrt{|\eta|}\ \ t) - \frac{E(W)}{\eta}, & \eta < 0, \quad d_2 = -1, \\ \frac{E(W)}{2}\ \ t^2 + C_1, & \eta = 0, \quad d_2 = -1, \\ C_1 e^{-2at} + \frac{E(W)}{2a}\ \ t, & \eta = 0, \quad d_2 \text{ close to } -1, \\ C_1 e^{-at} cos(\sqrt{|\eta| - a^2}\ \ t) - \frac{E(W)}{\eta}, & \eta < -a^2, \quad d_2 \text{ close to } -1, \\ C_1 e^{-at} cosh(\sqrt{\eta + a^2}\ \ t) - \frac{E(W)}{\eta}, & otherwise, \end{cases} \tag{2.2}$$

where the constant C_1 is chosen appropriately to match the initial condition of the approximated AR(2) process (as given in the theorems of Appendix A). The approximation of $\{Z_k\}$ by the solution $Z(t)$ holds in the following sense: Given any $T > 0$ and $\delta > 0$

$$P\left(\sup_{0 \leq k \leq \frac{T}{\mu^\alpha}} |Z_k - Z(k\mu^\alpha)| > \delta \right) \to 0$$

as $\mu \to 0$, where $\alpha = 1$ if $Z(t)$ is the solution of a first order ODE and $\alpha = 1/2$ if it is the solution of a second order ODE.

When d_2 is close but not equal to -1, two ODEs approximate the same AR(2) process. This is an important case and results when a second order AR process approximates a fading channel with a U-shaped band limited spectrum. It is obtained for small values of $f_d T$.

For example if f_dT equals 0.04, 0.01 or 0.005 the channel is approximated by an AR(2) process with (d_1, d_2, μ) equal to $(1.9707, -0.9916, 0.00035)$, $(1.9982, -0.9995, 1.38e^{-6})$ and $(1.9995, -0.9999, 8.66e^{-8})$ respectively ([2], [28])). One could approximate such an AR(2) process with the first order ODE of (2.1). However this approximation will not be very accurate and will require μ to be very small. In this case, the second order ODE approximates the channel trajectory better. We will plot these approximations in Section 2.3.

In the following, we summarize the above ODE approximations relating them to the stability properties of the AR(2) process.

Stable AR process

A stable AR(2) process has all the poles inside the unit circle ([25]) and hence satisfies, $d_1 + d_2 < 1$, $d_1 - d_2 > -1$ and $|d_2| < 1$. In this case, $\eta = (d_1 + d_2 - 1)/\mu < 0$ and the solution of the first order channel ODE of (2.1) will be an exponential curve for some constant vectors C_1, C_2,

$$Z(t) = C_1 e^{\frac{\eta}{1+d_2} t} + C_2.$$

This has a unique global attractor $Z_* = C_2$.

As discussed in the previous section, stable channels with d_2 close to -1 are better approximated by,

$$C_1 e^{-at} cos(\sqrt{|\eta| - a^2} \;\; t) + C_2.$$

It is easy to see that even this ODE has C_2 as an unique global attractor (note that for a stable channel $a > 0$).

Unstable channels

An unstable channel has the poles outside the unit circle. The AR process in this case is approximated by,

$$Z(t) = C_1 r(t) + C_2$$

where with $p = 1$ or 2, $r(t)$ is given by,

$$r(t) = \begin{cases} e^{\eta t}, & d_1 + d_2 > 1, \\ t^p, & d_1 + d_2 = 1. \end{cases}$$

Note that the hyperbolic cosine curves are also unstable. These curves are equal to $C\left(e^z + e^{-z}\right)$, for some z and hence their behavior can be understood by studying the dominating term, the raising exponential.

Marginally stable channels

It has all its poles on the unit circle. An AR process with $d_2 = -1$, $d_1 < 2$ results in a marginally stable channel. The AR process in this case is approximated by,

$$Z(t) = C_1 cos(\sqrt{\eta}\, t) + C_2.$$

AR(p) process :

Following a similar procedure, one can try to show that an AR(p) process given by (when $1 + \sum_{i=2}^{n}(i-1)d_i \neq 0$),

$$Z_{k+1} = \sum_{i=1}^{p} d_i Z_{k+1-i} + \mu W_k \tag{2.3}$$

can be approximated for small values of μ, with the solution of the ODE,

$$\dot{Z}\,(t) = \frac{1}{1 + \sum_{i=2}^{n}(i-1)d_i}\left[E(W) + \frac{\sum_{i=1}^{n} d_i - 1}{\mu} Z(t)\right].$$

Complex Channels

A time varying complex channel can be written in the equivalent double dimensional real domain (a double dimensional real vector is formed from any complex vector by stacking all the real components first followed by the complex components). The complex channel modeled by an AR(2) process, can now be approximated by ODEs of this chapter after converting it to the equivalent real double dimensional vector.

2.3 Examples

In this section we illustrate the theory developed so far using some examples. We consider a three-tap channel. We assume that $W_k \sim \mathcal{N}(Mn, 1)$ for different values of mean Mn.

In Figure 2.1, we plot the actual trajectory and the ODE solution of a stable AR(2) process, with $d_1 = 0.6$, $d_2 = 0.3995$ and step size $\mu = 0.001$. It is clear from the figure that, the ODE solutions approximate the actual trajectories well.

In Figure 2.2, we plot the channel trajectories for $d_1 + d_2 = .8$, which is away from 1. Now we have taken $\mu = 0.01$. We can see that when $1 - (d_1 + d_2)$ is large, the AR process converges faster to the attractor and this channel will be like a channel without drift. In this case, it is very close to an IID Gaussian random variable. This is also evident from the theory, as, in this case $|\eta|$ will be larger. Also it is evident from the figures that the approximation is accurate even for $d_1 + d_2$ away from 1.

In Figure 2.3, we plot an unstable AR(2) process. We set $d_1 = 0.9$, $d_2 = 0.1001$ and $\mu = 0.001$. We see that the AR process is unstable, as $d_1 + d_2 = 1.001 > 1$. We observe that the AR(2) parameters are diverging to infinity exponentially, as shown by the theory. We can also see that the ODE approximation is very accurate even in this case.

In Figure 2.4, we consider a special case, when $d_2 = -1$. We set $d_1 < 2$, to get a marginally stable AR(2) process, whose trajectory is approximated by a cosine waveform. The channel ODE once again well approximates the actual trajectories.

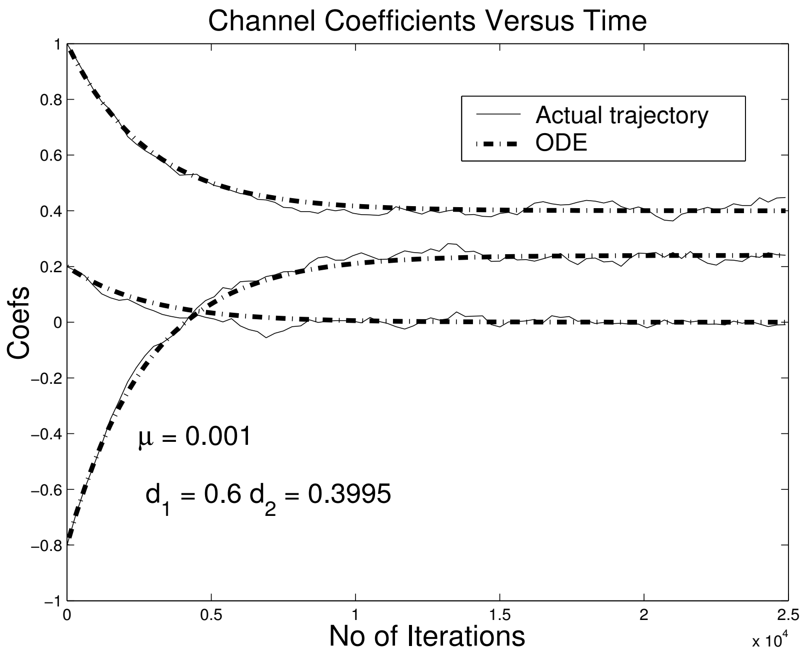

Figure 2.1: Trajectories of a stable AR(2) process

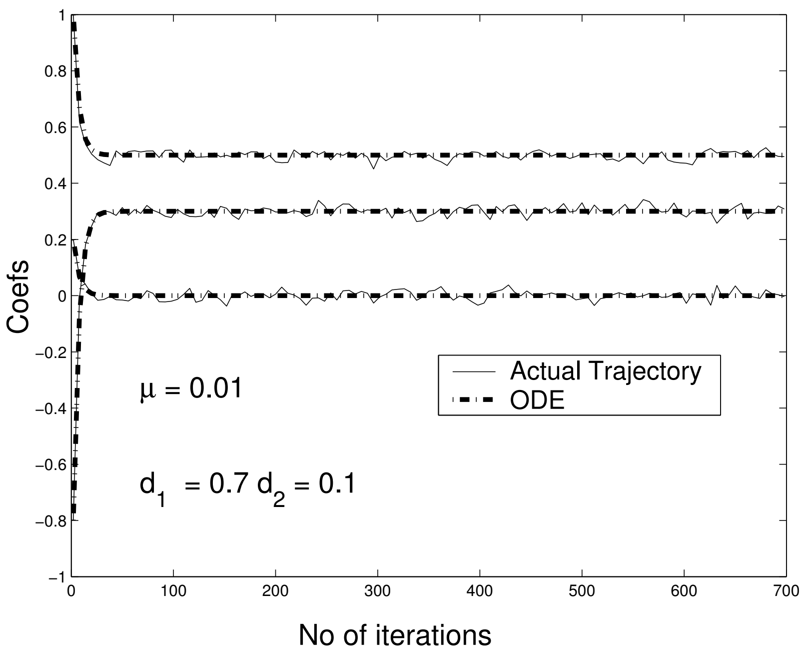

Figure 2.2: Trajectories of a stable AR(2) process with $d_1 + d_2 = 0.8$

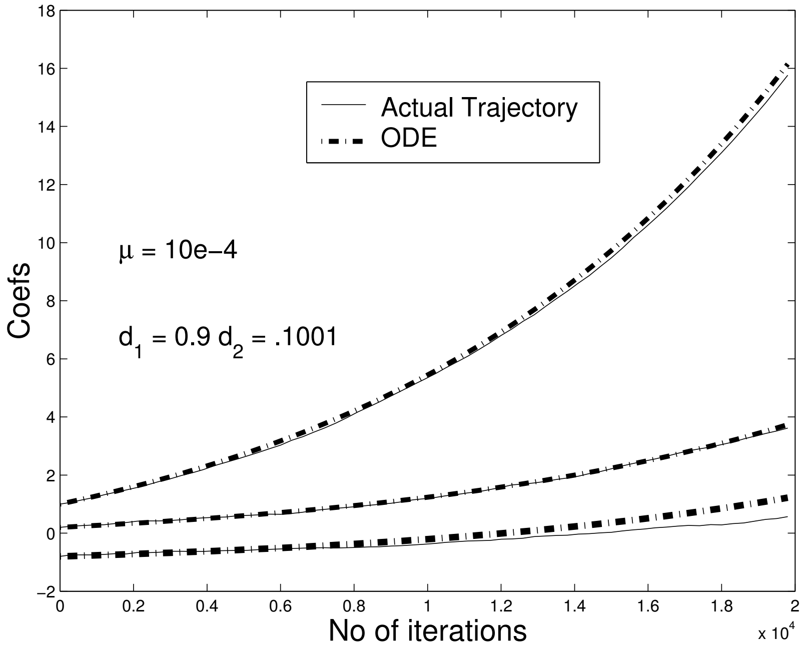

Figure 2.3: Trajectories of an AR(2) process when $d_1 + d_2 > 1$, i.e. unstable

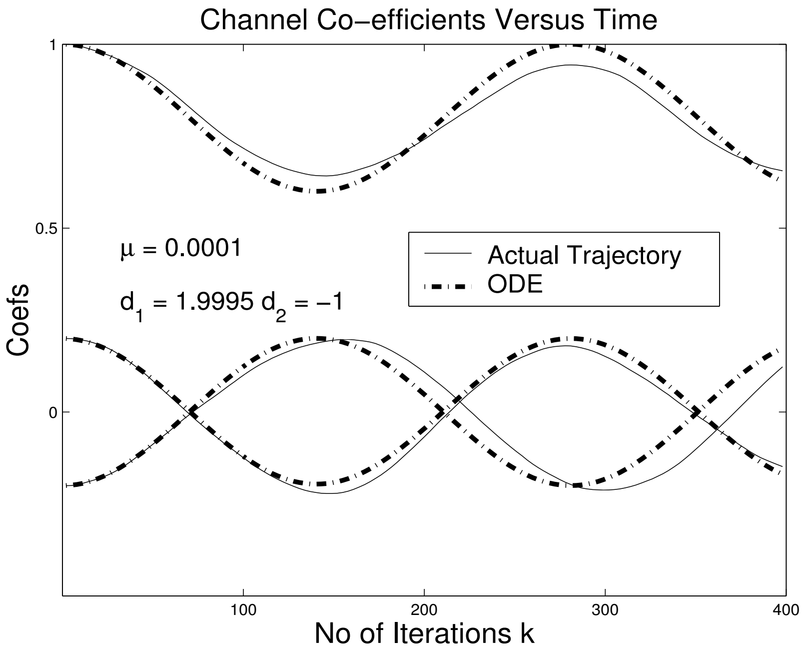

Figure 2.4: Trajectories of an AR(2) process with $d_2 = -1$ and $d_1 < 2$

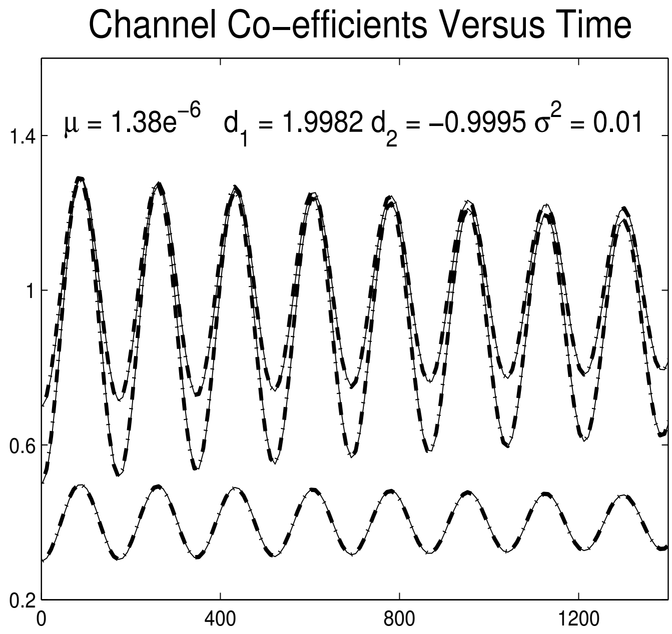

Figure 2.5: Trajectories of an AR(2) process with d_2 close to -1 and $d_1 < 2$ for $f_d T = 0.01$.

In Figures 2.5, 2.6, we consider another special case, when d_2 is close to -1. In this case, as is shown theoretically, a better ODE approximation is obtained by a second order ODE. We set $f_d T = 0.01$, $f_d T = 0.04$ (e.g., symbol time $T = 100\mu$s, at 2.4-GHz transmission with mobile speeds 45 Km/h or 150 Km/h) respectively. We get a stable AR(2) process, whose trajectory is approximated by an exponentially decaying cosine waveform. The second order ODE well approximates the actual trajectories for $f_d T$ upto 0.04.

In Figure 2.7, we are plot the channel coefficients of a complex channel. We are considering a three-tap channel. In these figures, we plot both the real and imaginary parts of the coefficients separately. We see that the ODE approximation is very close even for a complex channel.

Finally, we plot an AR(4) process in Figure 2.8. The parameters of the process are provided in the figure. We see that the ODE solution once again approximates the actual

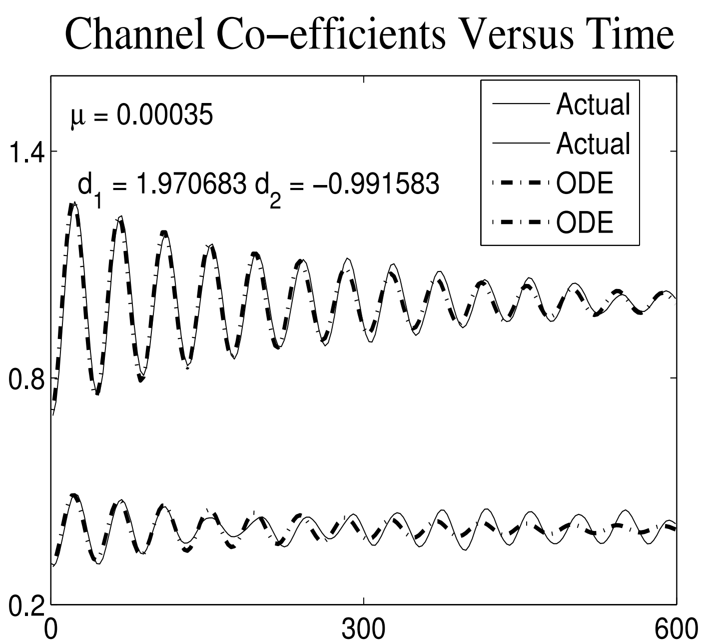

Figure 2.6: Trajectories of an AR(2) process with d_2 close to -1 and $d_1 < 2$ with $f_d T = 0.04$.

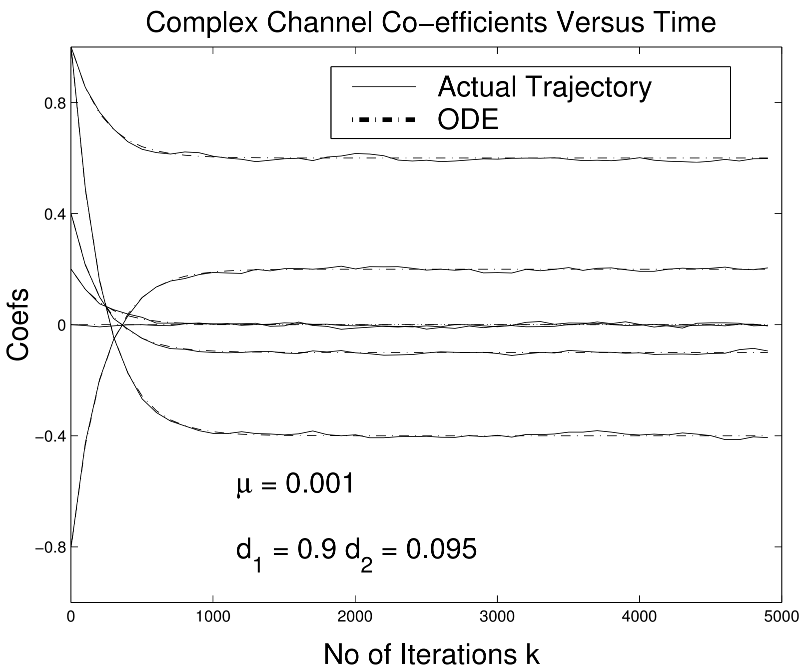

Figure 2.7: Trajectories of a Complex Channel.

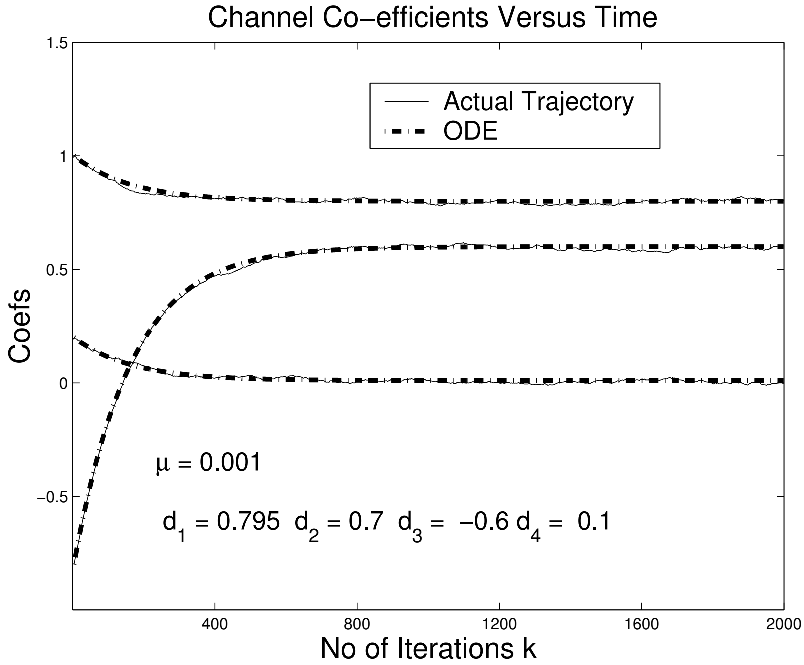

Figure 2.8: Trajectories of an AR(4) process

trajectories well. Do note that, as in the AR(2) process case (e.g., when $d_2 = -1$), there will be different values of d_1, d_2, d_3, d_4 for which the exponential curve will not be approximating the channel trajectory.

Conclusions

In this chapter we studied a wireless channel modeled by an AR(2) process. We first obtained different ODEs for a general system whose components depend upon 2 previous values. We then obtained a system of ODEs whose solutions approximate the trajectory of an AR(2) process based on the value of d_2. We showed that (Theorem 2.1) with $d_2 \in (-1, 1]$ (all stable and a class of unstable channels), the AR(2) process can be approximated by a first order ODE. The solutions of this ODE are either exponential or linear curves. When $d_2 = -1$, the AR(2) process either will have poles on the unit circle (when $d_1 \leq 2$) or outside the unit circle ($d_1 > 2$). We showed that (Theorem 2.2) a second order ODE approximates this class of AR(2) processes. With $d_2 = -1$, it is shown that the AR(2) process can be approximated by a cosine curve when $d_2 < 2$, by a square curve when $d_2 = 2$ and by a hyperbolic cosine curve when $d_2 > 2$.

We also showed that with d_2 close to -1 (a very important case resulting when a second order AR process approximates the well-known U-shaped band limited spectrum of the received fading signal with lower Doppler rates [2]), all classes of channels, stable/unstable channels, are better approximated by a second order ODE (Theorem 2.3). An exponentially raising/decaying, cosine/hyperbolic-cosine curve approximates this kind of a channel.

We suggested a system of ODEs, whose solution can approximate an AR(p) process with $p > 2$.

For simplicity, we have only shown the theory for the case when all the taps of the channel are given by the same AR(2) process. But in all the examples of our simulations we have used different AR(2) processes for different channel taps (we only varied the mean $E(W)$ in our simulations, one can also use different d_1, d_2 for different taps). The theory will easily go through even for the general case.

Appendix A

We study a general system given by,

$$Z_{k+1} = (1 - d_2)Z_k + d_2 Z_{k-1} + \mu H(W_k, Z_k). \tag{2.4}$$

We will approximate it with three different ODEs depending upon the value of d_2 under the assumptions **A**.1-**A**.3. Note that under the assumption **A**.2, the ODE, $\dot{Z} = h(Z(t))$, has a unique local solution for any initial condition.

When $d_2 \in (-1, 1]$ **:**

Under this condition, we will show that the above general system (2.4) can be approximated by the solution of the ODE,

$$\dot{Z}(t) = \frac{1}{1+d_2} E\left[H\left(W, Z(t)\right)\right] = \frac{1}{1+d_2} h(Z(t)). \tag{2.5}$$

We represent the solution of ODE (2.5) with initial condition $Z(t_0) = A$ by $Z(t, t_0, A)$ for any $t \geq t_0$.

Let Q_1 and Q_2 be any two compact sets such that $Q_1 \subset Q_2$ and we choose $T > 0$ such that there exists a $\delta_0 > 0$ satisfying,

$$d\left(Z(t, 0, A), Q_2^c\right) \geq \delta_0, \tag{2.6}$$

for all $A \in Q_1$ and for all $t \leq T$. If the ODE (2.5) has a unique solution for any initial condition and if it is bounded for all time then for any $T > 0$, $\delta_0 > 0$ and a compact Q_1, one can always find a compact Q_2 that satisfies (2.6) for all $t \leq T$ and all $A \in Q_1$. We show *the existence of unique bounded solutions for all the ODEs considered in this thesis and conditions similar to (2.6) are then satisfied using this logic* .

The general system $\{Z_k\}$ depends upon two previous values Z_{k-1} and Z_{k-2}. An ODE approximation is obtained for the first time for such an adaptive system. We use the following small trick to achieve this. The ratio

$$\frac{Z_k - (1-d_2)Z_{k-1} - d_2 Z_{k-2}}{\mu} = (1-d_2)\frac{Z_k - Z_{k-1}}{\mu} + 2d_2 \frac{Z_k - Z_{k-2}}{2\mu}$$

is shown to converge to $(1+d_2)\,\dot{Z}(t)$.

Theorem 2.1 *With the above Assumptions* $\mathbf{A}.1$*–*$\mathbf{A}.3$ *and with* $d_2 \in (-1, 1]$, *for any two compact sets* $Q_1 \subset Q_2$, *for all* $T > 0$ *satisfying condition (2.6), for all* δ, *with* $(1-d_2)Z_0 + d_2 Z_{-1} = A$,

$$P\left\{\sup_{1\le k\le\left\lfloor\frac{T}{\mu}\right\rfloor}|Z_k - Z(k\mu,0,A)|\ge\delta\right\}\to 0$$

as $\mu\to 0$ *uniformly for all* $A\in Q_1$.

Proof : Please refer to Appendix B.

When $d_2=-1$ **:**

With $d_2=-1$, we will show that the solution of the second order ODE,

$$\frac{d^2Z(t)}{dt^2}=E\left[H(W,Z(t))\right]=h(Z(t)), \tag{2.7}$$

approximates the general system (2.4). With the rest of the notations as above, we have the following theorem. Do note that in this case the time scale is given by $k\sqrt{\mu}$ in place of the usual $k\mu$. Here we show that the ratio,

$$\frac{Z_k-2Z_{k-1}+Z_{k-2}}{\mu}=\frac{\frac{Z_k-Z_{k-1}}{\sqrt{\mu}}-\frac{Z_{k-1}-Z_{k-2}}{\sqrt{\mu}}}{\sqrt{\mu}}$$

converges to $\frac{d^2Z}{dt^2}(t)$.

Theorem 2.2 *With the above Assumptions* $\mathbf{A}.1$*–*$\mathbf{A}.3$*,* $d_2=-1$*, for any two compact sets* $Q_1\subset Q_2$*, for all* $T>0$ *satisfying condition (2.6), for all* δ*, with* $Z_0=Z_{-1}=A$*,* $\dot{Z}\,(0)=0$

$$P\left\{\sup_{1\le k\le\left\lfloor\frac{T}{\sqrt{\mu}}\right\rfloor}|Z_k - Z(k\sqrt{\mu},0,A)|\ge\delta\right\}\to 0$$

as $\mu\to 0$ *uniformly for all* $A\in Q_1$.

Proof : Please refer to Appendix C.

When d_2 is close to -1 :

When d_2 is close but not equal to -1, we will consider the following modified general system,

$$Z_k = (2 - \sqrt{\mu}\eta_1)Z_{k-1} - (1 - \sqrt{\mu}\eta_1)Z_{k-2} + \mu H(Z_{k-1}, W_k), \tag{2.8}$$

where η_1 is a fixed constant. By Theorem 2.3, we once again approximate the above trajectory by the following second order ODE,

$$\frac{d^2Z}{dt^2} = -\eta_1 \frac{dZ}{dt} + h(Z(t)), \tag{2.9}$$

where $h(Z) = E_W[H(Z, W)]$ as before. Do note that the time scale is once again $k\sqrt{\mu}$.

Theorem 2.3 *With the Assumptions* **A**.*1*–**A**.*3, for the system (2.8) and the ODE (2.9), for any two compact sets $Q_1 \subset Q_2$, for all $T > 0$ satisfying condition (2.6), for all δ, with $Z_0 = Z_{-1} = A$, $\dot{Z}(0) = 0$*

$$P\left\{\sup_{1 \leq k \leq \left\lfloor \frac{T}{\sqrt{\mu}} \right\rfloor} |Z_k - Z(k\sqrt{\mu}, 0, A)| \geq \delta\right\} \to 0$$

as $\mu \to 0$ uniformly for all $A \in Q_1$.

Proof : Please refer to Appendix D.

Appendix B

Proof of Theorem 2.1: Let $t_n \stackrel{\triangle}{=} n\mu$. Fix $A \in Q_1$ and let $Z(t)$ represent the solution $Z(t, 0, A)$ of ODE (2.5). We will limit ourselves to $n\mu \leq T$. We prove the above theorem in the following 3 steps.

Step 1: Approximating the solution $Z(t_{k+1})$ in terms of $Z(t_k)$ and $Z(t_{k-1})$.

By assumption **A**.2, there exist constants $L_1(Q_2), L_2(Q_2)$ such that,

$$\begin{aligned} |h(Z) - h(Z_1)| &\leq L_1(Q_2)|Z - Z_1| \quad \text{for all } Z, Z_1 \in Q_2, \\ |h(Z)|^2 &\leq L_2(Q_2) \quad \text{for all } Z \in Q_2. \end{aligned} \tag{2.10}$$

For any $k \geq 1$, $Z(t_k)$ can be expanded with respect to $Z(t_{k-1})$ using Taylor's series expansion as,

$$Z(t_k) = Z(t_{k-1}) + \mu \frac{h(Z(t_k))}{1+d_2} - \bar{\alpha}_k, \text{ where } |\bar{\alpha}_k| \leq \bar{L}\mu^2$$

for some constant $\bar{L}$. Therefore, by (2.10),

$$|Z(t_k) - Z(t_{k-1})| \leq \mu \tilde{L},$$

for some other constant $\tilde{L}$. For any other $k > 1$, $Z(t_{k+1})$ can also be expanded with respect to both $Z(t_k), Z(t_{k-1})$ as,

$$\begin{aligned} Z(t_{k+1}) &= (1-d_2)Z(t_{k+1}) + d_2 Z(t_{k+1}) \\ &= (1-d_2)\left[Z(t_k) + \mu \frac{h(Z(t_k))}{1+d_2} + \alpha_k^{'}\right] \\ &\quad + d_2\left[Z(t_{k-1}) + 2\mu \frac{h(Z(t_{k-1}))}{1+d_2} + \alpha_k^{''}\right]. \end{aligned}$$

Therefore, for all $t_1 < t_k < T$, the solution $Z(t_k)$ satisfies,

$$Z(t_{k+1}) = (1-d_2)Z(t_k) + d_2 Z(t_{k-1}) + \mu h(Z(t_k)) - \alpha_k, \tag{2.11}$$

where

$$\begin{aligned} |\alpha_k| &= \left|(1-d_2)\alpha_k^{'} + d_2\alpha_k^{''} + \frac{2d_2\mu}{1+d_2}\left[h(Z(t_{k-1})) - h(Z(t_k))\right]\right| \\ &\leq \frac{1}{1+d_2}\left[(1-d_2)L_2(Q_2) + d_2 L_2(Q_2) + 2d_2 L_1(Q_2)\tilde{L}\right]\mu^2 \leq L\mu^2, \end{aligned}$$

and constant L is chosen such that,

$$L \geq \frac{1}{1+d_2}[L_2(Q_2) + 2d_2 L_1(Q_2)\tilde{L}].$$

Therefore $|\bar{\alpha}_k| \quad \leq \quad L\mu^2 \quad$ for all k.

Step 2: Approximating the trajectory Z_{k+1} in terms of Z_k and Z_{k-1}.

Define error process, ϵ_k for any $k \geq 0$, by

$$Z_{k+1} = (1 - d_2)Z_k + d_2 Z_{k-1} + \mu h(Z_k) + \epsilon_k, \tag{2.12}$$

where,

$$\epsilon_k = \mu[H(Z_k, W_k) - h(Z_k)].$$

Define $\tau := \left\{ \inf_{1 \leq n \leq \lfloor \frac{T}{\mu} \rfloor} : Z_n \in Q_2^c \right\}$. In **Step 3**, we will need to bound,

$$\mathbf{1}\{k < \tau\} \sum_{i=2}^{k-1} \epsilon_i \left(1 - [-d_2]^{i+1}\right).$$

Defining,

$$K_k \stackrel{\triangle}{=} \sum_{i=2}^{k-1} \mathbf{1}\{i < \tau\} \epsilon_i \left(1 - [-d_2]^{i+1}\right),$$

one gets,

$$\mathbf{1}\{k < \tau\} \left| \sum_{i=2}^{k-1} \epsilon_i \left(1 - [-d_2]^{i+1}\right) \right| \leq |K_k|,$$

where $\{K_k\}$ is a martingale wrt the sigma algebra $F_k \stackrel{\triangle}{=} \sigma\left(W_i; i < k\right)$ as shown below.

Since Z_{k-1} is measurable F_{k-1} and W_{k-1} is independent of F_{k-1}, $\mathbf{1}\{k - 1 < \tau\}$ is measurable F_{k-1} and hence $E[\epsilon_{k-1}|F_{k-1}] = 0$. Therefore,

$$E\left[K_k|F_{k-1}\right] = \sum_{i=2}^{k-2} \mathbf{1}\{i < \tau\} \epsilon_i \left(1 - [-d_2]^{i+1}\right) = K_{k-1}.$$

Thus $\{K_k\}$ is a martingale. Therefore, by Doob's martingale inequality,

$$E\left\{\sup_{k\leq m}|K_k|^2\right\}\leq 4\sup_{k\leq m}E|K_m|^2.$$

Thus we have,

$$\begin{aligned}
&E\left[\left\{\sup_{k\leq\frac{T}{\mu}}\ \mathbf{1}\{k<\tau\}\left|\sum_{i=2}^{k-1}\epsilon_i\left(1-[-d_2]^{i+1}\right)\right|\right\}^2\right]\\
&\qquad\leq\ 4\sup_{k\leq\frac{T}{\mu}}E|K_k|^2\\
&\qquad\overset{a}{=}\ 4\sup_{k\leq\frac{T}{\mu}}\sum_{i=2}^{k-1}E\left|\ \mathbf{1}\{i<\tau\}\epsilon_i\left(1-[-d_2]^{i+1}\right)\right|^2\\
&\qquad\leq\ 8\sum_{i=2}^{\lfloor\frac{T}{\mu}\rfloor-1}E|\ \mathbf{1}\{i<\tau\}\epsilon_i|^2\\
&\qquad\leq\ 8\mu T[C(Q_2)+L_2(Q_2)],
\end{aligned}$$

where equality a follows because for any $j<k$,

$$\begin{aligned}
E\left[\ \mathbf{1}\{j<\tau\}\ \mathbf{1}\{k<\tau\}\epsilon_j^T\epsilon_k\right] &= E\left[E\left[\ \mathbf{1}\{j<\tau\}\ \mathbf{1}\{k<\tau\}\epsilon_j^T\epsilon_k|F_k\right]\right]\\
&= E\left[\ \mathbf{1}\{j<\tau\}\ \mathbf{1}\{k<\tau\}\epsilon_j^T E\left[\epsilon_k|F_k\right]\right]\\
&= 0.
\end{aligned}$$

Step 3: Error between the solution $Z(t_k)$ and the trajectory Z_k.

The difference between the ODE solution $Z(t_k)$ and the actual system value Z_k, for $k=1$, is given by,

$$\begin{aligned}
Z_1-Z(t_1) &= (1-d_2)Z_0+d_2Z_{-1}+\mu h(Z_0)+\epsilon_0-\left[Z(0)+\mu\frac{h(Z(0))}{1+d_2}-\alpha_0\right]\\
&= \mu h(Z_0)-\mu\frac{h(Z(0))}{1+d_2}+\epsilon_0+\alpha_0,
\end{aligned}$$

because of the initial conditions given in the hypothesis of the theorem. By Assumption **A**.2 and the bound on α_0, we get,

$$|Z_1 - Z(t_1)| \leq \mu M(1 + |H(Z_0, W_0)|). \tag{2.13}$$

For any $k > 1$, the above difference is obtained by subtracting equation (2.11) from (2.12),

$$\begin{aligned} Z_k - Z(t_k) &= (1 - d_2)\left(Z_{k-1} - Z(t_{k-1})\right) + d_2\left(Z_{k-2} - Z(t_{k-2})\right) + b_{k-1} \\ &= Z_{k-1} - Z(t_{k-1}) + d_2\left(Z_{k-2} - Z(t_{k-2})\right) \\ &\quad - d_2\left(Z_{k-1} - Z(t_{k-1})\right) + b_{k-1}, \end{aligned} \tag{2.14}$$

where

$$b_k \stackrel{\triangle}{=} \mu\left[h(Z_k) - h(Z(t_k)\right] + \epsilon_k + \alpha_k \quad \text{for all } k \geq 0, \tag{2.15}$$

is a composite term consisting of all the errors. By replacing $(Z_{k-1} - Z(t_{k-1}))$ in the equation (2.14) with the same equation for $k-1$ and continuing till $k = 3$, one gets after some cancellations,

$$Z_k - Z(t_k) = d_2(Z_1 - Z(t_1)) + (Z_2 - Z(t_2)) + (-d_2)\left(Z_{k-1} - Z(t_{k-1})\right) + \sum_{i=2}^{k-1} b_i.$$

Further, iteratively, from $(Z_{k-1} - Z(t_{k-1}))$ one gets (continuing till $k = 3$),

$$\begin{aligned} Z_k - Z(t_k) &= d_2 \sum_{j=0}^{k-3} [-d_2]^j (Z_1 - Z(t_1)) + \sum_{j=0}^{k-2} [-d_2]^j (Z_2 - Z(t_2)) \\ &\quad + \sum_{i=0}^{k-3} b_{k-1-i} \sum_{j=0}^{i} [-d_2]^j \\ &= d_2 \frac{1 - [-d_2]^{k-2}}{1 + d_2}(Z_1 - Z(t_1)) + \frac{1 - [-d_2]^{k-1}}{1 + d_2}(Z_2 - Z(t_2)) \\ &\quad + \sum_{i=0}^{k-3} b_{k-1-i} \frac{1 - [-d_2]^{i+1}}{1 + d_2}. \end{aligned}$$

With $d_2 \in (-1, 1]$, $|1 - [-d_2]^i| \leq 2$ for all i and hence we get,

$$|Z_k - Z(t_k)| \leq \frac{\left[2|Z_1 - Z(t_1)| + 2|Z_2 - Z(t_2)| + \left|\sum_{i=0}^{k-3} b_{k-1-i}\left(1 - [-d_2]^{i+1}\right)\right|\right]}{1 + d_2}$$

and therefore on the set $\{1 \leq k < \tau\}$, for any $2 \leq k \leq \frac{T}{\mu}$ from (2.15), using the bound (2.10) and the bound on α_k,

$$|Z_k - Z(t_k)| \;\; \leq \;\; \frac{2}{1+d_2}\sum_{i=2}^{k} a_i |Z_{i-1} - Z(t_{i-1})| + \mathcal{U} + \frac{2}{1+d_2} L\mu T \tag{2.16}$$

with

$$\mathcal{U} \;\; \stackrel{\triangle}{=} \;\; \frac{1}{1+d_2}\left[\sup_{m \leq \lfloor \frac{T}{\mu} \rfloor} \mathbf{1}\{m < \tau\} \left|\sum_{i=2}^{m-1} \epsilon_i (1 - [-d_2]^{i+1})\right|\right],$$

where $a_1 = 1$, $a_2 = 1 + L_1(Q_2)\mu$, $a_i = L_1(Q_2)\mu \quad$ for all $i \geq 3$.

Using the bound (2.13), from Lemma 2.1, on the set $\{1 \leq k < \tau\}$, for any $1 \leq k \leq \frac{T}{\mu}$

$$|Z_k - Z(t_k)| \leq \left[\mathcal{U} + \frac{2}{1+d_2} L\mu T + \mu M(1 + |H(Z_0, W_0)|)\right] e^{\frac{2(TL_1(Q_2)+a_1+a_2)}{1+d_2}}.$$

Therefore using $\left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2$, one gets,

$$\begin{aligned}\sup_{1 \leq k < \tau} |Z_k - Z(t_k)|^2 \;\; \leq \;\; & \left[3\mathcal{U}^2 + \frac{12L^2\mu^2T^2}{(1+d_2)^2} + 3\mu^2 M^2 (1 + |H(Z_0, W_0)|)^2\right] \\ & e^{\frac{4(TL_1(Q_2)+a_1+a_2)}{1+d_2}}.\end{aligned}$$

Thus using **Step** 2 and assumption **A**.3,

$$\begin{aligned}& E\left[\sup_{1 \leq k < \tau} |Z_k - Z(t_k)|^2\right] \\ & \quad \leq \;\; \left[\frac{24[L_2(Q_2) + C(Q_2)]\mu T + 12L^2\mu^2T^2}{(1+d_2)^2} + 3\mu^2 M_1^2\right] e^{\frac{4(TL_1(Q_2)+a_1+a_2)}{1+d_2}}.\end{aligned}$$

For any $\delta < \delta_0$ (δ_0 defined in (2.6)), the set $\{\sup_{1 \leq k \leq \lfloor \frac{T}{\mu} \rfloor} |Z_k - Z(t_k)| \geq \delta\}$ is same as the set $\{\sup_{1\leq k<\tau} |Z_k - Z(t_k)| \geq \delta\}$ because, when $k = \tau < \left\lfloor \frac{T}{\mu} \right\rfloor$, $Z_k \in Q_2^c$ and hence $|Z_k - Z(t_k)| \geq \delta_0 > \delta$. The theorem follows by Chebyshev's inequality. ■

Lemma 2.1 *If for some positive constants* r_1, r_2, r_3, $\nu_1 \leq r_3$ *and* $\nu_r \leq r_1 \sum_{i=2}^{r} \mu_i \nu_{i-1} + r_2$ *for* $r = 2, 3, \cdots n$, *then with* $\mu_1 \stackrel{\triangle}{=} 0$, $\nu_r \leq (r_2 + r_3) e^{\left(r_1 \sum_{i=1}^{r} \mu_i\right)}$ *for all* $1 \leq r \leq n$.

Proof : Define $\nu_0 = 0$, $\mu_1 = 0$. The above statement becomes, $\nu_0 = 0$ and $\nu_r \leq r_1 \sum_{i=1}^{r} \mu_i \nu_{i-1} + (r_2 + r_3)$, for $r = 1, \cdots n$ and the result follows by Lemma 8 of page 231 in [4]. ■

Appendix C

Proof of Theorem 2.2: Let $t_n \stackrel{\triangle}{=} n\sqrt{\mu}$. Fix $A \in Q_1$ and let $Z(t)$ represent the solution of the ODE (2.7) with $Z(0) = A$ and $\dot{Z}(0) = 0$. We will limit ourselves to $n\sqrt{\mu} \leq T$. We prove the above theorem in the following 3 steps as in Theorem 2.1.

Step 1: Approximating the solution $Z(t_{k+1})$ in terms of $Z(t_k)$ and $Z(t_{k-1})$.

By assumption **A**.2, there exist constants $L_1(Q_2), L_2(Q_2)$ such that,

$$
\begin{aligned}
|h(Z) - h(Z_1)| &\leq L_1(Q_2)|Z - Z_1|, \quad \text{for all } Z, Z_1 \in Q_2, \\
|h(Z)|^2 &\leq L_2(Q_2), \quad \text{for all } Z \in Q_2.
\end{aligned} \tag{2.17}
$$

For any $k \geq 1$, the first derivative $\dot{Z}(t_k)$ can be expanded using Taylor's series expansion as,

$$
\dot{Z}(t_k) = \dot{Z}(t_{k-1}) + \sqrt{\mu} h(Z(t_{k-1})) - \bar{\alpha}_k, \ \text{where } |\bar{\alpha}_k| \leq \bar{L}\mu \tag{2.18}
$$

for some constant $\bar{L}$. Therefore using the upper bound (2.17) and the initial condition $\dot{Z}(0) = 0$,

$$
|\dot{Z}(t_1)| \leq \sqrt{\mu}\tilde{L}
$$

for some other constant $\tilde{L}$. For any $k > 1$, $Z(t_{k+1})$ can be expanded with respect to both $Z(t_k), Z(t_{k-1})$ as,

$$\begin{aligned}
Z(t_{k+1}) &= 2Z(t_{k+1}) - Z(t_{k+1}) \\
&= 2\left[Z(t_k) + \sqrt{\mu}\,\dot{Z}\,(t_k) + \frac{\mu}{2}h(Z(t_k)) + \alpha_k^{'}\right] \\
&\quad - \left[Z(t_{k-1}) + 2\sqrt{\mu}\,\dot{Z}\,(t_{k-1}) + 4\frac{\mu}{2}h(Z(t_{k-1})) + \alpha_k^{''}\right] \\
&= 2Z(t_k) - Z(t_{k-1}) + 2\sqrt{\mu}\left[\dot{Z}\,(t_k) - \dot{Z}\,(t_{k-1}) - \sqrt{\mu}h(Z(t_{k-1}))\right] \\
&\quad + \mu h(Z(t_k)) + \alpha_k^{'''}.
\end{aligned}$$

Using the equation (2.18), the solution $Z(t_k),$ for all $t_1 < t_k < T$ satisfies,

$$Z(t_{k+1}) = 2Z(t_k) - Z(t_{k-1}) + \mu h(Z(t_k)) - \alpha_k, \tag{2.19}$$

where $|\alpha_k| \leq L\sqrt{\mu}^3$, for some constant L.

Step 2: Approximating the trajectory Z_{k+1} in terms of Z_k and Z_{k-1}.

Here the error process ϵ_k for any $k \geq 0$ is defined as in **Step 2** of Theorem 2.1. Define $\tau := \left\{ \inf_{1 \leq n \leq \left\lfloor \frac{T}{\sqrt{\mu}} \right\rfloor} : Z_n \in Q_2^c \right\}$. Following similar steps as in the **Step 2** of Theorem 2.1, we can show that,

$$E\left[\sup_{m \leq \left\lfloor \frac{T}{\sqrt{\mu}} \right\rfloor} \mathbf{1}\{m < \tau\} \left|\sum_{i=0}^{m-2} (i+1)\epsilon_{m-1-i}\right|\right]^2 \leq 8\sqrt{\mu}T^3C.$$

Step 3: Error between the solution $Z(t_k)$ and the trajectory Z_k.

The difference between the ODE solution $Z(t_k)$ and Z_k, for $k = 1$, is given by,

$$\begin{aligned}
Z_1 - Z(t_1) &= 2Z_0 - Z_{-1} + \mu h(Z_0) + \epsilon_0 \\
&\quad - \left[Z(0) + \sqrt{\mu}\,\dot{Z}\,(0) + \frac{\mu}{2}h(Z(0)) - \alpha_0\right]
\end{aligned}$$

$$= \mu h(Z_0) - \frac{\mu}{2} h(Z(0)) + \epsilon_0 + \alpha_0,$$

because of the initial conditions given in the hypothesis of the theorem. By Assumption **A**.2, we get,

$$|Z_1 - Z(t_1)| \leq \mu M(1 + |H(Z_0, W_0)|). \tag{2.20}$$

For any $k > 1$, the above difference is obtained by subtracting equation (2.19) from (2.12),

$$\begin{aligned} Z_k - Z(t_k) &= 2\left(Z_{k-1} - Z(t_{k-1})\right) - \left(Z_{k-2} - Z(t_{k-2})\right) + b_{k-1} \\ &= Z_{k-1} - Z(t_{k-1}) - \left(Z_{k-2} - Z(t_{k-2})\right) \\ &\qquad + \left(Z_{k-1} - Z(t_{k-1})\right) + b_{k-1}, \end{aligned}$$

where

$$b_k \stackrel{\triangle}{=} \mu \left[h(Z_k) - h(Z(t_k)\right] + \epsilon_k + \alpha_k \quad \text{for all } k \geq 0, \tag{2.21}$$

is a composite term consisting of all errors. Continuing as in **Step 3** of Theorem 2.1 till $k = 2$ and since $Z_0 = Z(0)$, we get,

$$Z_k - Z(t_k) = (Z_1 - Z(t_1)) + (Z_{k-1} - Z(t_{k-1})) + \sum_{i=1}^{k-1} b_i.$$

Continuing further we get,

$$Z_k - Z(t_k) = k(Z_1 - Z(t_1)) + \sum_{i=0}^{k-2} (i+1) b_{k-1-i}.$$

Using the upper bound (2.20),

$$\begin{aligned} |Z_k - Z(t_k)| &\leq \frac{T|Z_1 - Z(t_1)|}{\sqrt{\mu}} + \left|\sum_{i=0}^{k-2} (i+1) b_{k-1-i}\right| \\ &\leq TM(1 + |H(Z_0, W_0)|)\sqrt{\mu} + \left|\sum_{i=0}^{k-2} (i+1) b_{k-1-i}\right| \end{aligned}$$

and therefore on the set $\{1 \leq k < \tau\}$, for any $1 \leq k \leq \frac{T}{\sqrt{\mu}}$ from (2.21), using the bound (2.17) and the bound on α_k,

$$\begin{aligned} |Z_k - Z(t_k)| &\leq L_1(Q_2)\sqrt{\mu}T \sum_{i=1}^{k-1} |Z_i - Z(t_i)| + \mathcal{U} \\ &\quad + (LT^2 + TM(1 + |H(Z_0, W_0)|))\sqrt{\mu}, \qquad (2.22) \\ \text{with } \mathcal{U} &\triangleq \left[\sup_{m \leq \left\lfloor \frac{T}{\sqrt{\mu}} \right\rfloor} \mathbf{1}\{m < \tau\} \left| \sum_{i=0}^{m-2} (i+1)\epsilon_{m-1-i} \right| \right]. \end{aligned}$$

From Lemma 8 in page 231 of [4], on the set $\{1 \leq k < \tau\}$, for any $1 \leq k \leq \frac{T}{\sqrt{\mu}}$

$$|Z_k - Z(t_k)| \leq \left[\mathcal{U} + (LT^2 + MT(1 + |H(Z_0, W_0)|))\sqrt{\mu}\right] e^{\left(T^2 L_1(Q_2)\right)}.$$

Therefore using $\left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2$, one gets,

$$\begin{aligned} \sup_{1 \leq k < \tau} |Z_k - Z(t_k)|^2 &\leq \left[3\mathcal{U}^2 + 3L^2T^4\mu + 3M^2T^2(1 + |H(Z_0, W_0)|)^2\mu\right] \\ &\quad e^{2T^2 L_1(Q_2)}. \end{aligned}$$

Thus using the **Step 2** and assumption **A**.3,

$$\begin{aligned} & E\left[\sup_{1 \leq k < \tau} |Z_k - Z(t_k)|^2\right] \\ &\quad \leq \left[24CT^3\sqrt{\mu} + 3L^2T^4\mu + 3M_1^2T^2\mu\right] e^{2T^2 L_1(Q_2)}. \end{aligned}$$

The theorem follows by Chebyshev's inequality. ■

Appendix D

Proof of Theorem 2.3 : Let $t_n \triangleq n\sqrt{\mu}$. Fix $A \in Q_1$ and let $Z(t)$ represent the solution of the ODE (2.7) with $Z(0) = A$ and $\dot{Z}(0) = 0$. We will limit ourselves to $n\sqrt{\mu} \leq T$. We prove the above theorem in the following 3 steps just like in Theorems 2.1 and 2.2.

Step 1: Approximating the solution $Z(t_{k+1})$ in terms of $Z(t_k)$ and $Z(t_{k-1})$.

By assumption **A**.2, there exist constants $L_1(Q_2), L_2(Q_2)$ such that,

$$\begin{aligned} |h(Z)-h(Z_1)| &\leq L_1(Q_2)|Z-Z_1|, && \text{for all } Z, Z_1 \in Q_2, \\ |h(Z)|^2 &\leq L_2(Q_2), && \text{for all } Z \in Q_2. \end{aligned} \tag{2.23}$$

For any $k \geq 1$, the first derivative $\dot{Z}\,(t_k)$ can be expanded using Taylor's series expansion as,

$$\dot{Z}\,(t_k) = (1-\sqrt{\mu}\eta_1)\,\dot{Z}\,(t_{k-1}) + \sqrt{\mu}h(Z(t_{k-1})) - \bar{\alpha}_k, \text{ where } |\bar{\alpha}_k| \leq \bar{L}\mu \tag{2.24}$$

for some constant $\bar{L}$. Therefore using the upper bound (2.23) and the initial condition $\dot{Z}\,(0) = 0$,

$$|\,\dot{Z}\,(t_1)| \leq \sqrt{\mu}\tilde{L}$$

for some other constant $\tilde{L}$. For any $k > 1$, $Z(t_{k+1})$ can be expanded with respect to both $Z(t_k), Z(t_{k-1})$ as,

$$\begin{aligned} Z(t_{k+1}) &= (2-\sqrt{\mu}\eta_1)Z(t_{k+1}) - (1-\sqrt{\mu}\eta_1)Z(t_{k+1}) \\ &= (2-\sqrt{\mu}\eta_1)\left[Z(t_k) + \sqrt{\mu}(1-\frac{\sqrt{\mu}\eta_1}{2})\,\dot{Z}\,(t_k) + \frac{\mu}{2}h(Z(t_k)) + \alpha_k^{'}\right] \\ &\quad -(1-\sqrt{\mu}\eta_1)\left[Z(t_{k-1}) + 2\sqrt{\mu}(1-\sqrt{\mu}\eta_1)\,\dot{Z}\,(t_{k-1}) + 4\frac{\mu}{2}h(Z(t_{k-1})) + \alpha_k^{''}\right] \\ &= (2-\sqrt{\mu}\eta_1)Z(t_k) - (1-\sqrt{\mu}\eta_1)Z(t_{k-1}) + \mu h(Z(t_k)) \\ &\quad +2\sqrt{\mu}(1-\sqrt{\mu}\eta_1)\left[\dot{Z}\,(t_k) - (1-\sqrt{\mu}\eta_1)\,\dot{Z}\,(t_{k-1}) - \sqrt{\mu}h(Z(t_{k-1}))\right] + \alpha_k^{'''}, \end{aligned}$$

where

$$\alpha_k^{'''} = (2-\sqrt{\mu}\eta_1)\alpha_k^{'} - (1-\sqrt{\mu}\eta_1)\alpha_k^{''} + \dot{Z}\,(t_k)\left(\frac{\sqrt{\mu}(\sqrt{\mu}\eta_1)^2}{2}\right) - \frac{\mu\sqrt{\mu}\eta_1}{2}h(Z(t_k)).$$

Using the equation (2.24), the solution $Z(t_k)$, for all $t_1 < t_k < T$, satisfies,

$$Z(t_{k+1}) \quad = \quad (2-\sqrt{\mu}\eta_1)Z(t_k) - (1-\sqrt{\mu}\eta_1)Z(t_{k-1}) + \mu h(Z(t_k)) - \alpha_k, \qquad (2.25)$$

where $|\alpha_k| \leq L\sqrt{\mu}^3$, for some constant L.

Step 2: Approximating the trajectory Z_{k+1} in terms of Z_k and Z_{k-1}.

Here the error process ϵ_k for any $k \geq 0$ is defined as in **Step 2** of Theorem 2.1. Define $\tau := \left\{ \inf_{1 \leq n \leq \lfloor \frac{T}{\sqrt{\mu}} \rfloor} : Z_n \in Q_2^c \right\}$. Following similar steps as in the **Step 2** of Theorem 2.1, we can show that,

$$E\left[\sup_{m \leq \lfloor \frac{T}{\sqrt{\mu}} \rfloor} \mathbf{1}\{m < \tau\} \left| \sum_{i=0}^{m-2} \sum_{l=0}^{i} (1-\sqrt{\mu}\eta_1)^l \epsilon_{m-1-i} \right| \right]^2 \leq 8\sqrt{\mu}\ T^3 C.$$

Step 3: Error between the solution $Z(t_k)$ and the trajectory Z_k.

The difference between the ODE solution $Z(t_k)$ and Z_k, for $k = 1$, is given by,

$$\begin{aligned} Z_1 - Z(t_1) &= (2-\sqrt{\mu}\eta_1)Z_0 - (1-\sqrt{\mu}\eta_1)Z_{-1} + \mu h(Z_0) + \epsilon_0 \\ &\quad - \left[Z(0) + \sqrt{\mu}\ \dot{Z}(0) + \frac{\mu}{2}(h(Z(0)) - \eta_1\ \dot{Z}(0)) - \alpha_0 \right] \\ &= \mu h(Z_0) - \frac{\mu}{2} h(Z(0)) + \epsilon_0 + \alpha_0, \end{aligned}$$

because of the initial conditions given in the hypothesis of the theorem. By Assumption **A**.2, we get,

$$|Z_1 - Z(t_1)| \leq \mu M(1 + |H(Z_0, W_0)|). \qquad (2.26)$$

For any $k > 1$, the above difference is obtained by subtracting equation (2.25) from (2.12),

$$Z_k - Z(t_k) \quad = \quad (2-\sqrt{\mu}\eta_1)\left(Z_{k-1} - Z(t_{k-1})\right) - (1-\sqrt{\mu}\eta_1)\left(Z_{k-2} - Z(t_{k-2})\right) + b_{k-1}$$

$$
\begin{aligned}
&= (1-\sqrt{\mu}\eta_1)\left[Z_{k-1}-Z(t_{k-1})-(Z_{k-2}-Z(t_{k-2}))\right] \\
&\quad +(Z_{k-1}-Z(t_{k-1}))+b_{k-1},
\end{aligned}
$$

where

$$b_k \stackrel{\triangle}{=} \mu\left[h(Z_k)-h(Z(t_k)\right]+\epsilon_k+\alpha_k \quad \text{for all } k \geq 0, \tag{2.27}$$

is a composite term consisting of all errors. Continuing as in **Step 3** of Theorem 2.1 till $k=2$ and since $Z_0 = Z(0)$, we get,

$$Z_k - Z(t_k) = (1-\sqrt{\mu}\eta_1)^{k-1}(Z_1 - Z(t_1)) + (Z_{k-1} - Z(t_{k-1})) + \sum_{i=1}^{k-1} b_i (1-\sqrt{\mu}\eta_1)^{k-1-i}.$$

Continuing further we get,

$$Z_k - Z(t_k) \;=\; \sum_{i=0}^{k-1}(1-\sqrt{\mu}\eta_1)^i (Z_1 - Z(t_1)) + \sum_{l=0}^{k-2}\sum_{i=0}^{l}(1-\sqrt{\mu}\eta_1)^i b_{k-1-l}.$$

Using the upper bound (2.26),

$$
\begin{aligned}
|Z_k - Z(t_k)| &\leq \frac{T|Z_1 - Z(t_1)|}{\sqrt{\mu}} + \left|\sum_{l=0}^{k-2}\sum_{i=0}^{l}(1-\sqrt{\mu}\eta_1)^i b_{k-1-l}\right| \\
&\leq TM(1+|H(Z_0,W_0)|)\sqrt{\mu} + \left|\sum_{l=0}^{k-2}\sum_{i=0}^{l}(1-\sqrt{\mu}\eta_1)^i b_{k-1-l}\right|.
\end{aligned}
$$

Rest of the steps follow as in **Step 3** of Theorem 2.2 using the current theorem's **Step 2**. ■

Chapter 3

Blind/Semi-blind versus Training Equalizers

Semiblind/blind equalizers are believed to work unsatisfactorily in fading channels compared to training based methods due to slow convergence and or high computational complexity ([14], [15], [21]). In this chapter we revisit this issue for MIMO fading channels. We compare the blind, semi-blind and training based equalizers when the blind algorithm is CMA. We show that in Ricean (with Line of sight, LOS) environment semiblind/blind algorithms outperform training equalizers. We also find the optimum training length in training and semiblind methods. Furthermore, we show that by adapting the step size of the semiblind algorithm based on training based channel estimate, one can outperform the training method even for Rayleigh channels.

This chapter is organized as follows. Section 3.1 describes the problems involved and our approach to the comparison. Section 3.2 describes our model and our assumptions. Section 3.3 considers the training based channel estimation/equalization and Section 3.4 studies the blind algorithm (CMA). Section 3.5 combines the two approaches to obtain a semiblind algorithm. Section 3.6 compares the three algorithms using few examples and Section 3.7 concludes the chapter. The proofs of the lemmas are available in Appendices A and B.

3.1 The issues and our approach

While comparing training based methods with blind algorithms, one encounters the problem of comparing the loss in BW (Bandwidth) in training based methods (due to training symbols) with the gain in BER (due to better channel estimation/equalization accuracy) as compared to the blind algorithms. We overcome this problem by comparing these methods via the channel capacity they provide. Towards this goal, we combine the channel, the equalizer and the decoder to form a composite channel (Figure 3.1). The input to this channel will be symbols from a finite alphabet and the output of this channel will be the decoded symbols. Hence this is a discrete channel. The capacity of this composite channel will be a good measure for comparison. In this chapter we use a hard decoder after the equalizer. But the analysis will go through even for a soft decoder, as long as the output of the decoder comes from a discrete alphabet. We use hard decoder for ease of notation.

Consider a frame involving N channel uses with N_t training symbols. In a training based method, the channel is estimated from these symbols, an equalizer is designed using the channel estimate and then the information symbols are decoded (hard/soft decoder is used after the equalizer) using the equalizer. Using the probability of error provided by this method one can compute the capacity $C(N_t)$ of the composite channel per frame. Optimum training sequence length, N_t^*, would be the one that corresponds to the maximum value of $C(N_t)$.

In a blind algorithm (say CMA) $N_t = 0$, but using the general statistics of the input/output symbols of the channel, one estimates the channel (or may directly obtain an equalizer). After all the N symbols of a frame arrive (for delay constrained applications one can use only a fraction of the N symbols for blind estimation), we obtain an equalizer. We use this equalizer to estimate the transmitted information symbols of the frame. The resulting probability of error can be used to obtain the capacity of the composite channel C_{CMA}. Comparing $C(N_t^*)$ with C_{CMA} provides a reasonable comparison. One can compare these capacities with the capacity obtained by a semi-blind method, which is a combination of the two methods. In this thesis, training based equalizer is improved upon (after the training sequence) using the blind CMA to obtain a semi-blind equalizer.

Obtaining channel capacity of the composite channel, for training based methods in our model, is not difficult. But for the blind and semiblind methods, one needs to know the equalizer value at the end of the frame (or till the point the algorithm is updated). At that point the equalizer will often be away from the equilibrium point. Thus, we need the value of the equalizer at a specific time under transience. It depends upon the initial conditions, the value of the fading channel and the receiver noise and input realizations. Thus, it is not practically feasible to obtain the capacity of such a composite channel. We circumvent this problem by using the result in [47], where, given the initial conditions and a realization of the fading channel, the equalizer value at any time (even under transience) can be approximated by the solution of an ODE. Also, this value is independent of the receiver noise and input realizations.

Computation of the exact capacity of the composite channels for training based methods is intensive. Thus, we will approximate it by a lower bound obtained by limiting the input to a kth order Markov chain. This bound is tight when the equalizer compensates the channel reasonably well or if the channel has small ISI. For high SNR regions, we obtain a much simpler lower bound (obtained using uniform input distribution) on capacity. We use the same lower bounds for semiblind as well as blind algorithms.

Notations

The following notation is used throughout the chapter. The notations in this chapter are quite different from that in the remaining chapters as we are dealing directly with the complex signals (in the other chapters the results are obtained for real signals and then are extended for complex signals).

All bold capital letters represent complex matrices, capital letters represent real matrices and small letters represent the complex scalars. Bold small letters with bar on top represent complex column vectors, while the same bold small letters without bar represent their real counter parts given by $\mathbf{u} \stackrel{\triangle}{=} Real(\bar{\mathbf{u}}) := [\ \bar{\mathbf{u}}_{1,r} \quad \bar{\mathbf{u}}_{2,r} \quad \cdots \quad \bar{\mathbf{u}}_{n,r} \quad \bar{\mathbf{u}}_{1,i} \quad \cdots \quad \bar{\mathbf{u}}_{n,i}\ ]^T$ (for an n dimensional complex vector $\bar{\mathbf{u}}$). Here $\bar{\mathbf{u}}_{k,r} + i\bar{\mathbf{u}}_{k,i}$ is the k^{th} element of $\bar{\mathbf{u}}$. $\mathbf{A}^H$,

$\mathbf{A}^T$, $\mathbf{A}^{HT}$ represent Hermitian, transpose and conjugate of matrix $\mathbf{A}$ respectively. Special symbol θ is used for an equalizer and has the following special notation. A complex vector is represented by $\bar{\theta}$, while $\theta = Real(\bar{\theta})$ is its real counterpart. The complex matrix in this case is represented by $\underline{\bar{\theta}}$, while a real matrix is represented by $\underline{\theta}$.

Two more real vectors corresponding to a complex vector are defined as, $\tilde{\mathbf{u}}$=$Real(\bar{\mathbf{u}}^{HT})$ and $\check{\mathbf{u}} = Real(i\bar{\mathbf{u}}^{HT})$. Note that $\theta^T\tilde{\mathbf{u}}$, $\theta^T\check{\mathbf{u}}$ are the real and imaginary parts of the complex inner product $\bar{\theta}^H\bar{\mathbf{y}}$ respectively. For any complex matrix $\mathbf{Z}$, $vect(\mathbf{Z})$ represents $\mathbf{Z}$ in vector form by concatenating elements of all row vectors one after the other. We will represent $vect(\mathbf{Z})$ by $\bar{\mathbf{z}}$ (and $\mathbf{z}$ represents real counterpart as explained above). For a family of matrices $\{\mathbf{Z}_l\}_{l=0}^{L-1}$, $vect(\{\mathbf{Z}_l\}_{l=0}^{L-1})$ denotes $vect([\mathbf{Z}_0\ \mathbf{Z}_1\ \cdots \mathbf{Z}_{L-1}])$. $\mathbb{E}(.)$ represents expectation.

For any vector (matrix) $\bar{\mathbf{u}}$ ($\mathbf{Z}$), $\hat{\bar{\mathbf{u}}}$ ($\hat{\mathbf{Z}}$) represents its estimate, obtained either by MMSE criterion or minimum distance criterion. Complex sequence $\{\bar{\mathbf{s}}_l :\ n_1 \leq l \leq n_2\}$ is represented by $\bar{\mathbf{s}}_{n_1}^{n_2}$. Same notation is used for a real sequence. Θ_l represents l length convolution matrix of dimension $lm\times(l+M-1)n$, formed from $m\times n$ dimensional complex matrices $\{\underline{\bar{\theta}}_p\}_{p=0}^{M-1}$ as given by,

$$\Theta_l = \begin{bmatrix} \underline{\bar{\theta}}_0 & \underline{\bar{\theta}}_1 & \cdots & \underline{\bar{\theta}}_{M-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \underline{\bar{\theta}}_0 & \cdots & \underline{\bar{\theta}}_{M-2} & \underline{\bar{\theta}}_{M-1} & \cdots & \mathbf{0} \\ & . & . & . & & & \\ \mathbf{0} & \mathbf{0} & . & \underline{\bar{\theta}}_0 & \underline{\bar{\theta}}_1 & \cdots & \underline{\bar{\theta}}_{M-1} \end{bmatrix}.$$

Similarly, let $\mathcal{Z}_l$ represent the convolutional matrix corresponding to the channel matrices $\{\mathbf{Z}_l\}_{l=0}^{L-1}$. L and M denote channel and equalizer lengths respectively and $M_L \stackrel{\triangle}{=} M+L-1$. N_t is the number of training sequences. Remaining notations are introduced as and when required.

3.2 The Model and Our Assumptions

Consider a wireless channel with m transmit antennas and n receive antennas with $m \leq n$ (see Fig. 3.1). The time axis is divided into frames; each frame consisting of N channel uses. The transmitted symbols are chosen from a finite alphabet, $\mathcal{S} = \{\int_1, \int_2, \cdots, \int_{n_S}\}$. At time k, vector $\bar{\mathbf{s}}(k) \varepsilon \mathcal{S}^m$ is transmitted from the m transmit antennas. We use $\mathcal{S}^{(N_t)}$ to represent the set $\mathcal{S}^{m(N-N_t)}$. Note that information sequence $\bar{\mathbf{s}}_{N_t+1}^{N} \in \mathcal{S}^{m(N-N_t)}$. We represent the elements of $\mathcal{S}^{(N_t)}$ by $\{\bar{\int}_i; 1 \leq i \leq (n_S)^{m(N-N_t)}\}$ i.e., each $\bar{\int}_i$ is a $m(N-N_t)$ length complex vector formed from elements of $\mathcal{S}$. The channel $\{\mathbf{Z}_l\}_{l=0}^{L-1}$is assumed constant during a frame (quasi-static channel). We assume it to vary independently from frame to frame. We further assume that the training length $N_t \geq L$. For the blind/semiblind algorithm if required, we do not use the information symbols present in the first M_L symbols of the frame (Note that all the symbols in the frame are information symbols for blind algorithms, while all the symbols after the first N_t symbols are information symbols in the semiblind method). This assumption is required for designing an equalizer directly as in the CMA. This will ensure that equalizer estimation is independent of previous frame channel realization. Thus, the channel estimate and/or equalizer, for a frame depends upon the received symbols during that frame only. Therefore, we will consider a single frame in this chapter.

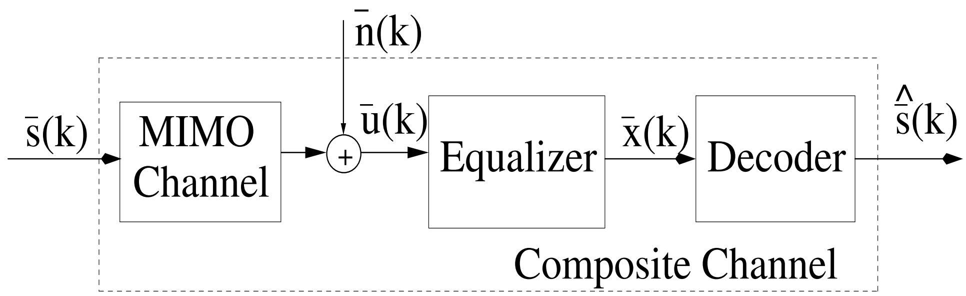

Figure 3.1: Composite channel used for capacity comparison

The vectors received at the n receive antennas in any frame are,

$$\bar{\mathbf{u}}(k) = \sum_{l=0}^{L-1} \mathbf{Z}_l \bar{\mathbf{s}}(k-l) + \bar{\mathbf{n}}(k), \quad \text{for all } k = 1, \cdots, N$$

where $\bar{\mathbf{n}}(k)$ is an IID sequence of complex Gaussian vectors with mean 0 and co-variance $\sigma_n^2 I$ (denoted by $C\mathcal{N}(0, \sigma_n^2 I)$) and $\bar{\mathbf{z}} \sim C\mathcal{N}(\mu_{\bar{\mathbf{z}}}, C_z)$ (Note that $\bar{\mathbf{z}} = vect(\{\mathbf{Z}_l\}_{l=0}^{L-1})$). This is a Rayleigh/Ricean channel. At the receiver, one wants to estimate $\bar{\mathbf{s}}(k)$, for all $k > N_t$. One common way is to use an equalizer at the receiver to nullify the effect of $\{\mathbf{Z}_l\}_{l=0}^{L-1}$and then use symbol by symbol detection to detect all the transmitted symbols in the frame. We assume hard decoding here, although soft decoding with discretization can be handled in a similar way.

We assume that the channel statistics ($\mu_{\bar{\mathbf{z}}}, C_z$) is available both at the transmitter and the receiver, but, the receiver and the transmitter do not know the actual channel state $\{\mathbf{Z}_l\}_{l=0}^{L-1}$. The receiver tries to estimate $\{\mathbf{Z}_l\}_{l=0}^{L-1}$and then obtain an equalizer as the channel estimate's inverse filter, or, directly obtain an equalizer to estimate/detect the information symbols transmitted. For this, the most common method used in wireless channels is to send a known training sequence in the frame. This is used by the receiver to estimate $\{\mathbf{Z}_l\}_{l=0}^{L-1}$(say via Minimum Mean Square Error (MMSE) estimator) and then obtain an equalizer. In the rest of the frame, information symbols are transmitted and are decoded at the receiver using the equalizer. This results in a training based equalizer. Alternatively, one can estimate a blind equalizer, using only the statistics of the received and transmitted signal. One can also use a combination of the blind and training based methods to obtain a semi-blind equalizer.

The 'composite' channel, used for comparison of various equalizers, is formed by combing the channel, equalizer (either training, blind or semi-blind) and the decoder (Fig. 3.1). Its input is the transmitted information sequence corresponding to one complete frame $\bar{\mathbf{s}}_{N_t+1}^N$, while the corresponding decisions $\hat{\bar{\mathbf{s}}}_{N_t+1}^N$form the output. Thus the composite channel is a finite input-output alphabet channel. Further *it forms a time invariant channel.* This is because the channel state is not known to the transmitter and hence the transmitter would experience average behavior in every frame. Using capacity of this composite channel, we obtain the required systematic comparison of the three methods.

3.3 Training Based Channel Equalizer

The MMSE estimator (Wiener filter) of the channel is ([27]) obtained by using mN_t training symbols. A MMSE equalizer is then designed using the channel estimate. The symbols obtained from the equalizer are decoded using minimum distance criterion (equivalent to ML decoding in Gaussian channels) to output hard decisions.

Let $\bar{\mathbf{u}}_{TS} \stackrel{\triangle}{=} \bar{\mathbf{u}}_L^{N_t} = \mathbf{A}_{TS}\bar{\mathbf{z}} + \bar{\mathbf{n}}_L^{N_t}$. This represents received samples corresponding to the last $m(N_t-L+1)$ training symbols. Here $\mathbf{A}_{TS}$ is an $n(N_t-L+1)\times Lnm$ matrix formed appropriately using training symbols $\bar{\mathbf{s}}_1^{N_t}$. Note that only the last N_t-L+1 samples of the output corresponding to training symbols can be used for channel estimation. The MMSE channel estimator ([27]) is given by $\hat{\bar{\mathbf{z}}} = \mu_{\bar{\mathbf{z}}} + C_z\mathbf{A}_{TS}^H(\mathbf{A}_{TS}C_z\mathbf{A}_{TS}^H + \sigma_n^2 I)^{-1}(\bar{\mathbf{y}}_{TS} - \mathbf{A}_{TS}\mu_{\bar{\mathbf{z}}})$, and $(\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$ are jointly Gaussian with mean $(\mu_{\bar{\mathbf{z}}}, \mu_{\bar{\mathbf{z}}})$. It is interesting to note that, for uncorrelated channels (i.e. when C_z is diagonal), the channel co-efficients corresponding to each one of the receive antennae signals can be estimated independently to reduce the computational complexity and the same applies for real and imaginary parts of the channel.

The corresponding M length left MMSE equalizer (of dimension $m\times Mn$) equals, $\underline{\bar{\theta}}(\hat{\bar{\mathbf{z}}}) = \left[\left(\hat{\mathcal{Z}}_M\hat{\mathcal{Z}}_M^H + \sigma_n^2 I\right)^{-1}\hat{\mathcal{Z}}_M\mathcal{I}_{M_Ln,m}\right]^H$, where $\mathcal{I}_{M_Ln,m}$ is the matrix formed by first m columns of the identity matrix of dimension $M_Ln\times M_Ln$, and $\hat{\mathcal{Z}}_M$ represents the $Mn\times M_Lm$ dimensional convolution matrix formed using $\hat{\bar{\mathbf{z}}}$. Using $\bar{\mathbf{z}}$ and $\underline{\bar{\theta}}(\hat{\bar{\mathbf{z}}})$, define convolution matrices, $\mathcal{Z}_{N-N_t+M-1}$ and Θ_{N-N_t}, each of dimensions $(N-N_t+M-1)n\times(N-N_t+M_L-1)m$, and $(N-N_t)m\times(N-N_t+M-1)n$ respectively. The equalizer output corresponding to transmitted input vector $\bar{\mathbf{s}}_{N_t+1}^N$ will be,

$$\bar{\mathbf{x}}_{N_t+1}^N = \Theta_{N-N_t}\left(\mathcal{Z}_{N-N_t+M-1}\bar{\mathbf{s}}_{N_t-M_L+2}^N + \bar{\mathbf{n}}_{N_t-M+2}^N\right). \tag{3.1}$$

It is clear that the vector $\bar{\mathbf{s}}_{N_t-M_L+2}^{N_t}$, corresponding to the tail of the training sequence is common to all the frames and is known. The output, $\hat{\bar{\mathbf{s}}}_{N_t+1}^N$, of the decoder is obtained from symbol by symbol decoding of $\bar{\mathbf{x}}_{N_t+1}^N$. We compute the overall transition probabilities, $\{P(\hat{\bar{\mathbf{s}}}_{N_t+1}^N/\bar{\mathbf{s}}_{N_t+1}^N)\}$, of the composite channel by first computing transition probabilities,

$\{P(\hat{\bar{\mathbf{s}}}^N_{N_t+1}/\bar{\mathbf{s}}^N_{N_t+1}, \bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})\}$, given $\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$ and then averaging over all values of $\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$. ($\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$ are Gaussian with known joint distribution). By defining B_i as,

$$B_i \stackrel{\triangle}{=} \{\mathbf{x}^N_{N_t+1} \in R^{2m(N-N_t)} : \|\bar{\jmath}_l - \bar{\mathbf{x}}^N_{N_t+1}\|^2 \geq \|\bar{\jmath}_i - \bar{\mathbf{x}}^N_{N_t+1}\|^2 \ \text{ for all } \ l \neq i\}$$

the transition probabilities given $(\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$, from (3.1), are

$$P(\hat{\bar{\mathbf{s}}}^N_{N_t+1} = \bar{\jmath}_i/\bar{\mathbf{s}}^N_{N_t+1} = \bar{\jmath}_j; \bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}) = Prob(\mathbf{x}^N_{N_t+1} \in B_i/\bar{\mathbf{s}}^N_{N_t+1} = \bar{\jmath}_j; \bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}).$$

It is easy to see that the overall transition probabilities of the composite channel, $\{P(\hat{\bar{\mathbf{s}}}^N_{N_t+1}/\bar{\mathbf{s}}^N_{N_t+1})\}$, can be computed at its receiver and transmitter once the statistics of the original channel is known.

Hence for a given N_t, the composite channel becomes a time invariant channel with channel state information known at the transmitter and receiver and hence its capacity is given by ([12]),

$$C(N_t) = \sup_{P(\bar{\mathbf{s}}^N_{N_t+1}) \in \mathcal{P}(\mathcal{S}^{(N_t)})} I(\hat{\bar{\mathbf{s}}}^N_{N_t+1}, \bar{\mathbf{s}}^N_{N_t+1}).$$

Here $I(\hat{\bar{\mathbf{s}}}^N_{N_t+1}, \bar{\mathbf{s}}^N_{N_t+1})$ represents the mutual information with input pmf (probability mass function) $P(\bar{\mathbf{s}}^N_{N_t+1})$ and transition probabilities $\{P(\hat{\bar{\mathbf{s}}}^N_{N_t+1}/\bar{\mathbf{s}}^N_{N_t+1})\}$. $\mathcal{P}(\mathcal{S}^{(N_t)})$ is the set of probability mass functions on $\mathcal{S}^{(N_t)}$ and is compact (Note $\bar{\mathbf{s}}^N_{N_t+1} \in \mathcal{S}^{(N_t)}$). Throughout, we consider $P(\bar{\mathbf{s}}^N_{N_t+1})$ as an element of an Euclidean space as the set $\mathcal{S}^{(N_t)}$ has finite elements. Since $P(\hat{\bar{\mathbf{s}}}^N_{N_t+1}/\bar{\mathbf{s}}^N_{N_t+1})$ is independent of the input pmf $P(\bar{\mathbf{s}}^N_{N_t+1})$, the mutual information $I(\hat{\bar{\mathbf{s}}}^N_{N_t+1}, \bar{\mathbf{s}}^N_{N_t+1})$ is a strictly concave function of $P(\bar{\mathbf{s}}^N_{N_t+1})$ ([12], p.31) and hence optimization over the convex set results in a global maximum. Capacity for training equalizer equals $C(N_t^*)$, where N_t^* is the optimum training length.

3.3.1 Computation of $C(N_t)$

As in GSM, the frame length N can be as large as 140 or even more. In such cases direct computation of $C(N_t)$ may be difficult. Hence we calculate a lower bound on the capacity by restricting the input vector $\bar{\mathbf{s}}_{N_t+1}^N$ to be a Markov chain. Given a specific Markov chain, $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N)$, is computed recursively (note that output $\hat{\bar{\mathbf{s}}}(i)$ is Hidden Markov) similar to the way in [48]. We show below that this provides a tight lower bound.

Let $\beta_z := [\ \mu_{\bar{\mathbf{z}}}^T \quad vect(C_z)^T\ ]^T$. This parametric vector specifies the distribution of the complex Gaussian random variable $\bar{\mathbf{z}}$ completely. Let $g(\beta_z)$ represent the conditional probabilities of the composite channel for a given β_z and let $f(P(.|.))$ be a capacity achieving input distribution for given conditional probabilities $P(.|.)$.

Lemma 3.1 *g and f are continuous functions. Hence $f_o g$ is continuous in β_z.*

Proof : Please refer to the Appendix B.

For channels with zero ISI (say with β_z^0 as parametric vector), it is easy to see that $g(\beta_z^0)$ is memoryless (i.e. $P(\hat{\bar{\mathbf{s}}}_{N_t+1}^N / \bar{\mathbf{s}}_{N_t+1}^N) = \Pi_i P(\hat{\bar{\mathbf{s}}}(i)/\bar{\mathbf{s}}(i)))$ and $f_o g(\beta_z^0)$ will be an IID sequence of m length vectors. By the above lemma, for channels with small ISI or for systems with equalizers compensating the ISI to a good extent, a tight lower bound on the capacity can be achieved by restricting the input distributions to l-step Markov chains. Further for noiseless systems with ISI and ICI eliminated completely (with sufficient N_t), $g(\beta_z^0)$ will be an identity matrix and $f_o g(\beta_z^0)$ will be uniform IID, i.e. an IID sequence with each m length vector uniformly distributed. Thus under high SNR (Signal to Noise ratio) conditions, mutual information with IID and uniform distribution will itself form a tight lower bound. We compute this lower bound (in place of capacity) in Section 3.6. In fact, while computing capacity for examples in Section 3.6, we noticed that the composite channel transition matrix is nearly symmetric and that this bound is very tight even for low SNR conditions and also for blind and semiblind algorithms.

Computing $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N)$ with Markov chain input

Given a specific Markov chain, $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N)$, is computed recursively (note that output $\hat{\bar{\mathbf{s}}}(i)$ is Hidden Markov), as explained below. This algorithm is similar to the methods used in [48]. The Markov chain can have any finite memory, i.e. we take an l-step Markov chain with l being finite.

Let $M_d \stackrel{\triangle}{=} \max(M_L - 1, l)$ and let $\bar{\mathbf{s}}^i$ represent $\bar{\mathbf{s}}^i_{N_t+1}$. We have

$$I\left(\hat{\bar{\mathbf{s}}}^i; \bar{\mathbf{s}}^i\right) = H(\bar{\mathbf{s}}^i) + H(\hat{\bar{\mathbf{s}}}^i) - H(\hat{\bar{\mathbf{s}}}^i, \bar{\mathbf{s}}^i).$$

Here $H(.)$ represents entropy. For a Markov chain, $H(\bar{\mathbf{s}}^i)$ can be calculated easily.

Next we consider $H(\hat{\bar{\mathbf{s}}}^i)$. Clearly, $P(\hat{\bar{\mathbf{s}}}(i)/\bar{\mathbf{s}}^i_{i-M_d}, \bar{\mathbf{n}}^i_{i-M+1})$, the probability of the current output symbol given the inputs and noise symbols on which it directly depends, is independent of step i, (for initial steps it also depends upon the tail of training sequence) and can be computed by first conditioning on $(\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$ and then averaging in a way similar to that explained above. These conditional probabilities will be the basic elements in the calculation of $H(\hat{\bar{\mathbf{s}}}^i)$. Now,

$$H(\hat{\bar{\mathbf{s}}}^N) = \sum_{i=N_t}^{N} H(\hat{\bar{\mathbf{s}}}(i)/\hat{\bar{\mathbf{s}}}^{i-1}) = -\sum_{i=N_t}^{N} \mathbb{E}[log(p_i)]$$

where $p_i \stackrel{\triangle}{=} P(\hat{\bar{\mathbf{s}}}(i)/\hat{\bar{\mathbf{s}}}^{i-1})$. At any step i, p_i equals,

$$p_i = \int_{\mathbb{C}^{nM}} \sum_{\bar{\mathbf{s}}^i_{i-M_d}} P(\hat{\bar{\mathbf{s}}}(i)/\bar{\mathbf{s}}^i_{i-M_d}, \bar{\mathbf{n}}^i_{i-M+1}) dF(\bar{\mathbf{n}}^i_{i-M+1}, \bar{\mathbf{s}}^i_{i-M_d}/\hat{\bar{\mathbf{s}}}^{i-1}).$$

The above step follows as the output $\hat{\bar{\mathbf{s}}}(i)$ is conditionally independent of $\hat{\bar{\mathbf{s}}}^{i-1}$ given inputs $\bar{\mathbf{s}}^i_{i-M_d}$ and noise $\bar{\mathbf{n}}^i_{i-M+1}$ (Note that the output $\hat{\bar{\mathbf{s}}}(i)$ is a Hidden Markov chain depending on the Markov chain $(\bar{\mathbf{s}}^i_{i-M_d}, \bar{\mathbf{n}}^i_{i-M+1})$). One can easily show that the conditional distributions $F(\bar{\mathbf{n}}^i_{i-M+1}, \bar{\mathbf{s}}^i_{i-M_d}/\hat{\bar{\mathbf{s}}}^{i-1})$ can be computed recursively as,

$$\begin{aligned}
&F(\bar{\mathbf{n}}^i_{i-M+1}, \bar{\mathbf{s}}^i_{i-M_d}/\hat{\bar{\mathbf{s}}}^{i-1}) \\
&= Prob(\bar{\mathbf{s}}^i_{i-M_d}, \mathbb{N}^i_{i-M+1} \leq \bar{\mathbf{n}}^i_{i-M+1}/\hat{\bar{\mathbf{s}}}^{i-1}) \\
&= \frac{F(\bar{\mathbf{n}}(i))}{p_{i-1}} \sum_{\bar{\mathbf{s}}(i-M_d-1)} \int_{\bar{\mathbf{n}}(i-M)\in\mathbb{C}^n} P(\bar{\mathbf{s}}(i)/\bar{\mathbf{s}}^{i-1}_{i-1-M_d}) P(\hat{\bar{\mathbf{s}}}(i-1)/\bar{\mathbf{s}}^{i-1}_{i-1-M_d}, \bar{\mathbf{n}}^{i-1}_{i-M}) \\
&\qquad dF(\bar{\mathbf{n}}^{i-1}_{i-M}, \bar{\mathbf{s}}^{i-1}_{i-1-M_d}/\hat{\bar{\mathbf{s}}}^{i-2}).
\end{aligned}$$

Here $\mathbb{N}^i_{i-M+1}$ represents the noise at steps i to $i-M+1$ and $F(.)$ represents the cumulative distribution.

$H(\hat{\bar{\mathbf{s}}}_{N_t+1}^N, \bar{\mathbf{s}}_{N_t+1}^N)$ can be computed in a similar way. Similar computations will be used in the case of blind and semi-blind methods also.

3.4 Blind CMA Equalizer

In this section, we use the CMA algorithm to obtain the equalizer, without using any training sequence. We first explain the algorithm and then provide the procedure we use to obtain the capacity for this system. In fact, as mentioned before, it is practically infeasible to obtain the exact capacity in this case. Thus, we find a lower bound on the capacity by restricting the input distribution to be ergodic and stationary. We further assume it to be a Markov chain and use the procedure provided at the end of last section for computing the lower bound. The same applies to the semi-blind method.

The CMA equalizer is obtained by using the LMS algorithm ([15], [22]) to minimize the following cost function. For a single user MIMO channel with same source alphabet for all m transmit antennae, the CMA cost function is ([15])

$$\Sigma_{l=1}^m \mathbb{E}\left(|\bar{\theta}_l^H \bar{\mathbf{u}}_{k-M+1}^k|^2 - R_2^2\right)^2 .$$

This is equivalent to minimizing m independent cost functions (since the terms in the summation are positive),

$$\mathbb{E}\left(|\bar{\theta}_l^H \bar{\mathbf{u}}_{k-M+1}^k|^2 - R_2^2\right)^2, \; l = 1, 2, \ldots m \tag{3.2}$$

where $\bar{\theta}_l^T$, an nM length vector, represents the l^{th} row equalizer (to decode l^{th} transmit antenna symbol) and $R_2 = \mathbb{E}|\bar{\mathbf{s}}_{N-M_L+1}^N|^4 / \mathbb{E}|\bar{\mathbf{s}}_{N-M_L+1}^N|^2$.

To obtain the above optimum, the corresponding m update equations in the LMS algorithm for a given value of $\bar{\mathbf{z}}$ are ($1 \le l \le m$, $M_L \le k \le N$),

$$\theta_l(k+1) = \theta_l(k) + \mu H_{CMA}\left(\theta_l(k), \mathbf{z}(k), \mathbf{n}_{k-M+1}^k\right) \tag{3.3}$$

where $\theta_l(k)$ is the l^{th} equalizer at time k, $\bar{\mathbf{z}}(k) \stackrel{\triangle}{=} \bar{\mathbf{u}}_{k-M+1}^k - \bar{\mathbf{n}}_{k-M+1}^k = \mathcal{Z}_M \bar{\mathbf{s}}_{k-M_L+1}^k$ and

$$H_{CMA}(\theta, \mathbf{z}, \mathbf{n}) \stackrel{\triangle}{=} ((\theta^T \mathbf{u})^2 + (\theta^T \check{\mathbf{u}})^2 - R_2^2) \left((\theta^T \tilde{\mathbf{u}})\tilde{\mathbf{u}} + (\theta^T \check{\mathbf{u}})\check{\mathbf{u}}\right)$$

with $\bar{\mathbf{u}} = \bar{\mathbf{z}} + \bar{\mathbf{n}}$ and $\tilde{\mathbf{u}}$, $\check{\mathbf{u}}$ defined in notations.

We observe that $H_{CMA}(\mathbf{0}, \mathbf{z}, \mathbf{n}) = 0$ for all $\mathbf{z}, \mathbf{n}$, but $\mathbf{0}$ is not a minimizer of the cost function in (3.2). Thus, $\theta_l(0)$ should not be initialized with $\mathbf{0}$ in (3.3) for any l.

A close look at (3.3) shows that all m sub cost functions are same and the different equalizers should be initialized appropriately to extract the desired antenna's source symbols. In [15] a new joint CMA algorithm is proposed that ensures that the MIMO CMA separates all the sources successfully irrespective of the initial conditions. In this work, we choose the initial condition $\underline{\overline{\theta}}_0^*$ (which will be used in all frames) such that the channel capacity is optimized, where $\underline{\overline{\theta}}_0^*$ is an m-row matrix with its l^{th} row corresponding to the optimal initial condition for l^{th} row equalizer θ_l. This solves initialization problem to a good extent (at-least for channels with good line of sight signal) in blind case with the original CMA itself. The problem will be solved to a greater extent in the semiblind algorithm, as here a rough estimate of the training based equalizer forms the initializer.

The equalizer adaptation (3.3) can be started only after leaving out the first $M_L - 1$ received samples. This ensures that the equalizer adaptation is independent of previous frame symbols. Hence initializer $\underline{\overline{\theta}}_0$ actually corresponds to the m-row equalizer at step $k = M_L$. One may also update the equalizer tap only for a fraction of the symbols in the frame. This reduces delay in processing.

For real constellations like BPSK, a more suitable CMA cost function would be $((\theta^T \mathbf{u})^2 - R_2^2)^2 + (\theta^T \check{\mathbf{u}})^4$. All our analysis extends easily to this cost function also. This modified cost function forces the imaginary part of the equalizer output to zero. While computing the capacity for some examples numerically (in Section 3.6), we used this cost function with BPSK over complex channels.

While computing the capacity of the composite channel for this system, one faces a problem that, at the end of the frame, the CMA algorithm might not have converged (see more detailed comments on convergence in ([47]). Thus the equalizer is random even for fixed $\bar{\mathbf{h}}$ and computing its distribution for a practical frame length is almost impossible. In the next subsection, we approximate the value of the CMA equalizer at any time t with a deterministic quantity and then proceed with obtaining the channel capacity with that equalizer.

3.4.1 CMA Equalizer approximated by ODE

Now we assume that the input is stationary and ergodic. Each of the m update equations in (3.3) is similar to the CMA update equation for SISO. Therefore, it is easy to see that all the proofs in [47] for convergence of the CMA trajectory to the solution of an ODE (Ordinary differential equation) hold. As a result, we have (for any $0 < T < \infty$ and for any $1 \leq l \leq m$),

$$\sup_{0\leq t\leq T} \|\theta_l(\lfloor t/\mu \rfloor) - \theta_{ODE}(t)\| \xrightarrow{p} 0 \text{ as } \quad \mu \to 0$$

where $\theta_{ODE}(t)$, henceforth written as $\theta(t)$ (for notational simplicity), satisfies

$$\dot{\theta}_l(t) = \hat{H}_{CMA}(\theta_l(t)) \stackrel{\triangle}{=} \mathbb{E}_{\mathbf{z}}[\mathbb{E}_{\mathbf{n}}(H_{CMA}(\theta_l(t), \mathbf{z}, \mathbf{n}))] \tag{3.4}$$

where $\bar{\mathbf{z}} \stackrel{\triangle}{=} \mathcal{Z}_M \bar{\mathbf{s}}_{N-M_L+1}^N$. The initial conditions for both the ODE and the CMA are same. Thus the update equation (3.3) for any given $\bar{\mathbf{z}}$, can be approximated by the trajectory of the above ODE. The approximation can be made accurate with high probability by taking step size μ small. Note that the distribution of $\bar{\mathbf{z}}$ can be defined with respect to $\bar{\mathbf{s}}_{k-M_L+1}^k$, for any step k, as the input is assumed to have stationary distribution.

We obtain the capacity of the composite channel approximately by obtaining the capacity of the channel using the solution of the ODE as equalizer. The actual practical system may not use ODE. It should use (3.3). This computation is for offline comparison of performances.

We can solve (3.4) numerically for each l and obtain the equalizer $\underline{\overline{\theta}}(T)$ at time $T = \mu(N - M_L + 1)$ (a smaller T has to be taken if the equalizer is updated only for a fraction of the frame), which approximates the CMA equalizer at the end of the frame. This equalizer is used for decoding the entire frame as in previous section. It is easy to see that, $\underline{\overline{\theta}}(T)$ can be computed at the transmitter and the receiver (of the approximate composite channel) once the original channel statistics are known and hence the overall transition probabilities can be calculated as in the previous section. The only difference being that these transition probabilities depend upon the input distribution $P(\bar{\mathbf{s}}_{N_t+1}^N)$ and the common initial equalizer setting $\underline{\overline{\theta}}_0$. (Actually $N_t = 0$. But we use the same notation to maintain uniformity).

Given a value of $\bar{\mathbf{z}}$ equation (3.4) becomes ([47]),

$$\begin{aligned}
\dot{\theta}_l(t) &= \mathbb{E}_{\bar{\mathbf{s}}_{N-M_L+1}^N}[f(\tilde{\mathbf{z}}, \theta_l(t)) + f(\check{\mathbf{z}}, \theta_l(t))] \\
&- \mathbb{E}_{\bar{\mathbf{s}}_{N-M_L+1}^N}[(\theta_l^T(t)\tilde{\mathbf{z}})(\theta_l^T(t)\check{\mathbf{z}})\left(\tilde{\mathbf{z}}\check{\mathbf{z}}^T + \check{\mathbf{z}}\tilde{\mathbf{z}}^T\right)]\theta_l(t) \\
&+ 2R_2^2\sigma_n^2\theta_l(t) - 8\sigma_n^4\|\theta_l(t)\|^2\theta_l(t) - 6\sigma_n^4\theta_l(t)^{(2,1)} - 6\sigma_n^4\theta_l(t)^{(3)} \qquad (3.5)
\end{aligned}$$

where, $f(\mathbf{z},\theta) \stackrel{\triangle}{=} R_2^2\theta^T\mathbf{z}\mathbf{z} - (\theta^T\mathbf{z})^3\mathbf{z} - 4\sigma_n^2((\theta^T\mathbf{z})^2\theta + \|\theta\|^2\theta^T\mathbf{z}\mathbf{z})$. Matrix $R_{\bar{\mathbf{s}}_{N-M_L+1}^N}$ is the source covariance matrix, $\theta_l(t)^{(3)}$ is the vector formed by taking cube of the individual terms and $\|.\|$ represents the norm of the vector. Also, $\theta_l(t)^{(2,1)}$ is the vector formed by taking square of the individual terms in $\check{\theta}_l(t)$ and then multiplying term by term with vector $\theta_l(t)$.

It is clear that $\underline{\overline{\theta}}(T)$ is a function of $\bar{\mathbf{z}}, \underline{\overline{\theta}}_0$ and $P(\bar{\mathbf{s}}_{N-M_L+1}^N)$. Define $\Theta(T) \stackrel{\triangle}{=} \Theta_{N-N_t}(T)$, the $N-N_t$ length convolution matrix constructed using $\underline{\overline{\theta}}(T)$. Let $\Theta_R \stackrel{\triangle}{=} \begin{bmatrix} Re(\Theta(T)) & -Im(\Theta(T)) \\ Im(\Theta(T)) & Re(\Theta(T)) \end{bmatrix}$. Here $Re(\Theta(T))$ and $Im(\Theta(T))$ represent the matrices formed by keeping only the real or imaginary part, respectively, of each component of the matrix $\Theta(T)$.

As in the previous section, conditioned on $\bar{\mathbf{z}}, P(\bar{\mathbf{s}}_{N_t+1}^N)$ and $\underline{\overline{\theta}}_0$, the transitional probabilities of the approximate composite channel obtained by solving the ODE are (the probabilities of the initial $L-1$ symbols will be different, but the number is very small with respect to the frame length and hence it can be neglected),

$$Q(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j; \bar{\mathbf{z}}) = Q(\Theta_R \mathbf{u}_{N_t-M+2}^N \in B_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j; \bar{\mathbf{z}}) \qquad (3.6)$$

where $Q(.) \stackrel{\triangle}{=} P(./\underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$, represents the conditional distribution given $\underline{\overline{\theta}}_0$ and $P(\bar{\mathbf{s}}_{N_t+1}^N)$

and B_i is defined in the previous section. The overall transition probabilities for a given $P(\bar{\mathbf{s}}_{N_t+1}^N)$ and $\underline{\overline{\theta}}_0$ are now obtained by averaging over $\bar{\mathbf{z}}$. The conditional mutual information $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / \underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$, is a bounded function of $(\underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$ as the cardinality of the input alphabet, $\bar{\mathbf{s}}_{N_t+1}^N$and the output alphabet $\hat{\bar{\mathbf{s}}}_{N_t+1}^N$is finite. Thus, capacity (lower bound as input is restricted to be stationary and ergodic)

$$C(\underline{\overline{\theta}}_0) \stackrel{\triangle}{=} \sup_{P(\bar{\mathbf{s}}_{N_t+1}^N)} I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / \underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$$

of the approximate channel for a given $\underline{\overline{\theta}}_0$, is finite. Note that the approximate composite channel for a given $\underline{\overline{\theta}}_0$, is a discrete memoryless channel as in the case of the training based equalizer. The overall capacity of the blind algorithm $C_{CMA} \approx \sup_{\underline{\overline{\theta}}_0} C(\underline{\overline{\theta}}_0)$, is also finite.

The following lemma establishes the continuity of the mutual information with respect to input distribution $P(\bar{\mathbf{s}}_{N_t+1}^N)$ for any given $\underline{\overline{\theta}}_0$ and hence the achievability of $C(\underline{\overline{\theta}}_0)$. It also establishes the continuity of $C(\underline{\overline{\theta}}_0)$ with respect to $\underline{\overline{\theta}}_0$ and hence the achievability of C_{CMA} (when $\underline{\overline{\theta}}_0$ is restricted to a compact domain).

Lemma 3.2 *$I\left(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / \underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N)\right)$ is a continuous function of $\underline{\overline{\theta}}_0$ and $P(\bar{\mathbf{s}}_{N_t+1}^N)$. Also $C(\underline{\overline{\theta}}_0)$ is continuous in $\underline{\overline{\theta}}_0$.*

Proof : Please refer to the Appendix A.

Limiting $\underline{\overline{\theta}}_0$ to a compact domain can be justified as follows. Since $\bar{\mathbf{z}}$ has finite variance, for any given $\epsilon > 0$, by Tchebyshev's inequality, we find an $U_\epsilon < \infty$ such that $\|\bar{\mathbf{z}} - \mu_{\bar{\mathbf{z}}}\| < U_\epsilon$ with probability greater than $1 - \epsilon$. Let $\underline{\overline{\theta}}^*(\bar{\mathbf{z}})$ represent a CMA (also ODE) attractor for channel realization $\bar{\mathbf{z}}$. One can show using arguments similar to Lemma 7 in [47] that $\underline{\overline{\theta}}^*(\bar{\mathbf{z}})$ is continuous in $\bar{\mathbf{z}}$. Thus there exists $U_\epsilon^* < \infty$ such that $\|\underline{\overline{\theta}}^*(\bar{\mathbf{z}}) - \underline{\overline{\theta}}^*(\mu_{\bar{\mathbf{z}}})\| < U_\epsilon^*$ with probability greater than $1 - \epsilon$. Thus, we can limit $\underline{\overline{\theta}}_0$ to the compact disk $\{\underline{\overline{\theta}} : \|\underline{\overline{\theta}} - \underline{\overline{\theta}}^*(\mu_{\bar{\mathbf{z}}})\| \leq U_\epsilon^*\}$.

The compact domain can be located more easily (which can be used in computing numerical examples) as follows. It is shown in [65] that CMA attractor lies close to

MMSE equalizer (channel estimation is assumed perfect here) under reasonable conditions. Hence, $\underline{\overline{\theta}}_0$ can be constrained to a compact region around $\underline{\overline{\theta}}_{0\mu_{\bar{\mathbf{z}}}}:=\underline{\overline{\theta}}(\mu_{\bar{\mathbf{z}}})$ (MMSE equalizer for mean $\mu_{\bar{\mathbf{z}}}$ of the channel, defined in a way similar to $\underline{\overline{\theta}}(\hat{\bar{\mathbf{z}}})$ of the previous section) instead of $\theta^*(\mu_{\bar{\mathbf{z}}})$.

Thus, when the receiver and the transmitter have channel statistics, one can choose $\underline{\overline{\theta}}_0^*$ and $P^*(\bar{\mathbf{s}}_{N_t+1}^N)$, such that $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N;\bar{\mathbf{s}}_{N_t+1}^N/\underline{\overline{\theta}}_0^*, P^*(\bar{\mathbf{s}}_{N_t+1}^N))$ is as close as possible to C_{CMA}. However, it may be computationally intensive. Thus, in Section 3.6 we *compute a lower bound by estimating the mutual information for IID uniform input distribution and using* $\underline{\overline{\theta}}_0^* = \underline{\overline{\theta}}_{0\mu_{\bar{\mathbf{z}}}}$ to simplify the computations. We have *established the tightness of this lower bound in case of training based methods for high SNR regions.* Hence, *comparing the blind/semiblind methods with training based methods using this lower bound, establishes that the improvement of the blind/semiblind algorithms would be better by at least that much* (even if the bound is not tight in case of blind/semiblind methods). In fact, while computing the numerical examples of Section 3.6, we have seen that this bound is very tight for all three; blind, semiblind and training methods, for all most all the cases we studied.

3.5 Semi-Blind CMA Algorithm

In this section we study a semi-blind algorithm obtained by combining the previous two algorithms. We would like to see if this can improve the overall performance of the system.

We use the MMSE equalizer of the training based channel estimator $\underline{\overline{\theta}}(\hat{\bar{\mathbf{z}}})$ obtained in Section 3.3 as the initializer for the CMA algorithm. The equalizer co-efficients obtained from the CMA at the end of the frame are used for decoding the data for the whole frame. Once again we use the ODE approximation of the CMA trajectory in the capacity analysis. Now $T = \mu(N-N_t)$ if $N_t \geq M_L$. Otherwise, $T = \mu(N-M_L)$ and we start the CMA adaptation only after the first M_L-1 samples as in previous section. The conditional probabilities are obtained by first conditioning on and then averaging with respect to $\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$ as in Section 3.3.

As in the previous section, given N_t, the capacity (lower bound),

$$C_{SB}(N_t) = \sup_{P(\bar{\mathbf{s}}_{N_t+1}^N)} I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / P(\bar{\mathbf{s}}_{N_t+1}^N))$$

exists and the approximate capacity C_{SB} for the semi-blind algorithm is $C_{SB}(N_t^*)$, where N_t^* is the optimal training length. Once again, we have the following lemma, which establishes the achievability of the capacity.

Lemma 3.3 $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / P(\bar{\mathbf{s}}_{N_t+1}^N))$ *is a continuous function of* $P(\bar{\mathbf{s}}_{N_t+1}^N)$ *and hence the supremum can be achieved.*

Proof : Please refer to Appendix B.

Having obtained the capacity of the channel with training based, blind and semiblind methods, one can compare them for any MIMO wireless (block fading) channel. As mentioned above, we use the IID uniform distribution lower bound to compare the three. Then one can obtain the optimal scheme (say a semiblind channel with a given length N_t of the training sequence). In the next section we carry out this comparison for a few examples.

3.5.1 Modified Semi-blind Algorithm

We have modified the above semiblind algorithm to obtain the following improved algorithm. We get an estimate of the channel realization using training sequences and then one can estimate the channel and equalizer-initializer combined response. This indicates how close the initializer is to the attractor. We adapted the step size of the CMA algorithm based on this estimate. This modified algorithm improves the performance of the semi blind algorithms. The improvement is significant at low K-factors with good SNR conditions. The details of the modification and the corresponding gains for some examples are provided at the end of the next section.

3.6 Examples

We illustrate the theory developed with a few examples of practical interest. As a result we also draw some interesting conclusions. The three equalizers are compared over complex Gaussian channels with BPSK modulation. We consider 2×2 MIMO channels. The frame length N is 64 symbols. The channel lengths are 2, 3 and 4 and the equalizers are of length 2. We set $C_h = \sigma_h^2 I$. The channel gain is normalized to 1, for both the receive antennas. We follow the systematic approach explained above for calculating the capacity of the composite channel for all the equalizers. We observed that the equalizer value at the end of the frame was away from the equilibrium point in both the blind and semiblind algorithms.

In Figure 3.2 we plot the capacity of the three equalizers versus transmitted power with $\sigma_n^2 = 1$ and $L = 4$. Here, the first channel tap is Ricean, while the rest of the taps are Rayleigh. For this figure, all the four elements of the 2×2 mean matrix for the first tap have equal norm. The mean matrix of channel $\mu_{\mathbf{Z}}$ and the relative power ratios between the channel taps is given in Figure 3.2. We varied the K-factor (ratio of power in the mean component to that in the varying component of the first tap) of the channel during our experiments. For $K = 3$, (typical to terrain with moderate tree density ([18]) and some of the indoor channels), there is an improvement of up to 3 dB ($\approx 67\%$ improvement in TX power) in semiblind/blind (blind being the best) algorithms at around 12dB compared to the training method. At $K = 1.5$, the improvement is $\approx 20\%$ and the semiblind is the best. As the K-factor approaches 0 (Rayleigh channel), the improvement in semiblind diminishes, but, the blind becomes much worse. In fact, in Rayleigh channel, the capacity of the blind method is almost zero.

We have observed that for training and semiblind equalizers, the capacity increases with N_t, reaches a maximum and starts decreasing. From this, one can estimate the 'optimal' number of training symbols, N_t^* (see Table 3.1 for some examples). It is interesting to note that the same procedure can also be used for choosing the optimal training sequences (for a given N_t). This is possible mainly because the input alphabet is finite and hence a finite number of comparisons will do the job. This might be tedious and we have not conducted experiments in this regard.

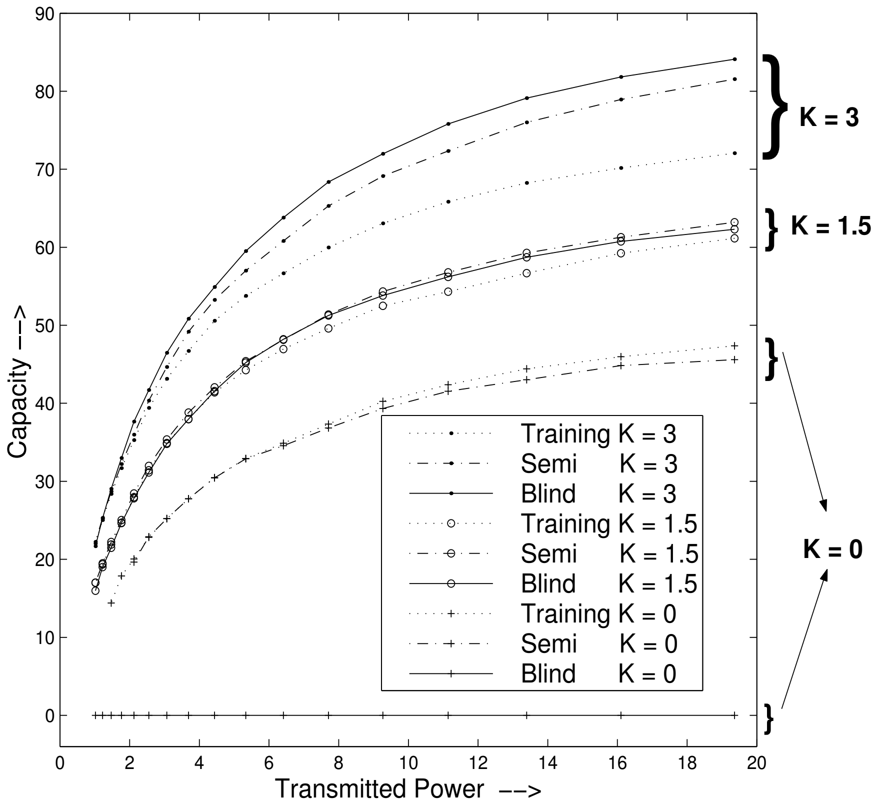

Figure 3.2: Capacity vs Txd power : $\sigma_n^2 = 1$ $L = 4$ $M = 2$
Relative Power ratios of channel taps in dB = $\left[\begin{array}{cccc} 0 & -5.4 & -10 & -13.7 \end{array}\right]$

$$\mu_{\mathbf{z}} = \sqrt{\frac{K}{1+K}} \left[\begin{array}{cccccccc} 0.42+j0.42 & -0.42+j0.42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.42+j0.42 & 0.42+j0.42 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Table 3.1 also shows comparison of various equalizers with respect to noise variance for two different K factors, for a fixed transmit power $\theta_A = 12dB$. Here also only the first tap is Ricean while the rest are Rayleigh. But the terms in the mean matrix of the first tap have unequal norms, the diagonal terms having bigger norm than the off diagonal ones. Further, the relative power ratios of the taps, given in the heading of the Table, are better than in Figure 3.2. Also the channel length $L = 3$. As expected, the capacity of the equalizers in this example is more than that in Figure 3.2. More importantly, semiblind/blind algorithms show more improvement over training methods compared to the example in Figure 3.2. Here we obtain a much better improvement of 29% at $K = 1.3$ and 55% at $K = 2$ by semiblind/blind methods over the training equalizers. Match σ_n^{*2} in the Table, represents the noise variance σ_n^2 of the system for the Blind (in the fifth column) or the Semiblind (in the last column) algorithm at which the capacity is within 0.1% of that of the training based equalizer.

As expected, the capacity of all three algorithms increases with decrease in noise variance and for high K systems the optimal number of training symbols is smaller. For high K systems, blind algorithm is the best at reasonable SNR's. This is because the other algorithms are loosing in capacity because of training symbols. But, we can see that at very low SNR's (near 0dB, not shown in the table, but this effect can be seen in Figure 3.2) the blind algorithm becomes worse eventually.

As commented in Section 3.4, we observed in case of high K, the blind algorithm was successfully separating all the sources (which also explains comparatively good performance of blind algorithms at high K). But for low K, the blind algorithm was not separating the sources resulting in bad performance (K = 1.3 case in Table 3.1). All the above observations are mainly because of the following reason. At high K values, the mean value of the channel is close to channel realizations with high probability. Hence taking the common initializer (common to all realizations) $\underline{\overline{\theta}}_0 = \underline{\overline{\theta}}_{0\mu_{\bar{\mathbf{h}}}}$ ensures that the initializer is close to the CMA attractor (and hence to Wiener Receiver ([65])) with high probability.

For the examples explained above, we performed optimization with respect to input distribution ($k-$step Markov chains) and found that uniform IID distribution gives a very tight lower bound even at low SNR's. The same is true for semiblind and blind algorithms. Similarly, for the blind algorithm, we performed optimization with respect to $\underline{\overline{\theta}}_0$ and found that in many cases $\underline{\overline{\theta}}_0 = \underline{\overline{\theta}}_{0\mu_{\bar{\mathbf{h}}}}$ gives a tight lower bound. These observations can be exploited to simplify the computations considerably. For all the examples, we used the steepest gradient method using forward differences as derivative to obtain the optimal point. While optimizing with the input distributions, the updated value was further projected on to the space of probability measures.

Table 3.1: Capacity Vs Noise Variance $\theta_A = 12dB$ $L = 3$ $M = 2$
Relative power ratios of the channel taps in $dB = \begin{bmatrix} 0 & -8.9 & -13.3 \end{bmatrix}$

$$\mu_{\bar{\mathbf{z}}} = \sqrt{K/(1+K)} \begin{bmatrix} 0.7+i0.525 & -0.206+i0.206 & 0 & 0 & 0 & 0 \\ -0.206+i0.206 & 0.7+i0.525 & 0 & 0 & 0 & 0 \end{bmatrix}$$

σ_n^2	K = 2				K = 1.3			
	Training (C, N_t^*)	Semi (C, N_t^*)	Blind C	Match σ_n^{*2}	Training (C, N_t^*)	Semi (C, N_t^*)	Blind C	Match σ_n^{*2}
1	(80.8, 5)	(86.3, 5)	89.4	1.76 B	(71.4, 5)	(75.4, 5)	74.1	1.35 S
2	(70.3, 5)	(76.7, 5)	78.4	2.78 B	(62.3, 5)	(65.9, 5)	64.3	2.47 S
4	(55.9, 7)	(60.2, 5)	60.9	4.75 B	(50.2, 7)	(53.1, 7)	49.5	4.35 S
6	(46.2, 7)	(48.9, 4)	49.2	6.67 B	(41.2, 7)	(43.6, 7)	40.8	6.50 S
8	(39.2, 7)	(41.1, 7)	41.2	8.80 B	(35.1, 7)	(36.5, 7)	34.7	8.50 S

Modified Semi-blind Algorithm

Here we divided the channel realizations into 7 bins based on the estimate of the channel and initializer combined response. Let $\bar{\mathbf{s}}_l$ represent the convolutional vector of the channel estimate with the training equalizer (initializer) for the l^{th} antenna. We used $\|\bar{\mathbf{s}}_{l_l}, r\|$, the real part of the l^{th} term of vector $\bar{\mathbf{s}}_l$, for dividing the channel realization in to 7 bins. The step size of the bin, with $\|\bar{\mathbf{s}}_{l_l}, r\|$ close to 1, is large. For bins with $\|\bar{\mathbf{s}}_{l_l}, r\|$ away from 1 (either below 0.05 or above 2) the step-size is almost zero. This is equivalent to using only the training based equalizer.

Table 3.2 gives the improvement of the modified algorithm with respect to the original semiblind algorithm. Match σ_n^{*2} gives the noise variance at which the modified-semiblind /semiblind algorithm has the capacity within 0.1% of that of the training capacity at $\sigma_n^2 = 1$ (similar to Table 3.1). In other words, it gives the SNR improvement of modified-semiblind / semiblind algorithms over training based methods at transmit power (equivalently SNR) equal to $12dB$. From the table one can see an improvement of up-to $30-46\%$, in transmit power, by the modified-semiblind algorithm over the original semiblind algorithm. The modified-semiblind algorithm shows significant SNR improvement of $30\%, 20\%$ and 8.6% at $K = 0.9, 0.1$ and 0 respectively over the training method. For these K-factors, the original semiblind is degraded compared to the training based methods as is evident from Table 3.2. Also, note that we have not shown the blind capacity in this table, as, at these K factors the blind performs worse than the original semiblind itself (evident also in Table 3.1).

These are initial experiments on the modifications possible. One can try working further in this direction and may achieve much better results. This algorithm is expected to give improvement in high SNR regions only, as the estimate of channel will be less noisy.

Table 3.2: Modified Semi-Blind Vs Semiblind $\theta_A = 12dB$, $L = 2$ $M = 2$
Relative power ratios of the channel taps in $dB = \begin{bmatrix} 0 & -7.5 \end{bmatrix}$

$$\mu_{\bar{\mathbf{z}}} = \sqrt{K/(1+K)} \begin{bmatrix} 0.52{+}i0.39 & -0.46 + i0.46 & 0 & 0 \\ -0.46 + i0.46 & 0.52{+}i0.39 & 0 & 0 \end{bmatrix}$$

K	Training	ModSemi		Semi	
	(C, N_t^*) $\sigma_n^2 = 1$	(C, N_t^*) $\sigma_n^2 = 1$	Match σ_n^{*2}	(C, N_t^*) $\sigma_n^2 = 1$	Match σ_n^{*2}
0.9	(65.55, 8)	(70.02, 6)	1.35	(65.42, 6)	0.99
0.5	(61.09, 8)	(64.79, 8)	1.28	(59.00, 8)	0.83
0.1	(57.74, 8)	(60.48, 8)	1.23	(54.32, 8)	0.76
0.01	(58.25, 8)	(60.38, 8)	1.12	(54.53, 8)	0.68
0.0	(58.46, 8)	(60.45, 8)	1.09	(55.20, 8)	0.71

3.7 Conclusions

We compared blind/semiblind equalizers with training based algorithms. The difficulty is in comparing the loss in accuracy of the blind algorithms with that of loss in data rate in training based methods. Information capacity is the most appropriate measure for this comparison. We observed that the semiblind/blind methods perform superior to training methods in LOS conditions (≈50 to 70% improvement in transmit power) even when they have not converged to the equilibrium point. But for Rayleigh fading, the semiblind methods are bad compared to training based and the blind methods become completely useless. Our approach is also used to obtain the optimum number of training symbols.

We modified the semiblind algorithm, where the step size is adapted with respect to the training based channel estimate $\hat{\hat{\mathbf{h}}}$. Initial experiments using this approach show good improvement over the original semi-blind algorithm. The improvement is significant for low K-factors (including Rayleigh channels) under high SNR conditions. One may achieve better results by further investigation.

Appendix A

In this Appendix, we provide proof of Lemma 3.2, We will need several more lemmas. The proofs of remaining Lemmas 3.1, and 3.3 will be given in Appendix B. This is done to avoid repetition of arguments.

Proof of Lemma 3.2: Lemma 3.4 and 3.7 provide a proof of Lemma 3.2. ■

Lemma 3.4 $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / \underline{\bar{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$ *is a continuous function of* θ_0 *and* $P(\bar{\mathbf{s}}_{N_t+1}^N)$.

Proof of Lemma 3.4: Let $(\underline{\bar{\theta}}_{0_n}, P(\bar{\mathbf{s}}_{N_t+1}^N)_n) \to (\underline{\bar{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$.

Denote $\Theta_R(\underline{\bar{\theta}}_{0_n}, \bar{\mathbf{z}}, P(\bar{\mathbf{s}}_{N_t+1}^N)_n)$ and $\Theta_R(\underline{\bar{\theta}}_0, \bar{\mathbf{z}}, P(\bar{\mathbf{s}}_{N_t+1}^N))$ by $\underline{\theta}_n(\bar{\mathbf{z}})$ and $\underline{\theta}(\bar{\mathbf{z}})$ respectively. By Lemma 3.5 t $\underline{\theta}_n(\bar{\mathbf{z}}) \to \underline{\theta}(\bar{\mathbf{z}})$ for all $\bar{\mathbf{z}}$.

Let $X_{n,j}(\bar{\mathbf{z}})$ denote a random vector with the distribution equal to the conditional distribution of $\underline{\theta}_n(\bar{\mathbf{z}})\mathbf{u}_{N_t-M+2}^N$, given $\bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j$ and $\bar{\mathbf{z}}$. Let $\mathbf{u}_{\bar{\jmath}_j}(\bar{\mathbf{z}})$ denote the mean of $\mathbf{u}_{N_t-M+2}^N$with $\bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j$, for a given $\bar{\mathbf{z}}$. Then

$$X_{n,j}(\bar{\mathbf{z}}) \sim \mathcal{N}\left(\left(\underline{\theta}_n(\bar{\mathbf{z}})\mathbf{u}_{\bar{\jmath}_j}(\bar{\mathbf{z}})\right),\ \sigma_n^2\left(\underline{\theta}_n(\bar{\mathbf{z}})\underline{\theta}_n(\bar{\mathbf{z}})^T\right)\right).$$

Let $X_j(\bar{\mathbf{z}})$ be defined in a similar way for $\underline{\theta}(\bar{\mathbf{z}})\mathbf{u}_{N_t-M+2}^N$. Then from (3.6) for all n,

$$\begin{aligned} Q_n(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j; \bar{\mathbf{z}}) &\stackrel{\triangle}{=} P(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j; \underline{\bar{\theta}}_{0_n}, P(\bar{\mathbf{s}}_{N_t+1}^N)_n, \bar{\mathbf{z}}) \\ &= Prob(X_{n,j}(\bar{\mathbf{z}}) \in B_i). \end{aligned}$$

where $Q_n(.)$ denotes the conditional distribution given $\underline{\overline{\theta}}_{0_n}, P(\bar{\mathbf{s}}_{N_t+1}^N)_n$. $Q(.)$ is defined in a similar way for $\underline{\overline{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N)$. The characteristic function of $X_{n,j}(\bar{\mathbf{z}})$ converges pointwise to that of $X_j(\bar{\mathbf{z}})$. Thus, $X_{n,j}(\bar{\mathbf{z}}) \xrightarrow{w} X_j(\bar{\mathbf{z}})$ for every j and for all $\bar{\mathbf{z}}$.

Let ∂B_i represent the boundary of set B_i. For every value of $\bar{\mathbf{z}}$, $\nabla \hat{H}_{CMA}(\mathbf{0})$ (derivative of the function $\hat{H}_{CMA}$) is a positive definite matrix as seen from equation (3.5). Hence $\mathbf{0}$ is a repeller of the ODE(3.5). Further, as mentioned earlier, $\mathbf{0}$ is not taken as an initial condition. Therefore, none of the rows of $\underline{\theta}(\bar{\mathbf{z}})$ (formed from the ODE solution) equal $\mathbf{0}$. Hence, by Lemma 3.6, $Prob(X_j(\bar{\mathbf{z}}) \in \partial B_i) = 0$ for all $\bar{\mathbf{z}}, i, j$. Then using Portmanteau Theorem (Theorem 2.1, p.16 of [9])

$$Q_n(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j, \bar{\mathbf{z}}) \to Q(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j, \bar{\mathbf{z}}) \text{ for all } \bar{\mathbf{z}}. \tag{3.7}$$

Then by bounded convergence theorem,

$$Q_n(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j) \to Q(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\jmath}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\jmath}_j).$$

The lemma now follows by noting that mutual information is continuous in conditional probabilities and the input distribution $P(\bar{\mathbf{s}}_{N_t+1}^N)$. ■

Lemma 3.5 $\underline{\overline{\theta}}(T)$ *and thus* Θ_R *is a continuously differentiable function of* $\underline{\overline{\theta}}_0, \bar{\mathbf{z}}$ *and* $P(\bar{\mathbf{s}}_{N-M_L+1}^N)$ *and hence that of* $P(\bar{\mathbf{s}}_{N_t+1}^N)$.

Proof : It suffices to show the result for each individual row of $\underline{\overline{\theta}}(T)$, which are obtained from ODE(3.5). Since the result is independent of row number, l, we omit l for ease of notations.

From ODE (3.5), we observe that $\hat{H}_{CMA}(\theta; P(\bar{\mathbf{s}}_{N-M_L+1}^N), \bar{\mathbf{z}})$ is a continuously differentiable function of θ, $P(\bar{\mathbf{s}}_{N-M_L+1}^N)$, $\bar{\mathbf{z}}$. It then satisfies uniform Lipschitz condition with respect to θ, $P(\bar{\mathbf{s}}_{N-M_L+1}^N)$, $\bar{\mathbf{z}}$ in any compact domain. The required C^1 property, now follows from Theorem 7.5 of ([11], p. 30). ■

Lemma 3.6 *Let* $Y \sim \mathcal{N}(\mu, \sigma^2 I)$ *and* $X^T = EY$, *where none of the rows of the* $2p \times q$

real matrix E equal $\mathbf{0}$. In the above, $q>0$ is any arbitrary integer and $p = m(N - N_t)$. Then $Prob(X \in \partial B_i) = 0$ for all i.

Proof of Lemma 3.6: For all $\int_j \in \mathcal{S}$, the original source alphabet, define the two dimensional set $C_{\int_j}$ by,

$$C_{\int_j} \stackrel{\triangle}{=} \left\{\mathbf{x}=(x_1, x_2) \in R^2 : \|\int_j - (x_1 + ix_2)\|^2 \leq \|\int_l - (x_1 + ix_2)\|^2 \text{ for all } l \neq j, 1 \leq l \leq n_S \right\}.$$

Let $\bar{\int}_{i_k}$ represent the k^{th} element of vector $\bar{\int}_i$. We reproduce the definition of the set B_i of Section 3.3 here,

$$B_i \stackrel{\triangle}{=} \left\{\mathbf{x} \in R^{2p} : \|\bar{\int}_l - \bar{\mathbf{x}}\|^2 \geq \|\bar{\int}_i - \bar{\mathbf{x}}\|^2 \text{ for all } l \neq i \right\}.$$

We will show below that $B_i = \Pi_{l=1}^{p} C_{\bar{\int}_{i_l}}$. Clearly $\Pi_{l=1}^{p} C_{\bar{\int}_{i_l}} \subset B_i$. Now we show the converse by contradiction. Say $\mathbf{x} \in B_i$. Then $\|\bar{\mathbf{x}} - \bar{\int}_i\| \leq \|\bar{\mathbf{x}} - \bar{\int}_l\|$ for all $l \neq i$. Let x_k represent the k^{th} element of vector $\mathbf{x}$. Say for some k, $x_k \in C_{\int_l}$ where $\int_l \neq \bar{\int}_{i_k}$. Let $\bar{\int}_j$ represent the vector formed from $\bar{\int}_i$ by only replacing the k^{th} element with $\int_l$. Then $\mathbf{x} \in B_j$, which will be a contradiction.

Let $\partial C_{\int_j}$ represent the boundary of set $C_{\int_j}$. Then boundary ∂B_i, is contained in the set,

$$\cup_{l=1}^{p} \Pi_{k=1}^{p} D_{i,l,k} \quad \text{where} \quad D_{i,l,k} = \begin{cases} R^2 & k \neq l \\ \partial C_{\bar{\int}_{i_l}} & k = l \end{cases}.$$

Therefore, it suffices to show that $Prob(X \in \Pi_{k=1}^{p} D_{i,l,k}) = 0$ for all i, l.

Define $\mathbf{x}_k \stackrel{\triangle}{=} Real\,(X_k + iX_{k+p})$ where X_k represents the k^{th} coordinate of the random vector X. The above condition is now equivalent to $Prob(\mathbf{x}_k \in \partial C_{\int_l}) = 0$ for all l, k.

Finally, it suffices to prove that $Prob(\mathbf{x}_k \in \partial C_{\int_i, \int_l}) = 0$ for all $i \neq l, k$, where

$$\begin{aligned} \partial C_{\int_j, \int_l} &\stackrel{\triangle}{=} \left\{\mathbf{x} \in R^2 : \|\int_j - (x_1 + ix_2)\|^2 = \|\int_l - (x_1 + ix_2)\|^2 \right\} \\ &= \left\{\mathbf{x} \in R^2 : (x_1 + ix_2) = \frac{\|\int_l\|^2 - \|\int_j\|^2}{2(\int_j - \int_l)^*} \right\}. \end{aligned}$$

The two dimensional Gaussian random vector $\mathbf{x}_k$ has non-zero variance as none of the rows of E are zero. Thus the above probability equals zero. ■

Lemma 3.7 $C(\underline{\bar{\theta}}_0)$ *is a continuous function of* $\underline{\bar{\theta}}_0$.

Proof : $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / \underline{\bar{\theta}}_0, P(\bar{\mathbf{s}}_{N_t+1}^N))$ is a continuous function of $\underline{\bar{\theta}}_0$ and $P(\bar{\mathbf{s}}_{N_t+1}^N)$. For every $\underline{\bar{\theta}}_0$, the constraint set $D(\underline{\bar{\theta}}_0) = \mathcal{P}(\mathcal{S}^{(N_t)})$ is compact. Thus the correspondence $\underline{\bar{\theta}}_0 \mapsto D(\underline{\bar{\theta}}_0)$ is compact and constant and hence continuous. Now the required continuity follows from the Maximum Theorem ([51] p. 235). ■

Appendix B

In the following we provide proofs of Lemmas 3.1, and 3.3. Here we refer to the arguments used in Lemma 3.2 repeatedly.

Proof of Lemma 3.1 : Let $\mathbb{Z}$ represent the distribution of $(\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$ with parameters β_z. Similarly, let $\mathbb{Z}_n$ correspond to that of β_{z_n}, where $\beta_{z_n} \to \beta_z$. Then the characteristic function of $\mathbb{Z}_n$ converges pointwise to that of $\mathbb{Z}$. Thus, $\mathbb{Z}_n \xrightarrow{w} \mathbb{Z}$.

Define Θ_R in the same way as in Section 3.4, now using $\underline{\bar{\theta}}(\hat{\bar{\mathbf{z}}})$ (the MMSE equalizer defined in Section 3.3 corresponding to the MMSE channel estimate $\hat{\bar{\mathbf{z}}}$). That is, first form complex convolutional matrix using the MMSE equalizer $\underline{\bar{\theta}}(\hat{\bar{\mathbf{z}}})$ and then obtain its real counter part Θ_R as in Section 3.4. It is easy to see that, for almost all $\hat{\bar{\mathbf{z}}}$, none of the rows of Θ_R equal $\mathbf{0}$. Following steps as in Lemma 3.2 we can show that, $P(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\int}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\int}_j; \bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$ is continuous for almost all $\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$.

With $P_n(.)$ ($P(.)$) representing the conditional probabilities averaged over $(\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$ which are distributed as $\mathbb{Z}_n$ ($\mathbb{Z}$),

$$P_n(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\int}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\int}_j) \to P(\hat{\bar{\mathbf{s}}}_{N_t+1}^N = \bar{\int}_i / \bar{\mathbf{s}}_{N_t+1}^N = \bar{\int}_j)$$

by the mapping theorem 2.7 in ([9] p. 21). Thus g is a continuous function.

Hence the mutual information $I(\hat{\bar{\mathbf{s}}}_{N_t+1}^N; \bar{\mathbf{s}}_{N_t+1}^N / P(\bar{\mathbf{s}}_{N_t+1}^N))$ is a continuous function of $\beta_z, P(\bar{\mathbf{s}}_{N_t+1}^N)$ and a strictly concave function of $P(\bar{\mathbf{s}}_{N_t+1}^N)$ for every given β_z. For every β_z, the constraint set $D(\beta_z) = \mathcal{P}(\mathcal{S}^{(N_t)})$ is compact and convex. Thus the correspondence $\beta_z \mapsto D(\beta_z)$ is compact, convex and constant and hence continuous. The lemma now follows by the Maximum Theorem under Convexity ([51] p.237). ■

Proof of Lemma 3.3 : The proof of this lemma is similar to that of Lemma 3.2. Here the initial condition $\underline{\bar{\theta}}_o(\hat{\bar{\mathbf{z}}})$ is fixed by $\hat{\bar{\mathbf{z}}}$ and then the conditional probabilities, now conditioned also on $\hat{\bar{\mathbf{z}}}$, will be averaged with the joint Gaussian distribution of $\bar{\mathbf{z}}, \hat{\bar{\mathbf{z}}}$. The overall conditional probabilities and hence mutual information now depends only upon $P(\bar{\mathbf{s}}_{N_t+1}^{N})$. By following steps similar to Lemma 3.2, one can show the required continuity (by just removing $\underline{\bar{\theta}}_o$ in all steps). We are also using the fact that $\underline{\bar{\theta}}_o(\hat{\bar{\mathbf{z}}})$ is a continuously differentiable function of $\hat{\bar{\mathbf{z}}}$ and that none of the rows of $\underline{\bar{\theta}}_0(\hat{\bar{\mathbf{z}}})$ equal $\mathbf{0}$ with probability one. ■

Chapter 4

LMS-LE versus LE-WF for a Wireless Channel

In this chapter we consider a time varying wireless fading channel, equalized by an LMS linear equalizer. We study how well this equalizer tracks the optimal Wiener equalizer. We model the channel by an Auto-regressive (AR) process studied in Chapter 2.

We obtain an ODE approximation of the LMS equalizer and the AR(2) process, using the ODE approximation of the general system obtained in the Appendix I at the end of the thesis. Using these ODEs, the error between the LMS equalizer and the Wiener filter is shown to decay exponentially to zero for a stable AR process and a class of unstable AR(2) processes. For another class of unstable channels the error decays linearly. The error remains bounded for a marginally stable channel.

The Mean Square Error (MSE), between the transmitted symbol and the equalizer output, also converges towards the minimum MSE (MMSE) exponentially, for a stable channel. We further show that the difference between the MSE and the MMSE does not explode with time even when the channel is unstable. Furthermore, we obtain a step size which provides optimal tracking performance whenever the error decays exponentially.

This Chapter is organized as follows. In Section 4.1 we explain our model. Section 4.2 provides the ODE approximation for the processes of interest. Section 4.3 uses the ODE approximation to provide exponential/inverse linear decay of the difference between the

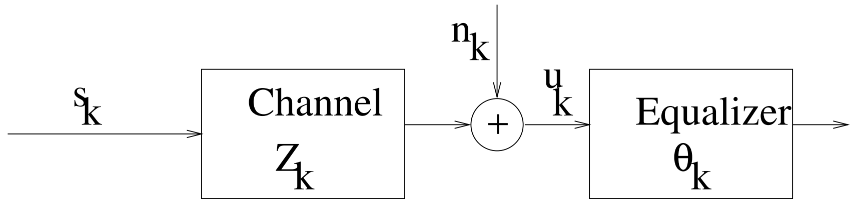

Figure 4.1: Block Diagram of Wireless channel followed by a Linear Equalizer (LE)

LMS equalizer and the Wiener filter. Section 4.4 extends the results to Complex channels and input signals. Section 4.5 provides examples to demonstrate the theory obtained. The appendices contain the proofs.

4.1 System Model, Notations and Assumptions

We consider a system consisting of a wireless channel followed by an adaptive linear equalizer (see Figure 4.1). The input of the channel s_k comes from a finite alphabet and is a zero mean IID process. The channel is a time varying finite impulse response filter (FIR) $\{Z_k\}$ of length L followed by white Gaussian noise $\{n_k\}$. We assume $n_k \sim \mathcal{N}(0, \sigma_n^2)$. Also, we assume that $\{s_k\}$ and $\{n_k\}$ are independent of each other. The channel output at time k is

$$u_k = \sum_{i=0}^{L-1} Z_{k,i} s_{k-i} + n_k,$$

where $Z_{k,i}$ is the i^{th} component of Z_k. The adaptive equalizer at time k is an FIR linear filter θ_k of length M. The linear equalizer update equation (the LMS-LE algorithm) is,

$$\theta_{k+1} = \theta_k - \mu U_k(\theta_k^T U_k - s_k), \tag{4.1}$$

where the length M channel output vector U_k can be written as,

$$U_k = \pi_k S_k + N_k,$$

where S_k, N_k are the appropriate length input and noise vectors respectively. We assume $ES_kS_k^T = I$. The convolutional matrix π_k depends upon channel values $Z_k, \cdots, Z_{k-M+1}$ and is given by

$$\begin{bmatrix} Z_{k,1} & Z_{k,2} & \cdots & Z_{k,L} & 0 & \cdots & 0 \\ 0 & Z_{k-1,1} & \cdots & Z_{k-1,L-1} & & \cdots & 0 \\ 0 & \vdots & & & & & \\ 0 & 0 & \cdots & Z_{k-M+1,1} & & \cdots & Z_{k-M+1,L} \end{bmatrix}.$$

We model the channel update process by an AR(2) process,

$$Z_{k+1} = d_1 Z_k + d_2 Z_{k-1} + \mu W_k \tag{4.2}$$

where W_k is an IID sequence, independent of the processes $\{s_k\}, \{n_k\}$. We assume $E|W_k|^4 < \infty$.

4.2 ODE Approximation

In Appendix I given at the end of the thesis, we obtain an ODE approximation for a general system whose components may depend upon two previous values. Using this, we present an ODE approximation for the linear equalizer (4.1), when the channel is modeled as an AR(2) process. We obtain the performance analysis of the LMS-LE via the approximating ODE.

A better ODE (second order) is shown to approximate the channel trajectory when d_2 is close, but not equal to -1 (as in Chapter 2 and Appendix I) .

The linear equalizer (4.1) tracking an AR(2) process (4.2) can be rewritten in the setup of Appendix I. Defining $G_{k+1} \stackrel{\triangle}{=} [U_k^T, \;\; S_k^T]^T$, one can rewrite the AR(2) process and the equalizer adaptation as,

$$\begin{aligned} Z_{k+1} &= (1-d_2)Z_k + d_2 Z_{k-1} + \mu(W_k + \eta Z_k), \\ \theta_{k+1} &= \theta_k + \mu H_1(\theta_k, G_{k+1}), \end{aligned}$$

$$H_1(\theta_k, G_{k+1}) \triangleq -U_k(\theta_k^T U_k - s_k),$$

where $\eta \triangleq \frac{d_1+d_2-1}{\mu}$. Theorem A.1 of Appendix I can be applied to this system if assumptions **B.**1-**B**.4 of Appendix I and **A.**1-**A**.3 of Chapter 2 are satisfied. Assumptions **A.**1-**A**.3 are satisfied as in Chapter 2. The rest is done in the following.

Verification of assumptions B.1-B.4 of Appendix I :

It is easy to see that $\{G_k\}$ is a process whose transition probabilities $P_{Z_k}(G, \mathbf{A})$ are function of Z_k alone. Thus condition **B**.1 is satisfied. **B**.2 is also satisfied as for any compact set Q and for any $\theta \in Q$,

$$|H_1(\theta, G)| \leq 2 \left[max \left\{ 1, \sup_{\theta \in Q} |\theta| \right\} \right] (1 + |G|^2).$$

Fixing channel $Z_k = Z$ for all k, we obtain the transition kernel $P_Z(.,.)$ for $\{G_k\}$ which is a function of Z alone. We write this process as $\{G_k(Z)\}$. It is easy to see that $\{G_k(Z)\}$ is a Markov chain and has a stationary distribution given by

$$\Pi_Z(\mathbf{A_1} \times [s_1, s_2, \cdots s_n]) = Prob(S = [s_1, s_2, \cdots s_n]) Prob(N \in \mathbf{A_1} - \pi_Z [s_1, s_2, \cdots s_n]^T)$$

where π_Z is the $M \times M + L - 1$ length convolutional matrix formed from vector Z and S, N are the input and noise vectors of length $M + L - 1$, M respectively. Define,

$$\begin{aligned}
R_{uu}(Z) &\triangleq E_Z\left[U(Z)U(Z)^T\right] = (\pi_Z \pi_Z^T + \sigma_n^2 I), \\
R_{us}(Z) &\triangleq E_Z\left[U(Z)s\right] = \pi_Z [1\ 0 \cdots 0]^T, \\
h_1(Z, \theta) &\triangleq E_Z(H_1(\theta, G(Z)) = -R_{uu}(Z)\theta + R_{us}(Z), \\
\nu_Z(G) &\triangleq \sum_{k \geq 0} P_Z^k (H_1(\theta, G) - h_1(Z, \theta)).
\end{aligned}$$

Since h_1 is continuously differentiable, it is locally Lipschitz. Thus conditions **B**.3 a, b are met.

We now prove condition **B**.3.c using Proposition 4.1 of Appendix A provided at the end of this chapter. $\{G_k\}$ is a linear dynamic process depending upon channel realization Z (in this section, Z_l represents the l^{th} component of constant vector Z, where Z_k is set equal to Z for all time k) and can be written as,

$$\begin{aligned} G_{k+1} &= A(Z)G_k + B(Z)\mathcal{W}_{k+1}, \\ \text{where, } A(Z) &= \begin{bmatrix} \mathbf{J}_M & \mathbf{P} \\ \mathbf{0}_{L+M-1\times M} & \mathbf{J}_{L+M-1} \end{bmatrix}, \\ B(Z) &= \begin{bmatrix} Z_1 & 1 \\ \multicolumn{2}{c}{\mathbf{0_{M-1\times 2}}} \\ 1 & 0 \\ \multicolumn{2}{c}{\mathbf{0_{L+M-2\times 2}}} \end{bmatrix}, \\ \mathcal{W}_{k+1} &= [s_k, n_k]. \end{aligned}$$

In the above definitions, $\mathbf{0}_n$ is a $n \times n$ zero matrix, $\mathbf{J}_n$ is a $n \times n$ shift matrix given by,

$$\mathbf{J}_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and the matrix $\mathbf{P}$ is a $M \times L + M - 1$ matrix defined by,

$$\mathbf{P} = \begin{bmatrix} Z_2 \quad Z_3 \quad \cdots \quad Z_L \quad 0 \quad \cdots \quad 0 \\ \mathbf{0_{M-1\times L+M-1}} \end{bmatrix}.$$

It is easy to see that, $A^n(Z) = 0$ for all $n \geq max\{L, L+M-1\}$ for all Z as it involves the powers of shift matrices $\mathbf{J}_L$, $\mathbf{J}_{L+M-1}$, which satisfy $\mathbf{J}_n^n = 0$. It is easy to see that the function $H(\theta, G)$, linear in $\underline{\theta}$, is in $L_i^1(R^n)$ (defined in Appendix A). Now all other conditions of Proposition 4.1 are satisfied trivially (because $A(Z)$ and $B(Z)$ are linear in Z) and hence Proposition 4.1 of Appendix A holds and therefore, $\mathbf{B}$.3.c holds with $\lambda = 1$.

The condition **B**.4 is trivially met, as, for any $n > M + L - 1$, the expectation does not depend upon the initial condition X, but, is bounded based on the compact set Q and because of the Gaussian random variable N and discrete random variable S.

Thus the conditions **B**.1–**B**.4 and **A**.1–**A**.3 required in Theorem A.1 are met for the linear equalizer and the AR(2) process. Therefore, by Theorem A.1 of Appendix I, for any linear equalizer adaptively equalizing an AR(2) process with small μ, one can approximate their joint trajectory $\{(\theta_k, Z_k)\}$ by the solution of the system of ODEs,

$$\begin{aligned}
(1+d_2)\,\dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \in (-1,1], \\
\frac{d^2 Z(t)}{dt^2} &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 = -1, \\
\frac{d^2 Z(t)}{dt^2} + \eta_1\,\dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \text{ is close to } -1,
\end{aligned} \tag{4.3}$$

$$\dot{\theta}\,(t) = -R_{uu}(Z(t))\theta(t) + R_{us}(Z(t)) \tag{4.4}$$

where $\eta_1 \stackrel{\triangle}{=} \frac{d_2+1}{\sqrt{\mu}}$, if we show that the above ODE has bounded solution for every finite time.

We will show below that the above system of ODE's have a unique solution for any finite time. Further we will show that the solution is bounded for any finite time and hence that the ODE approximation is valid for any finite time. We already know that the channel ODE (4.3) has a unique solution given by (2.2) of Chapter 2. Rest of the job is done by the following Lemma. This Lemma also shows that the equalizer ODE (4.4) has a unique attractor for a stable channel (an ODE for a stable channel itself has a unique attractor).

Lemma 4.1 *For each initial condition, the ODE (4.4) has a unique bounded solution for any finite time. Also, for a stable AR(2) process (i.e. with $\eta < 0$, $d_2 \neq -1$), the ODE (4.4) has a unique global exponentially stable attractor.*

Proof : We first prove the Lemma for a $Z(t)$, which is either exponential or a linear curve. For this $Z(t)$, one can write

$$\begin{aligned} R_{uu}(Z(t)) &= \begin{cases} B_1 e^{2\frac{\eta}{1+d_2}t} + B_2 e^{\frac{\eta}{1+d_2}t} + B_3, & for \quad \eta \neq 0, \\ B_1 t^2 + B_2 t + B_3, & for \quad \eta = 0, \end{cases} \\ R_{us}(Z(t)) &= \begin{cases} C_1 e^{\frac{\eta}{1+d_2}t} + C_2, & for \quad \eta \neq 0, \\ C_1 t + C_2, & for \quad \eta = 0, \end{cases} \end{aligned}$$

for some suitable matrices B_1, B_2, B_3 and vector C_1, C_2.

Define the vector function,

$$K(t) = \theta(t) - \theta_* \text{ with } \theta_* = B_3^{-1} C_2.$$

Then,

$$\begin{aligned} \dot{K}(t) &= \dot{\theta}(t) \\ &= -R_{uu}(Z(t))K(t) - R_{uu}(Z(t))\theta_* + R_{us}(Z(t)). \end{aligned} \tag{4.5}$$

Define the scalar function $b(t)$,

$$b(t) = \begin{cases} c e^{\frac{m_0 \eta}{1+d_2}t}, & for \quad \eta \neq 0, \\ ct^2 + c^{'}, & for \quad \eta = 0, \end{cases}$$

where $m_0 = 2$ if $\eta > 0$ and $m_0 = 1$ otherwise. Choosing positive constants $c, c^{'}$ appropriately we get,

$$|-R_{uu}(Z(t))\theta_* + R_{us}(Z(t))| \leq b(t) \text{ for all } t.$$

The matrix $R_{uu}(Z(t))$ is positive definite for all t, and it's minimum eigen value is greater than σ_n^2 for all t. Thus, for any vector K, the inner product,

$$\left\langle \dot{K}(t), K \right\rangle \leq -\sigma_n^2 |K|^2 + b(t)|K|$$

$$= \left[-\sigma_n^2|K| + b(t)\right]|K|.$$

Therefore by Global existence theorem (pp 169 - 170 of [42]), the ODE (4.5) and hence the ODE (4.4), has a unique solution for any finite time. This solution is bounded by the solution of the ODE (after choosing the initial conditions properly),

$$\dot{k}(t) = -\sigma_n^2 k(t) + b(t),$$

if we show that the above scalar ODE has a unique solution for any finite time. But this is immediately seen as the solution of this ODE is given by,

$$k(t) = c_1 e^{-\sigma_n^2 t} + e^{-\sigma_n^2 t}\int_0^t e^{\sigma_n^2 \tau} b(\tau) d\tau,$$

where c_1 is a constant depending on the initial conditions. For $\eta \neq 0$, the solution is given by,

$$k(t) = c_1' e^{-\sigma_n^2 t} + c_2 e^{\frac{m_0 \eta}{1+d_2} t}.$$

For $\eta = 0$,

$$\begin{aligned} k(t) &\leq c_1 e^{-\sigma_n^2 t} + \int_0^t b(\tau) d\tau \\ &\leq c_1 e^{-\sigma_n^2 t} + c_2 t^3 + c_3 t. \end{aligned}$$

This proves the first statement of the Lemma.

For stable channels, i.e., with $\eta < 0$, from this upper bound it is easy to see that θ_* becomes the unique global exponentially stable attractor.

When $Z(t)$ is given by a hyperbolic cosine curve or by an exponential hyperbolic cosine curve (these are not stable channels), the proof goes through in a similar manner, because $Z(t)$ in that case will be sum of a constant term and two exponentials.

When $Z(t)$ is given by a polynomial curve ($Z(t) = C_1 t^2 + C_2$), the proof once again goes through in a similar manner (a higher degree polynomial upper bounds the solution now).

When $Z(t)$ is a cosine/ exponential cosine curve, the proof goes through also because a cosine term is always upper bounded. This case includes a stable channel (exponentially decaying cosine curve). In this case the exponential term can dominate the solution and one can easily see that the equalizer ODE once again has a unique global exponentially stable attractor. ■

AR(p) process for any $p \geq 1$:

One can also try to show that the equalizer tracking an AR(p) process (2.3) can be approximated by the solution of the system of ODEs,

$$\begin{aligned}
\dot{Z}(t) &= \frac{1}{1+\sum_{i=2}^{n}(i-1)d_i}\left[E(W_1)+\frac{\sum_{i=1}^{n}d_i-1}{\mu}Z(t)\right], \\
\dot{\theta}(t) &= -R_{uu}(Z(t))\theta(t)+R_{us}(Z(t)).
\end{aligned}$$

We have simulated some examples using an AR(4) process and one can see from Figure 4.12 in Section 4.5, that the solution of the above ODE approximates the equalizer and the AR(4) process well.

4.3 Performance Analysis

We will show that a linear equalizer tracks the instantaneous Wiener filter when the channel is modeled as an AR(2) process. That is, we will show that the error between the linear equalizer and the instantaneous Wiener filter decays to zero. We study this performance using the solutions of the ODE's (4.3), (4.4) which approximate the trajectory of the channel and equalizer adaptation closely for small values of μ. We will also analyze the time evolution of the difference between the instantaneous MSE of the linear equalizer and the instantaneous MMSE. At the end of this section we obtain an optimum step size for LMS, when the channel enjoys exponential decay of the error $E(t)$ (defined below).

We define the error process $E(t)$ as the error between the equalizer $\theta(t)$ and the Wiener filter corresponding to the channel value at t, i.e.,

$$\begin{aligned} E(t) &\triangleq \theta(t) - \theta_*(t) \quad \text{where} \\ \theta_*(t) &\triangleq [R_{uu}(Z(t))]^{-1} R_{us}(Z(t)). \end{aligned} \tag{4.6}$$

Let $M(t)$ define the difference between the MSE and the MMSE at time t, i.e.,

$$M(t) \triangleq g(\theta(t), t) - g(\theta_*(t), t),$$

where the MSE $g(\underline{\theta}, t)$ at time t with the equalizer $\underline{\theta}$ is given by,

$$g(\underline{\theta}, t) = \left[\theta^t R_{uu}(Z(t))\theta - 2R_{us}(Z(t))\theta + E(s_k^2)\right].$$

By direct computations,

$$M(t) = E(t)^t R_{uu}(Z(t)) E(t).$$

Hence,

$$|M(t)| \leq |R_{uu}(Z(t))| \, |E(t)|^2 . \tag{4.7}$$

We will show that the error process $E(t)$ decays to zero, exponentially for an AR(2) process with $d_1 + d_2 \neq 1$ and at rate $\frac{1}{t}$ for an AR(2) process with $d_1 + d_2 = 1$. We also show that $M(t)$ decays exponentially for a stable AR(2) process and that it does not explode with time for an unstable process. The processes $E(t)$, $M(t)$ remain bounded for a marginally stable channel. The stable process is dealt with in Subsection 4.3.1 and the unstable process in Subsection 4.3.2. A Marginally stable channel is considered in Subsection 4.3.3.

4.3.1 Stable Channels

Initially we will work with a stable AR(2) process. A stable AR(2) process has all the poles inside the unit circle and hence satisfies ([25]), $d_1 + d_2 < 1$, $d_1 - d_2 > -1$ and $|d_2| < 1$. In this case, $\eta = (d_1 + d_2 - 1)/\mu < 0$ and the solution of the channel ODE (4.3) will be an exponential curve

$$Z(t) = C_1 e^{-\gamma t} + C_2 \tag{4.8}$$

where the constants C_1 and C_2 depend upon the initial condition, and constant γ is given by $-\frac{\eta}{1+d_2}$ (note that $\gamma > 0$). When d_2 is close to -1, a stable channel is better approximated by an exponentially decaying cosine curve (solution of the second order ODE). However the exponential term dominates the behavior and hence the performance analysis of a stable channel can be understood by studying an exponentially decaying channel. In this case γ will be given by $a = \frac{1+d_2}{2\sqrt{\mu}}$ and hence we will see that the error decay rate will be given by $\frac{1+d_2}{2\sqrt{\mu}}$ (using $|e^{at} cos(bt)| \leq e^{at}$, by slight alteration of the proof of Theorem 4.1, which is given below, we once again obtain exponential error decay at rate a).

With π_C representing the $M \times L + M - 1$ convolutional matrix using vector C,

$$\begin{aligned} R_{uu}(Z(t)) &= e^{-2\gamma t}\pi_{C_1}\pi_{C_1}^T + e^{-\gamma t}[\pi_{C_2}\pi_{C_1}^T + \pi_{C_1}\pi_{C_2}^T] + \pi_{C_2}\pi_{C_2}^T + \sigma_n^2 I, \\ R_{us}(Z(t)) &= [e^{-\gamma t}\pi_{C_1} + \pi_{C_2}][1\ 0 \cdots 0]^T. \end{aligned}$$

It is clear that the ODE (4.3) has a unique global attractor $Z_* = C_2$. The Wiener filter $\underline{\theta}_*$ corresponding to Z_*, is

$$\theta_* = \left(\pi_{C_2}\pi_{C_2}^T + \sigma_n^2 I\right)^{-1} \pi_{C_2}[1\ 0 \cdots 0]^T.$$

By Lemma 4.1, θ_* is the unique exponentially stable global attractor for the equalizer ODE (4.4). The following theorem establishes the exponential decay of the processes $\{E(t)\}$ and $\{M(t)\}$. For the special case when $\sigma_n^2 = \gamma$, the term c_2 in the Theorem 4.1

would actually be $c_2 t$. Since $te^{-\gamma t} \to 0$ as $t \to \infty$ the decay is still ensured. For simplifying the explanations, we do not consider this case.

Theorem 4.1 *The processes $E(t)$, $M(t)$ decay exponentially with time:*

$$\begin{aligned} |E(t)| &\leq c_1 e^{-\sigma_n^2 t} + c_2 e^{-\gamma t}, \\ |M(t)| &\leq c_1' e^{-2\sigma_n^2 t} + c_2' e^{-2\gamma t}. \end{aligned}$$

Proof : We will show below exponential decay of the error process $E(t)$. Then from (4.7), the process $M(t)$ also decays exponentially, as, $R_{uu}(Z(t))$ is bounded by a constant for all t.

The derivative of the error $E(t)$ is given by,

$$\begin{aligned} \dot{E}(t) &= -R_{uu}(Z(t))\theta(t) + R_{us}(Z(t)) - \dot{\theta}_*(t) \\ &= -R_{uu}(Z(t))E(t) - \dot{\theta}_*(t). \end{aligned} \tag{4.9}$$

The matrix $R_{uu}(Z(t))$ is positive definite for all t, and it's minimum eigen value is greater than σ_n^2 for all t. Therefore $|[R_{uu}(Z(t))]^{-1}| \leq \sigma_n^{-2}$ for all t. Hence by direct calculations from (4.8),

$$\begin{aligned} \left|\dot{\theta}_*(t)\right| &= \left|[R_{uu}(Z(t))]^{-1}\left(\dot{R}_{uu}(Z(t))\theta_*(t) - \dot{R}_{us}(Z(t))\right)\right| \\ &\leq \frac{1}{\sigma_n^2}\left|\left(\dot{R}_{uu}(Z(t))\theta_*(t) - \dot{R}_{us}(Z(t))\right)\right| \\ &\stackrel{a}{\leq} \frac{1}{\sigma_n^2}\left(\tilde{c}_1 e^{-\gamma t}\frac{1}{\sigma_n^2} + \tilde{c}_2 e^{-\gamma t}\right) \\ &\leq c e^{-\gamma t}, \end{aligned}$$

for some appropriate scalar positive constants $\tilde{c}_1, \tilde{c}_2$ and c. For proving inequality a, we used $|R_{uu}(Z(t))]^{-1}| \leq \sigma_n^{-2}$, $e^{-2r} < e^{-r}$ for any positive r and that $|\mathbf{AB}C| \leq |\mathbf{A}||\mathbf{B}||C|$ for any matrices $\mathbf{A}, \mathbf{B}$ and vector C.

Define scalar function, $b(t) \stackrel{\triangle}{=} ce^{-\gamma t}$. Then using equation (4.9), for any vector E, the inner product

$$\begin{aligned} \left\langle \dot{E}(t), E \right\rangle &= \langle -R_{uu}(Z(t))E, E \rangle - \left\langle \dot{\theta}_*(t), E \right\rangle \\ &\leq -\sigma_n^2 |E|^2 + b(t)|E| \\ &= \left[-\sigma_n^2 |E| + b(t)\right] |E|. \end{aligned}$$

By Global Existence theorem ([42], pp 169, 170) the solution of the ODE (4.9) will be bounded by the solution of the scalar ODE,

$$\dot{e}(t) = -\sigma_n^2 e(t) + b(t)$$

when the initial condition $e(0) = |E(0)|$ and $E(0)$ is the initial condition of the error ODE (4.9). Thus the solution of the error ODE (4.9) satisfies

$$\begin{aligned} |E(t)| &\leq e(t) \\ &= c_1 e^{-\sigma_n^2 t} + c_2 e^{-\gamma t} \end{aligned}$$

with $c_1 + c_2 = |E(0)|$. ■

4.3.2 Unstable Channels

Now we consider unstable channels. An AR(2) process approximating such a channel is given by,

$$Z(t) = C_1 r(t) + C_2$$

where $r(t)$ is given by,

$$r(t) = \begin{cases} e^{\gamma t}, & d_1 + d_2 > 1, \\ t, & d_1 + d_2 = 1. \end{cases}$$

Note that $\gamma = \frac{\eta}{1+d_2}$. A hyperbolic cosine curve and a square curve ($r(t) = t^2$) are another class of unstable channels. But they behave similar to the above channels and hence it suffices to study the above unstable channels. For these channels,

$$\begin{aligned} R_{uu}(Z(t)) &= B_1 r(t)^2 + B_2 r(t) + B_3, \\ R_{us}(Z(t)) &= C_1 r(t) + C_2, \end{aligned} \tag{4.10}$$

for some suitable matrices B_1, B_2, B_3 and vectors C_1, C_2.

$E(t)$ process for an unstable channel :

The following theorem shows that the error $E(t)$ given by equation (4.6) decays to zero, exponentially for an AR(2) process with $d_1 + d_2 > 1$ and at rate $\frac{1}{t}$ for a process with $d_1 + d_2 = 1$. We will be dealing with the process $M(t)$ for an unstable channel at the end of this subsection.

Theorem 4.2 *The error process $E(t)$ satisfies:*

$$|E(t)| \leq c_1 e^{-\sigma_n^2 t} + 1_{\{t>T\}} \frac{c_2}{r(t)},$$

for some $T < \infty$.

Proof : The derivative of the error $E(t)$ is given by,

$$\begin{aligned} \dot{E}(t) &= -R_{uu}(Z(t))\theta(t) + R_{us}(Z(t)) - \dot{\theta}_*(t) \\ &= -R_{uu}(Z(t))E(t) - \dot{\theta}_*(t). \end{aligned} \tag{4.11}$$

The matrix $R_{uu}(Z(t))$ is positive definite for all t. We upper bound the norm of its inverse in the following. From (4.10)

$$|R_{uu}(Z(t))| \geq |B_1| r(t)^2 - |B_2| r(t) - |B_3|.$$

The matrix $B_1 = \pi_C \pi_C^T$ for some vector C and hence is a positive definite matrix unless $C = 0$. Therefore, $|B_1| > 0$. Thus, it is possible to choose $T \geq 1$ large enough such that,

$$\begin{aligned} |B_1| - \frac{|B_2|}{r(t)} - \frac{|B_3|}{r(t)^2} &> \beta > 0 \quad \text{for all } t \geq T, \\ \frac{e^{\sigma_n^2 t}}{t^2} &> \frac{e^{\sigma_n^2 t_1}}{t_1^2} \quad \text{for all } t \geq T, t > t_1. \end{aligned}$$

The second inequality is required for $r(t) = t$ and it's use will become evident in the last step of the proof. Defining,

$$f(t) \triangleq \begin{cases} \frac{\alpha'}{r(t)^2}, & t > T, \\ \\ \frac{\alpha'}{\sigma_n^2}, & 0 \leq t \leq T, \end{cases}$$

and choosing α' appropriately we can get $|[R_{uu}(Z(t))]^{-1}| \leq f(t)$ for all t. By direct calculations we get,

$$\begin{aligned} \left|\dot{\theta}_*(t)\right| &= \left|[R_{uu}(Z(t))]^{-1}\left(\dot{R}_{uu}(Z(t))\theta_*(t) - \dot{R}_{us}(Z(t))\right)\right| \\ &\leq f(t)\left|\left(\dot{R}_{uu}(Z(t))\theta_*(t) - \dot{R}_{us}(Z(t))\right)\right| \\ &\leq f(t)\left(\tilde{c}_1 r(t)\frac{dr(t)}{dt} f(t) r(t) + \tilde{c}_2 \frac{dr(t)}{dt} + \tilde{c}_3\right) \end{aligned}$$

for some appropriate scalar positive constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$. Defining,

$$b(t) \triangleq \begin{cases} ce^{-\gamma t}, & d_1 + d_2 > 1 \text{ and } t > T, \\ ct^{-2}, & d_1 + d_2 = 1 \text{ and } t > T, \\ c', & 0 \leq t \leq T, \end{cases}$$

and choosing the positive constants c, c' appropriately, we can ensure that $b(t)$ is a continuous function and

$$\left|\dot{\theta}_*(t)\right| \leq b(t)$$

for all $t \geq 0$. Using equation (4.11), for any vector E, the inner product

$$\begin{aligned}\left\langle \dot{E}\,(t), E \right\rangle &= \langle -R_{uu}(Z(t))E, E\rangle - \left\langle \dot{\theta}_*\,(t), E \right\rangle \\ &\leq -\sigma_n^2|E|^2 + b(t)|E| \\ &= \left[-\sigma_n^2|E| + b(t)\right]|E|.\end{aligned}$$

By Global Existence theorem ([42], pp 169, 170) the solution of the ODE (4.11) will be bounded by the solution of the scalar ODE,

$$\dot{e}\,(t) = -\sigma_n^2 e(t) + b(t)$$

when the initial condition $e(0) = |E(0)|$, where $E(0)$ is the initial condition of the error ODE (4.11). Thus the solution of the error ODE (4.11) is bounded as before by,

$$\begin{aligned}|E(t)| &\leq e(t) \\ &= c_1^{'} e^{-\sigma_n^2 t} + e^{-\sigma_n^2 t}\int_0^t e^{\sigma_n^2\tau} b(\tau)d\tau \\ &\leq c_1^{'} e^{-\sigma_n^2 t} + e^{-\sigma_n^2 t} c^{'} \int_0^{min\{T,t\}} e^{\sigma_n^2\tau}d\tau + 1_{\{t>T\}} e^{-\sigma_n^2 t}\int_T^t b(\tau)e^{\sigma_n^2\tau}d\tau \\ &\leq c_1^{''} e^{-\sigma_n^2 t} + 1_{\{t>T\}} e^{-\sigma_n^2 t}\int_T^t b(\tau)e^{\sigma_n^2\tau}d\tau.\end{aligned}$$

Thus for $r(t) = e^{\gamma t}$, the proof is complete.

For $r(t) = t$, $b(t) = ct^{-2}$ and hence the second term in the above expression is upper bounded for all $t \geq T$ by,

$$\begin{aligned}e^{-\sigma_n^2 t}\int_T^t b(\tau)e^{\sigma_n^2\tau}d\tau &= ct^{-2}\int_T^t \frac{\tau^{-2}e^{\sigma_n^2\tau}}{t^{-2}e^{\sigma_n^2 t}}d\tau \\ &\leq ct^{-2}\int_T^t d(\tau) \\ &= ct^{-2}(t-T) \leq ct^{-1}.\end{aligned}$$

The above inequality follows by the choice of T. ■

$M(t)$ **process for an unstable channel :**

By Theorem 4.2 the error process $E(t)$ decays exponentially/polynomially with time for any unstable AR(2) process. We also have,

$$|R_{uu}(Z(t))| \leq b_1 r(t)^2 + b_2$$

for some constants b_1, b_2. Hence from equation (4.7), one can upper bound the process $M(t)$ as

$$|M(t)| \leq c_1' e^{-2\sigma_n^2 t} r(t)^2 + c_2'.$$

Thus $M(t)$ does not explode with time as long as $e^{-2\sigma_n^2 t} r(t)^2$ does not. This will not happen unless $r(t) = e^{\gamma t}$ with $\gamma > \sigma_n^2$. In this case, by using the optimum step size μ_* (defined at the end of this Section) one can again ensure that $M(t)$ does not explode with time. Hence $M(t)$ does not explode with time for any unstable channel (by choosing the optimum step size μ_* if required).

4.3.3 Marginally stable Channels

A Marginally stable channel ($d_1 < 2$ and $d_2 = -1$) is approximated by a cosine waveform. For this channel, we are not able to show that the error decays with time. However, using similar steps as before, one can see that the processes $E(t)$ and $M(t)$ remain bounded with time. In fact, for this case, we can see in Section 4.5 that, the error may not decay to zero.

4.3.4 Optimum Step-size for channels with exponential error decay

Theorems 4.1 and 4.2 show that for an AR(2) process with $d_1 + d_2 \neq 1$, if the channel noise variance σ_n^2 and/or $\frac{|\eta|}{1+d_2}$ is large, the decay will be faster. Our examples in Section 4.5 will verify this dependence. But this does not imply that the system performance is improved with larger σ_n^2. With σ_n^2 increasing, the variation in the Wiener filter reduces (note that the performance of the Wiener filter itself worsens as σ_n^2 increases) and hence error between the Wiener filter and the actual equalizer decays faster. We will also see this in the simulations in Section 4.5. But below we will use this dependence to obtain a step size that optimizes the decay rate for any σ_n^2.

With $d_1+d_2 \neq 1$, we see that the error decays exponentially at a rate which is equal to the minimum of σ_n^2 and $\frac{|d_1+d_2-1|}{\mu(1+d_2)}$. By introducing a constant r in the equalizer adaptation

$$\theta_{k+1} \quad = \quad \theta_k - \mu r U_k(\theta_k^T U_k - s_k),$$

we see that the error $E_r(t)$ (we introduce the subscript r to show the dependency on factor r) decays as

$$|E_r(t)| \leq c_1 e^{-r\sigma_n^2 t} + c_2 e^{\frac{-|d_1+d_2-1|}{\mu(1+d_2)}t}.$$

This decay is obtained in the same way as in Theorem 4.1, the only difference being that $|rR_{uu}(Z(t))|$ is now lower bounded by $r\sigma_n^2$. Then the error decay rate is equal to the minimum of $r\sigma_n^2$ and $\frac{|d_1+d_2-1|}{\mu(1+d_2)}$. If $\sigma_n^2 << \frac{|d_1+d_2-1|}{\mu(1+d_2)}$, the error decay rate reduces considerably, which can be compensated by choosing an appropriate scaling factor r. Hence one can choose the optimum $r_* = max\left\{\frac{|d_1+d_2-1|}{\sigma_n^2\mu(1+d_2)}, 1\right\}$, to achieve the fastest decay in the error. But note that this decay only considers the first order analysis and hence if the value of r is very large such that the overall equalizer adaptation step-size $r_*\mu$ is large then, one must choose a smaller r. We will denote the optimal step size $r_*\mu$ by μ_*.

In Section 4.5, we will show that the improvement obtained by μ_* can be significant.

4.4 Complex Channel and inputs

In this section we will study a linear complex equalizer that adaptively equalizes a complex channel whose inputs are complex signals from a finite alphabet. We will see that the theorems and their proofs are simple extensions of the real case. The same will hold good for the other systems studied in this thesis. Thus, we will not explicitly provide this extensions in the rest of the thesis.

We introduce an extra notation in this section. For any complex matrix, $E = E_r + iE_i$, we denote the corresponding double dimensional real matrix by $f(E)$, which is defined as,

$$f(E) \triangleq \begin{bmatrix} E_r \\ E_i \end{bmatrix}.$$

We use E^* to denote the conjugate transpose of the complex matrix E. The rest of the notations remain same as before except that now all symbols will represent complex signals.

The channel, the channel output and the equalizer adaptation with complex inputs and signals can be written as,

$$\begin{aligned} Z_{k+1} &= d_1 Z_k + d_2 Z_{k-1} + \mu W_k, \\ U_k &= \pi_k S_k + N_k, \\ \theta_{k+1} &= \theta_k - \mu U_k \left(\theta_k^* U_k - s_k\right)^*. \end{aligned} \tag{4.12}$$

Here $N_k \sim C\mathcal{N}(0, \sigma_n^2 I)$, $W_k \sim C\mathcal{N}(0, I)$ are IID complex circularly symmetric Gaussian vectors independent of each other and independent of the complex IID vector sequence S_k coming from a finite alphabet.

We define $\tilde{\theta}_k := f(\theta_k)$, $\tilde{Z}_k := f(Z_k)$, which are real vectors. Defining, $G_{k+1}^T := [f(U_k)^T; f(S_k)^T]$, one can rewrite the complex equalizer adaptation (4.12) and the channel adaptation, in the equivalent real domain as,

$$\begin{aligned}
\tilde{Z}_{k+1} &= d_1\tilde{Z}_k + d_2\tilde{Z}_{k-1} + \mu f(W_k), \\
\tilde{\theta}_{k+1} &= \tilde{\theta}_k + \mu H_1(\tilde{\theta}_k, G_{k+1}) \text{ where} \\
H_1(\tilde{\theta}_k, G_{k+1}) &= f(U_k s_k^*) - A_k\tilde{\theta}_k \text{ where,}
\end{aligned}$$

$$A_k = \begin{bmatrix} U_{k,r}U_{k,r}^T + U_{k,i}U_{k,i}^T & U_{k,r}U_{k,i}^T - U_{k,i}U_{k,r}^T \\ \\ -U_{k,r}U_{k,i}^T + U_{k,i}U_{k,r}^T & U_{k,r}U_{k,r}^T + U_{k,i}U_{k,r}^T \end{bmatrix}.$$

Once again, $\{G_k\}$ is a process, whose transition probabilities $P_{Z_k}(G, \mathbf{A})$, are a function of $\tilde{Z}_k$ alone. Thus condition **B**.1 is satisfied. **B**.2 is also satisfied as in Section 4.2.

Fixing channel $Z_k = Z$ for all k, it is easy to see that $\{G_k(\tilde{Z})\}$, where $\tilde{Z} = f(Z)$, has a stationary distribution given by,

$$\begin{aligned}
\Pi_{f(Z)}(f(\mathbf{A_1}) \times f([s_1, s_2, \cdots s_n])) &= Prob(S = [s_1, s_2, \cdots s_n]) \\
&\qquad Prob(N \in \mathbf{A_1} - \pi_Z[s_1, s_2, \cdots s_n]^T).
\end{aligned}$$

Define,

$$\begin{aligned}
R_{uu}(\tilde{Z}) &\triangleq E_{\tilde{Z}}A_k(\tilde{Z}), \\
R_{us}(\tilde{Z}) &\triangleq E_{\tilde{Z}}f(U_k(\tilde{Z})s_k^*), \\
h_1(\tilde{Z}, \tilde{\theta}) &\triangleq E_{\tilde{Z}}(H_1(\tilde{\theta}, G(\tilde{Z})) = -R_{uu}(\tilde{Z})\tilde{\theta} + R_{us}(\tilde{Z}), \\
\nu_{\tilde{Z}}(G) &\triangleq \sum_{k\geq 0} P_Z^k(H_1(\tilde{\theta}, G) - h_1(\tilde{\theta}, \tilde{Z})).
\end{aligned}$$

Once again, we see that the conditions **B**.3 a, b are met using these definitions. Also, it is easy to see that,

$$[R_{uu}(f(Z))]\, f(\theta) = f\left(\left[\pi_Z\pi_Z^* + \sigma_n^2 I\right]\theta\right),$$

$$R_{us}(f(Z)) \quad = \quad f(\pi_Z[1\ 0 \cdots 0]^T),$$

which shows that the instantaneous Wiener filter is a zero of the gradient vector $h_1(.,.)$ at that time.

As in Section 4.2, using Proposition 4.1, we can prove condition **B.**3.c. Condition **B**.4 can also be verified as in Section 4.2. It is trivial to see that the conditions **A**.1–**A**.3 are also satisfied.

Thus the conditions **B**.1–**B**.4 and **A**.1–**A**.3 required in Theorem A.1 are met for the complex linear equalizer and the complex AR(2) process. Thus, for any complex linear equalizer adaptively equalizing a complex AR(2) process with a small μ, one can approximate their joint trajectory $\{(\tilde{Z}_k, \tilde{\theta}_k) = f((\theta_k, Z_k))\}$ by the solution of the system of ODEs,

$$\begin{aligned}
(1+d_2)\ \dot{\tilde{Z}}\ (t) \quad &= \quad \left[E(W) + \eta \tilde{Z}(t)\right], \quad \text{if } d_2 \in (-1, 1], \\
\frac{d^2 \tilde{Z}(t)}{dt^2} \quad &= \quad \left[E(W) + \eta \tilde{Z}(t)\right], \quad \text{if } d_2 = -1, \\
\frac{d^2 \tilde{Z}(t)}{dt^2} + \eta_1\ \dot{\tilde{Z}}\ (t) \quad &= \quad \left[E(W) + \eta \tilde{Z}(t)\right], \quad \text{if } d_2 \text{ is close to } -1,
\end{aligned} \tag{4.13}$$

$$\dot{\tilde{\theta}}\ (t) \quad = \quad -R_{uu}(\tilde{Z}(t))\tilde{\theta}(t) + R_{us}(\tilde{Z}(t)), \tag{4.14}$$

where $\eta \stackrel{\triangle}{=} \frac{d_1+d_2-1}{\mu}$ and $\eta_1 \stackrel{\triangle}{=} \frac{1+d_2}{\sqrt{\mu}}$. The approximation holds for all finite $T > 0$ because as we show below, this ODE has a unique bounded solution for all initial conditions and for all time.

Writing the ODE's directly in terms of the complex channel and equalizer co-efficients we get,

$$\begin{aligned}
(1+d_2)\ \dot{Z}\ (t) \quad &= \quad \left[E(W) + \eta Z(t)\right], \quad \text{if } d_2 \in (-1, 1], \\
\frac{d^2 Z(t)}{dt^2} \quad &= \quad \left[E(W) + \eta Z(t)\right], \quad \text{if } d_2 = -1, \\
\frac{d^2 Z(t)}{dt^2} + \eta_1\ \dot{Z}\ (t) \quad &= \quad \left[E(W) + \eta Z(t)\right], \quad \text{if } d_2 \text{ is close to } -1,
\end{aligned} \tag{4.15}$$

$$\dot{\theta}(t) = -\left[\pi_{Z(t)}\pi^*_{Z(t)} + \sigma_n^2 I\right]\theta(t) + \pi_{Z(t)} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T . \tag{4.16}$$

Once again, ODE (4.13) has a unique solution which is similar to the solution (2.2), the only difference being that it will be in a double dimensional space. As in Lemma 4.1, we obtain the following.

Lemma 4.2 *The ODE (4.14) and hence the ODE (4.16), has an unique bounded solution for all initial conditions and for all time. For a stable AR(2) process ($\eta < 0$ and $d_2 \neq -1$) the ODE (4.14) and hence the ODE (4.16), has an unique exponentially stable global attractor.*

Defining the error process $E(t)$ as in Section 4.3,

$$\begin{aligned} E(t) &\triangleq \tilde{\theta}(t) - \tilde{\theta}_*(t) \quad \text{where} \\ \tilde{\theta}_*(t) &\triangleq [R_{uu}(\tilde{Z}(t))]^{-1} R_{us}(\tilde{Z}(t)). \end{aligned}$$

But the process $\tilde{\theta}_*(t)$ is the double dimensional real vector formed from the real part and complex part of the instantaneous Wiener filter $\theta_*(t)$, i.e., $\tilde{\theta}_*(t) = f(\theta_*(t))$. Hence,

$$|E(t)| = |\theta(t) - \theta_*(t)|.$$

Repeating all the steps in Section 4.3, we get the following theorem.

Theorem 4.3 *The error process $E(t)$ decays exponentially/polynomially with time:*

$$|E(t)| \leq c_1 e^{-\sigma_n^2 t} + c_2 \begin{cases} e^{\frac{-|\eta|}{1+d_2}t}, & if \quad \eta \neq 0 \ and \ d_2 \neq -1, \\ \frac{1}{t}, & if \quad \eta = 0 \ and \ d_2 \neq -1, \\ \frac{1}{t^2}, & if \quad \eta = 0 \ and \ d_2 = -1. \end{cases}$$

The above theorem shows that for a complex linear equalizer adaptively equalizing a complex AR(2) process the error between the equalizer value and the instantaneous Wiener filter decays, exponentially if $d_1 + d_2 \neq 1$ and otherwise at the rate $\frac{1}{t}$ (Just like in Section 4.3). Once again as in Section 4.3 M(t), the difference between the MSE and MMSE at time t, reduces exponentially to zero for a stable channel and does not explode with time for an unstable channel.

4.5 Examples

In this section via examples we illustrate the theory developed so far. We consider a three-tap channel with a two tap equalizer. $W_k \sim \mathcal{N}(Mn, 1)$ for varying values of mean Mn. The input is BPSK.

In the first example, we set $d_1 = 0.6$, $d_2 = 0.3995$ and $\mu = 0.001$. This is a stable channel. The results are plotted in Figure 4.2. In first subfigure of Figure 4.2, we plot the actual trajectory and the ODE solution of the AR(2) process. In the second subfigure, we plot the corresponding equalizer trajectory and the ODE solution for two different values of σ_n^2. We also, plot the instantaneous Wiener filter in the same figure.

It is clear from both the sub figures, that the ODE solutions approximate the actual trajectories well. Figure 4.2 also shows that the error $E(t)$ decays faster with $\sigma_n^2 = 0.5$. This is also evident from Theorem 4.1. But as one can see from the second sub figure of Figure 4.2, this is mainly because the Wiener filter itself does not change much with time for the same channel if the noise variance is increased. We have also varied d_1, d_2 values keeping Mn, η fixed and seen that the decay rate increases with decrease in d_2, which is once again evident from Theorem 4.1. We can obtain a better performance, even with smaller σ_n^2 (or bigger d_2) by using an optimum step-size for the equalizer μ_* given in Section 4.3. We will show the improvement with μ_* in subsequent figures.

In Figure 4.3, we plot the equalizer and the channel trajectories for $d_1 + d_2 = .8$, which is away from 1 . Now we have taken $\mu = 0.01$. We can see that when $1 - (d_1 + d_2)$ is large, the AR(2) process converges faster to the attractor and this channel will be like a channel without drift. In this case, it is very close to an IID Gaussian random variable.

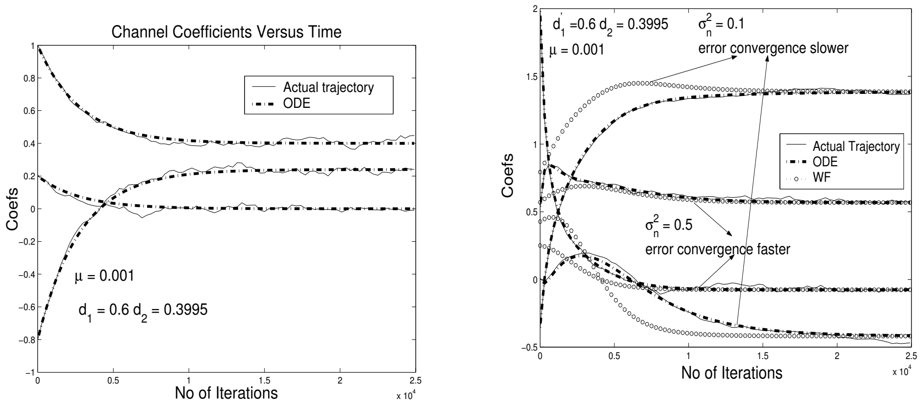

Figure 4.2: Trajectories of the Channel, the LE coefficients along with the Wiener filter for a stable channel.

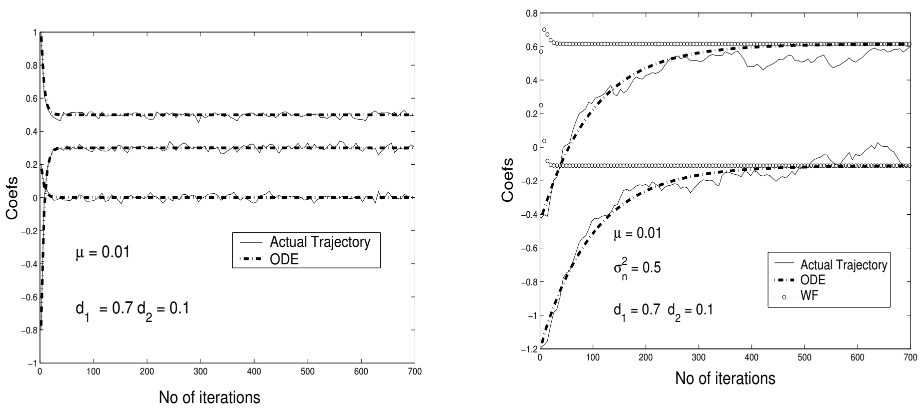

Figure 4.3: Trajectories of the Channel, the LE coefficients along with the Wiener filter for a stable channel with $d_1 + d_2 = 0.8 << 1$.

This is also evident from the theory developed as in this case $|\eta|$ will be larger. One can also see from figures that the approximation is accurate even for $d_1 + d_2$ away from 1. Furthermore, because of larger μ, the equalizer trajectory is more random.

In Figure 4.4, we plot the AR(2) process and the equalizer coefficients for an unstable AR(2) process. We set $d_1 = 0.9, d_2 = 0.1001$ and $\mu = 0.001$. We see that the AR(2) process is unstable, as $d_1 + d_2 = 1.001 > 1$. Thus the AR(2) parameters are converging towards infinity exponentially, as is shown by theory. We can also see that the ODE approximation is very accurate even in this case.

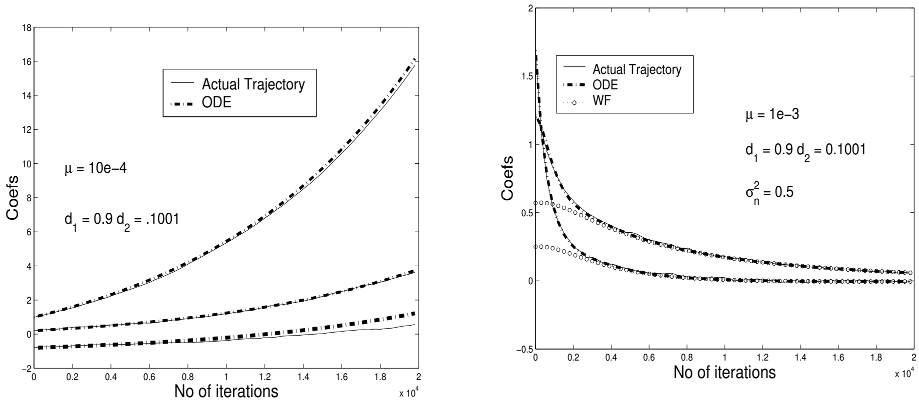

Figure 4.4: Trajectories of the Channel, the LE coefficients along with the Wiener filter for an unstable channel.

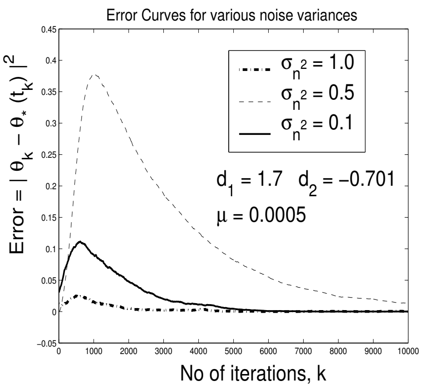

Figure 4.5: Error between the actual LE value and the Wiener filter for various values of σ_n^2.

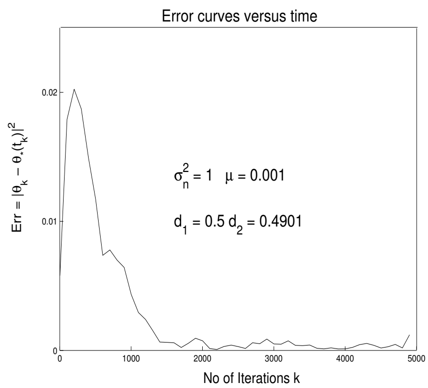

Figure 4.6: Error between the actual LE value and the Wiener filter for a complex channel.

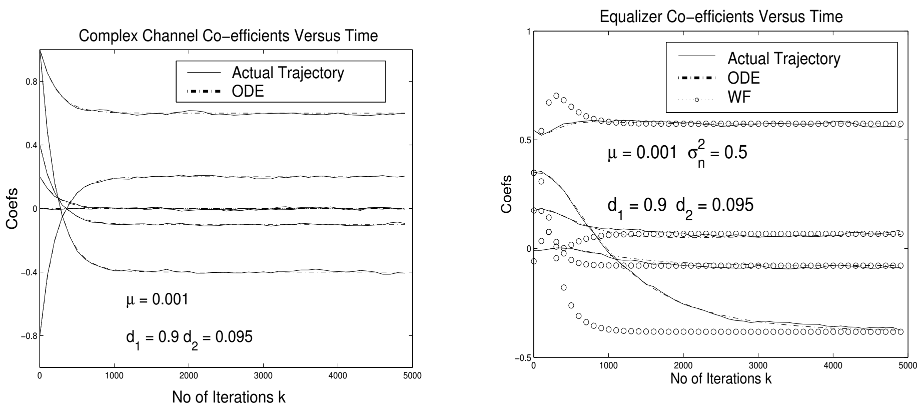

Figure 4.7: Trajectories of Complex Channel, LE coefficients.

In Figure 4.5, we plot the error between the actual equalizer trajectory and the Wiener filter, i.e., $|\theta_k - \theta_*(t_k)|^2$. In this figure, we varied the noise variance σ_n^2 keeping the remaining parameters fixed. The plots in the Figure 4.5 confirm the results of Theorem 4.1. They show that the error decays faster with larger σ_n^2. However, by using a bigger step size μ_* obtained in Section 4.3, we see that the error decays fast even for smaller σ_n^2 (Figure 4.8).

In Figure 4.7, we plot the channel coefficients and the equalizer coefficients for a complex channel and a complex linear equalizer. Once again, we are consider a three-tap channel and a two-tap equalizer. In these figures, we plot both the real and imaginary parts of the coefficients separately. In Figure 4.6, we plot the corresponding error curve $|\theta_k - \theta_*(t_k)|^2$. We see that the ODE approximation is very close even for a complex channel and equalizer. We also see that the error reduces to zero exponentially.

In Figure 4.8, we plot the equalizer coefficients and the ODE solution for different step sizes. The channel model remains the same for both the values of step sizes. We see that with a step size of μ_* (defined at the end of Section 4.3), the error between the Wiener filter and the ODE solution (hence the actual equalizer trajectory) decreases very fast.

In Figure 4.9, we consider a special case, when $d_2 = -1$. We set $d_1 < 2$, to get a marginally stable AR(2) process, whose trajectory is approximated by a cosine waveform. Even in this case, both the channel ODE and the equalizer ODE solutions well approximate the actual trajectories. For this case, we could not get a result on the optimal step size, μ_*, as in Section 4.3. We still choose a better step size for the equalizer by trial and error and obtained the equalizer ODE solution (and hence the actual equalizer trajectory) well approximating the Wiener filter.

In Figure 4.10, we consider a stable channel with d_2 is close to -1. In this case, as is shown theoretically, a better ODE approximation is obtained by a second order ODE. Here, the channel trajectory is approximated by an exponentially reducing cosine waveform. We considered an AR(2) process, which approximates the fading channel with band limited and U-shaped spectrum and received with $f_d T = 0.01$ ($f_d T$ represents the product of maximum Doppler frequency and the actual data sampling time). We once

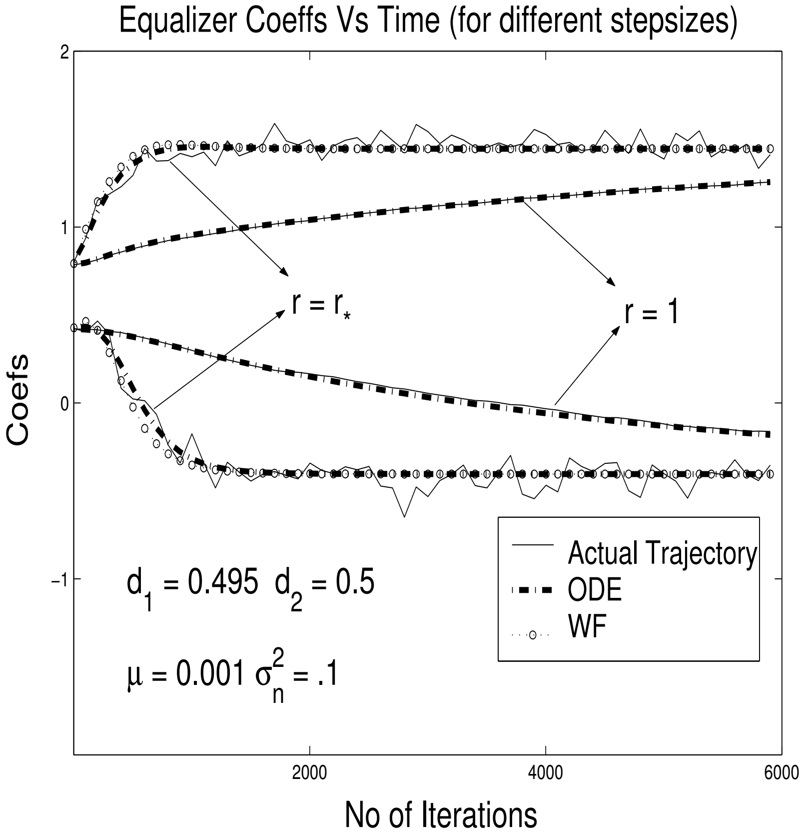

Figure 4.8: Error decay rate is minimum when the step size is equal to the optimum one, $\mu_* = r_*\mu$.

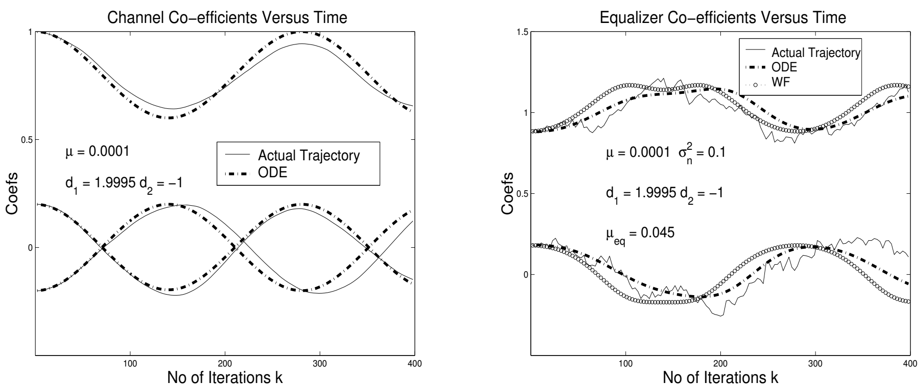

Figure 4.9: Trajectories of an AR(2) process, the LE coefficients with $d_2 = -1$ and $d_1 < 2$, i.e., for a marginally stable channel.

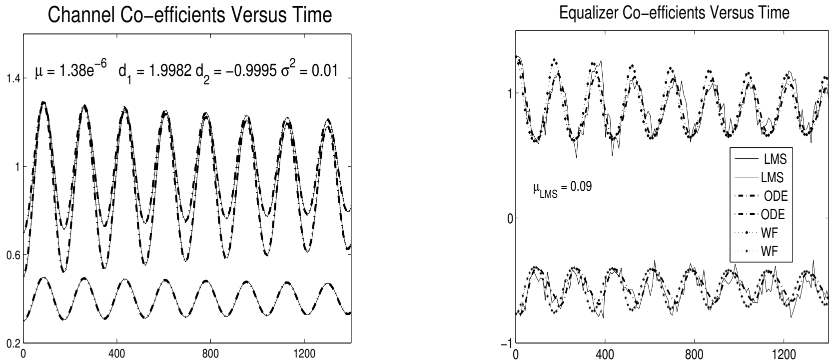

Figure 4.10: Trajectories of an AR(2) process, the LE coefficients with d_2 close to -1 and $d_1 < 2$, for a exponentially reducing cosine channel with $f_d T = 0.01$.

again obtained a better step size for the equalizer by trial and error and got the equalizer ODE solution (and hence the actual equalizer trajectory) providing good approximation to the Wiener filter.

We plot the instantaneous MSE versus time in Figure 4.11. We plot it for the channel of Figure 4.2 and for two values of noise variances σ_n^2, 0.1 and 1.0. We can see from the figure that the MSE converges exponentially to the steady state value as is given by Theorem 4.1.

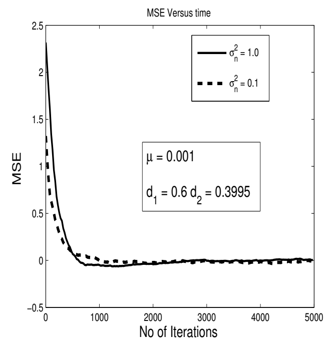

Figure 4.11: MSE Versus Time for a LE

Finally, we plot the AR(4) process and the corresponding equalizer trajectories in Figure 4.12. We see that the ODE solution once again approximates the actual trajectories well. We also used the optimal step size defined in Section 4.3. We see that the error

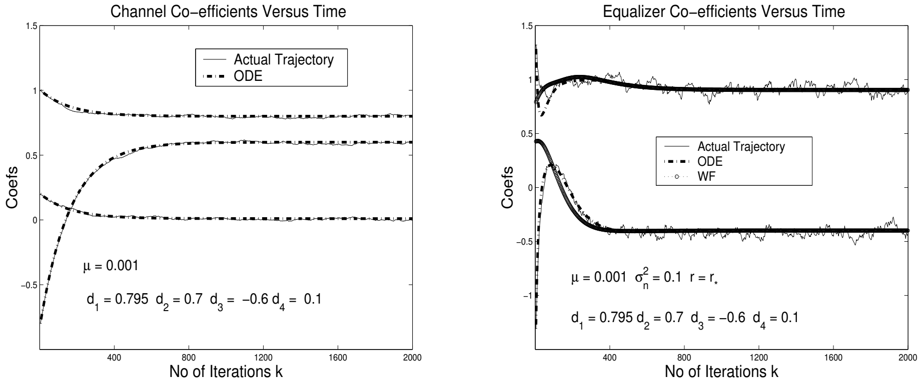

Figure 4.12: Trajectories of AR(4) process, LE coefficients with $r = r_*$, the optimum defined in Section 4.3.

between the instantaneous Wiener filter and the actual equalizer trajectories really decays fast. Do note that, as for an AR(2) process (e.g., when $d_2 = -1$), there will be different values of d_1, d_2, d_3, d_4 for which the exponential curve will not approximate the channel trajectory.

Also note that in some of the figures provided above, the equalizer ODE (and actual equalizer trajectory) is initialized away from the Wiener filter of initial channel state $Z(0)$.

Conclusions

In this chapter we study an LMS-linear equalizer tracking a time varying channel modeled by an AR(2) process (which is approximated by an ODE in Chapter 2). We obtain the first order analysis of the error, which is defined as the deviation of the equalizer value from the instantaneous Wiener filter. Towards this end we obtained the ODE of a general system whose components may depend upon 2 previous values, in Appendix I at the end of the thesis. Using this general system, we showed that one can obtain an ODE approximation for a linear equalizer while tracking an AR(2) process for any finite time T. For a stable channel ($d_2 \neq -1$ and $d_1 + d_2 < 1$), we showed that the error between the equalizer and the Wiener filter falls exponentially with time. For an unstable channel the error decays exponentially if $d_1 + d_2 \neq 1$. For an unstable channel with $d_1 + d_2 = 1$, the error decays at the rate $\frac{1}{t}$ if $d_2 \neq -1$ and at the rate $\frac{1}{t^2}$ if $d_2 = -1$. But for a Marginally stable channel ($d_2 = -1$ and $d_1 < 2$), the AR(2) process is approximated by a cosine waveform (Figure 4.9) and the error does not decay to zero but remains bounded in time.

The MSE is shown to approach the instantaneous MMSE exponentially with time when the channel is stable. It is also shown that the difference between the MSE and the MMSE does not explode with time even when the channel explodes (in case of an unstable channel).

We obtained an optimal step size for the linear equalizer tracking an AR(2) process when $d_2 \neq -1$ and $d_1 + d_2 \neq 1$ (i.e., when the error decay is exponential). This can substantially increase the rate of decay of the error.

We suggested a system of ODEs, whose solution can approximate an AR(p) process with $p > 2$ and the corresponding linear equalizer trajectory.

For simplicity, we have only shown the theory for the case when all the taps of the channel are given by the same AR(2) process. But in simulations we have used different AR(2) processes for different channel taps (we only varied the mean $E(W_1)$ in our simulations, one can also use different d_1, d_2 for different taps). The extension of the theory to the general case is very easy. The only thing to note here is that the error decay rate is governed by σ_n^2 as well as η of the tap for which $|\eta|$ is the smallest.

In this Chapter we have not considered the problem of designing an equalizer with delays greater than one. This is once again done to keep the discussions simple and the analysis can be easily extended.

Appendix A

In this section we prove the Proposition 4.1 that is used in this chapter as well as the next chapter. This is obtained after modifying the Proposition 10, p.270, [4] to suit our setup. We use the notations of Section 4.1.

Let the process $\{G_n\}$, taking values in $\mathcal{R}^p$ and parameterized by $(\underline{\theta}, \underline{Z})$, be defined by,

$$G_{n+1} = A(\underline{\theta}, \underline{Z})G_n + B(\underline{\theta}, \underline{Z})\mathcal{W}_{n+1}, \tag{4.17}$$

where $A(\underline{\theta}, \underline{Z})$ is a $p \times p$ matrix, $B(\underline{\theta}, \underline{Z})$ is a $p \times p'$ matrix and $\{\mathcal{W}_n\}$ is a sequence of IID random variables taking values in $\mathcal{R}^{p'}$.

The following definitions are used. For any given function g, for any $p \geq 0$ we set,

$$\begin{aligned}
||g||_{\infty,p} &= \sup_{G} \frac{|g(G)|}{1+|\underline{G}|^p}, \\
[g]_p &= \sup_{\underline{G} \neq \underline{G}', i} \frac{|g(G)-g(G')|}{|\underline{G}-\underline{G}'|\left(1+|\underline{G}|^p+|\underline{G}'|^p\right)}, \\
Li(p) &= \{g; [g]_p < \infty\}, \\
N_p(g) &= \sup\left\{||g||_{\infty,p+1}\, [g]_p\right\}.
\end{aligned}$$

A function $f(\underline{\theta}, \underline{Z}, G)$, differentiable wrt G, is of class $Li(Q)$ if there exist positive constants, L_1, L_2, p_1 and p_2 such that,

- For all $(\underline{\theta}, \underline{Z}) \in Q$, $N_{p_1}(f(\underline{\theta}, \underline{Z}, .)) \leq L_1$.
- For all $(\underline{\theta}, \underline{Z}), (\underline{\theta}', \underline{Z}') \in Q$, all G,

$$|f(\underline{\theta}, \underline{Z}, G) - f(\underline{\theta}', \underline{Z}', G)| \leq L_2 \left|(\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}')\right| (1+|G|^{p_2}).$$

A function $f(\underline{\theta}, \underline{Z}, G)$ is of class $Li^1(Q)$ if there exist positive constants, L_1, L_2, p_1 and p_2 such that,

- For all $(\underline{\theta}, \underline{Z}) \in Q$, $|f(\underline{\theta}, \underline{Z}, 0) + N_{p_1}(f'(\underline{\theta}, \underline{Z}))| \leq L_1$,
- For all $(\underline{\theta}, \underline{Z}), (\underline{\theta}', \underline{Z}') \in Q$, $|f(\underline{\theta}, \underline{Z}, 0) - f(\underline{\theta}', \underline{Z}', 0)| \leq L_2 \left|(\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}')\right|$,
- For all $(\underline{\theta}, \underline{Z}), (\underline{\theta}', \underline{Z}') \in Q$, all G,

$$|f'(\underline{\theta}, \underline{Z}, G) - f'(\underline{\theta}', \underline{Z}', G)| \leq L_2 \left|(\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}')\right| (1+|G|^{p_2}).$$

We assume the following:

X.1 For all $p \geq 0$, $||\mathcal{W}_n||_p \leq \mu_p < \infty$.

X.2 Let $(\underline{\theta}, \underline{Z})$ take values in a subset Q. There exist positive constants, α_1, α_2, β_1, β_2, M and $\rho \in (0,1)$ such that for all $(\underline{\theta}, \underline{Z}), (\underline{\theta}', \underline{Z}') \in Q$, and all $n \geq 0$,

(a) $|A(\underline{\theta}, \underline{Z})| \leq \alpha_1$,

(b) $|A(\underline{\theta}, \underline{Z}) - A(\underline{\theta}', \underline{Z}')| \leq \alpha_2 \, |(\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}')|$,

(c) $|A^n(\underline{\theta}, \underline{Z})| \leq M\rho^n$,

(d) $|B(\underline{\theta}, \underline{Z})| \leq \beta_1$,

(e) $|B(\underline{\theta}, \underline{Z}) - B(\underline{\theta}', \underline{Z}')| \leq \beta_2 \, |(\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}')|$.

We make a slight modification to the Proposition 10. Here the adaptive process $\{\underline{\theta}_k\}$ is modified as in Proposition 10, [4]. However the Markov chain $\{G_k\}$ is parametrized by $(\underline{\theta}, \underline{Z})$ in contrast to a single parameter $\underline{\theta}$ of the Proposition 10. Hence only the above assumptions have changed appropriately. One can easily see that the proof goes through even with this modification.

Proposition 4.1 *Let D be an open subset. Suppose the linear dynamical process (4.17), satisfies the assumptions X.1, X.2 for any compact set Q of D. Let,*

$$\underline{\theta}_{n+1} = \underline{\theta}_n + \gamma_{n+1} H(\underline{\theta}_n, G_{n+1}),$$

with $\{\gamma_{n+1}\}$ a sequence of positive steps. If H or $P_{\underline{\theta},\underline{Z}} H(\underline{\theta}, .) = \int H(\underline{\theta}, y) P_{\underline{\theta},\underline{Z}}(., dy)$ (the expectation of the function $H(\underline{\theta}, .)$ wrt the one-step transition measure $P_{\underline{\theta},\underline{Z}}$) is of class $Li(Q)$ ($Li^1(Q)$), for any compact set Q of D, then $\{\underline{\theta}_n, Z_n\}$ satisfies the following for all $\lambda < 1$ ($\lambda = 1$):

There exists a function h_1 on D, and for each $\theta, Z \in D$ a function $\nu_{Z,\theta}(.)$ such that

1. *h_1 is locally Lipschitz on D.*

2. *$(I - P_{\theta,Z})\nu_{\theta,Z}(G) = H_1(\theta, Z, G) - h_1(\theta, Z)$.*

3. *For all compact subsets Q of D, there exist constants C_3, C_4, q_3, q_4 and $\lambda \in [0.5, 1]$, such that for all $\theta, Z, \theta', Z' \in Q$*

(a) $|\nu_{\theta,Z}(G)| \leq C_3(1+|G|^{q_3})$,

(b) $|P_{\theta,Z}\nu_{\theta,Z}(G) - P_{\theta',Z'}\nu_{\theta',Z'}(G)| \leq C_4\,(1+|G|^{q_4})\left|(\theta,Z)-(\theta',Z')\right|^{\lambda}$.

Proof : It is easy to see that the Lemma 8 p.266 and Lemma 9 p.268 of [4] goes through with $\underline{\theta}$ replaced by ordered pair $(\underline{\theta}, \underline{Z})$ as the above assumptions are just the assumptions in Proposition 10 with $\underline{\theta}$ replaced with the ordered pair $(\underline{\theta}, \underline{Z})$. By modifying the statements of Theorem 5, p. 259 and Theorem 6, p. 262 (by replacing $\underline{\theta}$ with the ordered pair $(\underline{\theta}, \underline{Z})$) it is easy to see that the remaining proof of the Proposition goes through as in [4] (note that the $\{\underline{\theta}_n\}$ updation is similar to that of Proposition 10 p. 270). ■

Chapter 5

DD-LMS-LE versus WF for a Wireless Channel

We consider a time varying wireless fading channel, equalized by an LMS linear equalizer in decision directed mode (DD-LMS-LE). We study how well this equalizer tracks the optimal Wiener equalizer. We also examine when this equalizer can be used, i.e., when one can switch over to the DD mode.

We first study a DD-LMS-LE on a fixed channel. We obtain an ODE approximation for its trajectory. Using this ODE, we obtain the existence of DD attractors near the corresponding Wiener filter at high SNRs. We also show via some examples that for large noise variances (i.e., at low SNRs) the DD attractors will not be close to the WF.

Next we study a DD-LMS-LE tracking a time varying wireless channel, modeled by an AR(2) process of Chapter 2. We use the stochastic approximation theorem of Appendix I, provided at the end of the thesis, and obtain an ODE approximation for the DD-LMS-LE. Using this ODE approximation, we illustrate via some examples, that a DD-LMS-LE can indeed track an AR(2) process reasonably (the DD-LMS-LE trajectory is quite close to the instantaneous WFs) as long as the SNR is high (when properly initialized). With increase in noise variance the DD algorithm loses out.

This chapter is organized as follows. In Section 5.1 we explain our model. Section 5.2 studies the decision directed (DD) algorithm on a fixed channel. Section 5.3 obtains the ODE approximation for a time varying channel. Section 5.4 provides examples to demonstrate the ODE approximations and the proximity of the DD attractors to that of the WFs (at high SNRs). Section 5.5 concludes the chapter. The appendices contain some details on the proofs.

5.1 System Model, Notations and Assumptions

We consider a system consisting of a time varying (wireless) channel followed by an adaptive linear equalizer. The input of the channel s_k comes from a finite alphabet and forms a zero mean IID process. The channel is a time varying finite impulse response (FIR) linear filter $\{Z_k\}$ of length L followed by additive white Gaussian noise $\{n_k\}$. We assume $n_k \sim \mathcal{N}(0, \sigma^2)$. We also assume that $\{s_k\}$ and $\{n_k\}$ are independent of each other. The channel output at time k is

$$u_k = \sum_{i=0}^{L-1} Z_{k,i} s_{k-i} + n_k,$$

where $Z_{k,i}$ is the i^{th} component of Z_k. At the receiver, the channel output u_k passes through a linear equalizer θ_k and then through a hard decoder Q. The output of the hard decoder at time k is $\hat{s}_k$.

In this chapter we consider a linear equalizer updated using the LMS algorithm in decision directed mode (DD-LMS-LE). For this system, the LE $\underline{\theta}_k$, of length M at time k, is initially updated using a training sequence. After a while, the training sequence is replaced by the decisions made at the receiver about the current input symbol s_k. This is the decision directed mode.

The output of the hard decoder, $\hat{s}_k = Q(\underline{\theta}_k^t U_k)$, where $U_k = \pi_k S_k + N_k$ and S_k, N_k, U_k are the appropriate length input, noise and channel output vectors respectively. We assume $E[S_k S_k^T] = I$. Note that the convolutional matrix π_k depends upon the channel co-efficients $Z_k \cdots Z_{k-M+1}$ and is given by,

$$\begin{bmatrix} Z_{k,1} & Z_{k,2} & \cdots & Z_{k,L} & 0 & \cdots & 0 \\ 0 & Z_{k-1,1} & \cdots & Z_{k-1,L-1} & & \cdots & 0 \\ 0 & \vdots & & & & & \\ 0 & 0 & \cdots & Z_{k-M+1,1} & & \cdots & Z_{k-M+1,L} \end{bmatrix}.$$

In this chapter we assume the input to be BPSK, i.e., $s_k \in \{+1, -1\}$. This assumption is made to simplify the discussions and can easily be extended to any finite alphabet source. For BPSK, $Q(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}$.

In DD mode, the LE is updated using $(\hat{s}_k(\underline{\theta}) = Q(\underline{\theta}^t U_k))$,

$$\theta_{k+1} = \theta_k - \mu_k U_k(\theta_k^T U_k - \hat{s}_k(\underline{\theta}_k)) \tag{5.1}$$

where μ_k is a positive sequence of step-sizes.

Initially we study the DD system when the channel is fixed, i.e., $\underline{Z}_k = Z$ for all k. Later, we consider a time varying channel when the channel is modeled by an AR(2) process:

$$Z_{k+1} = d_1 Z_k + d_2 Z_{k-1} + \mu W_k. \tag{5.2}$$

Here W_k is an IID sequence, independent of the processes $\{s_k\}, \{n_k\}$. An AR(2) process can capture most of the channel dynamics of a wireless channel, required for the receiver design ([28]) and has been approximated by an ODE in Chapter 2. Using this ODE approximation, we obtain the required tracking performance analysis.

The fixed channel is studied in Section 5.2 while the time varying channel in Section 5.3.

5.2 DD-LMS-LE for a fixed channel

In this section, we assume that the channel is fixed, i.e., $Z_k = Z$ for all k. We first obtain an ODE approximation for it when the step-sizes $\mu_k \to 0$. We then obtain the existence of DD attractors (ODE) near the corresponding Wiener filters at high SNRs. We show that as noise variance σ^2 tends to zero, these DD attractors tend to the corresponding WFs.

5.2.1 ODE approximation

DD-LMS-LE for a fixed channel has been approximated by an ODE in [4]. (given by Theorem 13, p.278) . For convenience, the Theorem 13, p.278, [4] and its assumptions are reproduced in Appendix D as Theorem 5.3. We start our analysis with this ODE. Towards this goal, as a first step the DD-LMS-LE algorithm (5.1) is rewritten to fit in the setup of Appendix D (i.e., as an example of system (5.5))

$$\begin{aligned}
\underline{\xi}_k &:= \begin{bmatrix} \underline{S}_k^t & \underline{U}_k^t & \hat{s}_k \end{bmatrix}^t, \\
H(\underline{\theta}, \underline{\xi}) &:= \underline{U}^t \left(\underline{\theta}^t \underline{U} - \hat{s} \right), \\
\underline{\theta}_k &= \underline{\theta}_{k-1} - \mu_{k-1} H(\underline{\theta}_{k-1}, \underline{\xi}_k).
\end{aligned}$$

Let $\underline{\theta}(t, t_0, a)$ denote the solution of the following ODE with initial condition $\underline{\theta}(t_0) = a$ (π_Z is the convolutional matrix π_k of the previous section for a fixed channel Z),

$$\begin{aligned}
\dot{\underline{\theta}}\,(t) &= -R_{uu}\underline{\theta}(t) + R_{u\hat{s}}(\underline{\theta}(t)), \\
R_{uu} &= \pi_Z \pi_Z^t + \sigma^2 I, \\
R_{u\hat{s}} &= E\left[UQ\left(U^t \underline{\theta} \right) \right].
\end{aligned}$$

It is easy to see that the Markov chain $\{\xi_k\}$ has a unique stationary distribution for every $\underline{\theta}$ and that the DD-LMS satisfies all the required hypothesis of Theorem 5.3. Also, as shown below, the above ODE has a bounded unique solution for any finite time. Hence one can approximate its trajectory on any finite time scale with the solution of the above

ODE. We reproduce the precise result below.

For any initial condition $\underline{\theta}_0$ and for any finite time T, with $t(r) := \sum_{k=0}^{r} \mu_k$, $m(n,T) := max_{r \geq n}\{t(r) - t(n) \leq T\}$

$$\sup_{\{n \leq r \leq m(n,T)\}} |\underline{\theta}_r - \underline{\theta}\,(t(r),\ t(n),\ \underline{\theta}_0)| \xrightarrow{p} 0$$

as $n \to \infty$, whenever $\sum_k \mu_k = \infty$, $\sum_k \mu_k^{1+\delta} < \infty$ for some $\delta < 0.5$, $\mu_k \leq 1$ for all k and $\liminf_k \frac{\mu_{k+r}}{k} > 0$ for all r.

As in Lemma 5.1 of Appendix C one can show that, the above ODE has a unique global bounded solution for any finite time. We will also show the existence of attractors for this ODE, near WF, at least at high SNRs in the next subsection.

5.2.2 Relation between DD attractors and WFs

In the following we study the desired attractors in more detail. Using implicit function theorem ([5]), we obtain the existence of DD-LMS attractors close to the WFs at high SNRs. For this we make an extra assumption: $Q\left(\underline{\theta}_*(0)^t \pi_Z S_k\right) = s_k$ for all S_k and that $\pi_Z \pi_Z^t$ is invertible. In the above, $\underline{\theta}_*(\sigma_n^2) \stackrel{\triangle}{=} R_{uu}^{-1} R_{us}$, the WF at noise variance σ_n^2. These assumptions are generalization of the commonly made assumptions of perfect equalizability at $\sigma_n^2 = 0$ (see e.g., [16], [21]). Theorem 2 in [16] provides explicit conditions for perfect equalizability at $\sigma_n^2 = 0$.

Let (note that R_{uu}, $R_{u\hat{s}}$ depend on σ^2),

$$\begin{aligned} f(\underline{\theta}, \sigma^2) &\stackrel{\triangle}{=} -R_{uu}\underline{\theta} + R_{u\hat{s}}(\underline{\theta}), \\ \underline{\theta}^*(\sigma^2) &\stackrel{\triangle}{=} R_{uu}^{-1} R_{us}, \text{ where } R_{us} = E[Us]. \end{aligned}$$

Note that $\underline{\theta}^*(\sigma^2)$ represents the WF at noise variance σ^2, while a DD attractor is a zero of the function f. At $\sigma^2 = 0$, by the above mentioned assumptions, $R_{u\hat{s}}(\underline{\theta}^*(0))$ equals R_{us}. Hence $\underline{\theta}^*(0)$, the WF at zero noise variance, also becomes a DD attractor. Thus, $(\underline{\theta}^*(0), 0)$ is a zero of the function $f(.,.)$. One can easily verify the following :

- $f(\underline{\theta}, 0) = -R_{uu}\underline{\theta} + R_{us}$ whenever $\underline{\theta} \in B(\underline{\theta}^*(0), \epsilon)$ for some $\epsilon > 0$, where $B(x, r) = \{y : |x - y| \leq r\}$.

- Thus, $\frac{\partial f}{\partial \underline{\theta}}(\underline{\theta}^*(0), 0) = -R_{uu}$ and R_{uu} is invertible.

- Using Lemma 5.2 of Appendix C, f is continuously differentiable.

By implicit function theorem (Theorem 3.1.10, p. 115, [5]), there exists a $\sigma_0^2 > 0$ and a unique differentiable function g of σ^2 such that, for all $0 \leq \sigma^2 \leq \sigma_0^2$,

$$f(g(\sigma^2), \sigma^2) = 0.$$

Since $\frac{\partial f}{\partial \underline{\theta}}(\underline{\theta}^*(0), 0) = -R_{uu}$ is negative definite and $\frac{\partial f}{\partial \underline{\theta}}$ is continuous at $(\underline{\theta}^*(0), 0)$, $\frac{\partial f}{\partial \underline{\theta}}$ is negative definite over a small neighborhood around $(\underline{\theta}^*(0), 0)$. *Thus zeros, $g(\sigma^2)$ are DD attractors for all σ^2 small enough.* We represent these DD attractors at noise variance σ^2, by $\underline{\theta}_d^*(\sigma^2)$.

We will now relate the DD attractors, $\underline{\theta}_d^*(\sigma^2) = g(\sigma^2)$, to the corresponding WFs, $\underline{\theta}^*(\sigma^2)$, when σ^2 is close to zero. Define $h(\sigma^2) = R_{u\hat{s}}(\underline{\theta}_d^*(\sigma^2))$. Using dominated convergence theorem and continuity of the map g, one can see that $h(\sigma^2) \to h(0) = R_{us}$ whenever $\sigma^2 \to 0$. Define

$$m(\underline{\theta}, \sigma^2, \eta) = -R_{uu}\underline{\theta} + R_{us} + \eta.$$

At any noise variance, σ^2, $m(\underline{\theta}^*(\sigma^2), \sigma^2, 0) = 0$ as $\underline{\theta}^*(\sigma^2)$ is the unique WF at noise variance σ^2. Also, the function m is C^∞ (infinitely differentiable) in all parameters (note that R_{us} is a fixed vector independent of all the parameters whenever input is IID). Hence once again using implicit function theorem at any noise variance, σ_0^2 there exist $\alpha, \beta > 0$ and a continuous function $\gamma(.,.)$ such that,

$$m(\gamma(\sigma^2, \eta), \sigma^2, \eta) = 0 \text{ when } |\eta| \leq \beta, \text{ and } \left|\sigma^2 - \sigma_0^2\right| \leq \alpha.$$

Hence by continuity of the functions γ and h, the WF (which is also given by $\gamma(\sigma^2, 0)$)

will be close to the DD attractor, $\gamma\left(\sigma^2, [R_{us} - R_{u\hat{s}}(\underline{\theta}_d^*(\sigma^2))]\right)$ at low noise variances.

Above we have shown the existence of DD attractors at high SNRs (under zero noise perfect equalizability conditions), near the corresponding WF (the DD may have other attractors also, e.g., [38]). The following theorem shows that if the DD mode is started in the region of attraction of such an attractor, the DD-LMS-LE will converge to the attractor (and hence close to the WF) almost surely as the step sizes tend to zero (Proof is presented in the technical report no TR-PME-2007-1 downloadable from $http://www.pal.ece.iisc.ernet.in/PAM/tech\_rep07.html$).

Theorem 5.1 *Let $\mu_k = \mu\nu_k$ for all k, where $\sum_k \nu_k = \infty$ and $\sum_k \nu_k^{1+\delta} < \infty$. Then there exist $\sigma^2 > 0$ and $\epsilon > 0$ such that, for all $\sigma_n^2 \leq \sigma^2$ and for all $\underline{\theta}' \in B(\underline{\theta}_{*d}(\sigma_n^2), \epsilon)$,*

$$P_{\underline{\theta}_0=\underline{\theta}'}\left(\lim_{k\to\infty} \underline{\theta}_k = \underline{\theta}_{*d}(\sigma_n^2)\right) \geq 1 - \beta(\mu),$$

where $\beta(\mu) \to 0$ as $\mu \to 0$.

Therefore, the DD mode should be started when the LE is within the region of attraction of the above mentioned attractor (e.g., when the 'eye' has opened, as mentioned in chapter 11 of [21], [37], [43]). To reach the region of attraction, one starts with a 'good' initial condition and then uses a training sequence. The region of attraction of the desired attractor depends upon the channel Z, the input distribution and σ_n^2. However, for a given set of parameters it may be computed via the various available methods ([20]).

5.3 DD-LMS-LE tracking an AR(2) process

In this section we present the ODE approximation for the linear equalizer (5.1) in decision directed mode when the channel is modeled as an AR(2) process (5.2). Here we set the step-size $\mu_k = \mu$ for all k, to facilitate tracking of the time varying channel. We use Theorem A.1 of Chapter 4 to obtain the required ODE approximation.

We will show below that the trajectory $(\underline{\theta}_k, \underline{Z}_k)$ given by equations (5.1), (5.2) can be approximated by the solution of the following system of ODEs,

$$\begin{aligned}
(1+d_2)\,\dot{Z}(t) &= \left[E(W)+\eta Z(t)\right], \quad \text{if } d_2 \in (-1,1], \\
\frac{d^2 Z(t)}{dt^2} &= \left[E(W)+\eta Z(t)\right], \quad \text{if } d_2 = -1, \\
\frac{d^2 Z(t)}{dt^2} + \eta_1\,\dot{Z}(t) &= \left[E(W)+\eta Z(t)\right], \quad \text{if } d_2 \text{ is close to } -1,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\dot{\theta}(t) &= -R_{uu}(Z(t))\theta(t) + R_{u\hat{s}}(\underline{\theta}(t), \underline{Z}(t)), \qquad (5.4)\\
\eta &\triangleq \frac{d_1+d_2-1}{\mu}, \ \eta_1 = \frac{1+d_2}{\sqrt{\mu}} \\
R_{uu}(Z) &\triangleq E_Z\left[U(Z)U(Z)^T\right] = (\pi_Z\pi_Z^T + \sigma^2 I), \\
R_{u\hat{s}}(\underline{\theta}, \underline{Z}) &\triangleq E_Z\left[U(Z)\hat{s}(\underline{\theta})\right].
\end{aligned}$$

By Lemma 5.1, the above system of ODEs has unique bounded global solutions for any finite time. Let $Z(t,t_0,Z), \theta(t,t_0,\theta)$ represent the solutions of the ODEs (5.3), (5.4) with initial conditions $Z(t_0) = Z$, $\theta(t_0) = \theta$ and $\dot{Z}(t_0) = 0$ whenever the channel is approximated by a second order ODE.

We prove Theorem 5.2 using the Theorem A.1 of Appendix I, provided at the end of the thesis.

Theorem 5.2 *For any finite $T > 0$, for all $\delta > 0$ and for any initial condition $(G, \underline{\theta}, \underline{Z})$, with $d_2 Z_{-1} + d_1 Z_0 = Z$, $\dot{Z}(t_0) = 0$ whenever the channel is approximated by a second order ODE and $\theta_0 = \theta$,*

$$P_{G,Z,\theta}\left\{ \sup_{\left\{1\le k\le \frac{T}{\mu^\alpha}\right\}} \left|(Z_k, \underline{\theta}_k) - (Z(\mu^\alpha k, 0, Z), \theta(\mu k, 0, \underline{\theta}))\right| \ge \delta \right\} \to 0,$$

as $\mu \to 0$, uniformly for all $(Z,\theta) \in Q$, if Q is contained in the bounded set containing the solution of the ODEs (5.3), (5.4) till time T. In the above $\alpha = 1$ if $Z(.,.,.)$ is solution of a first order ODE and equals $1/2$ otherwise.

Proof : Please see the Appendix A.

The approximating ODE (5.4) suggests that, its instantaneous attractors will be same as the DD-LMS-LE attractors obtained in the previous section when the channel is fixed at the instantaneous value of the channel ODE (5.3). We have shown in the previous section that these attractors are close to the WF at high SNRs. We will verify the same behavior for tracking, using some examples, in the next section.

One of the uses of the above ODE approximation is that, one can study the tracking behavior of the DD-LMS (e.g., proximity of its trajectory to the instantaneous WFs) using this ODE. This is done in the next section. Further, one can also obtain instantaneous theoretical performance measures (approximate) like BER, MSE.

5.4 Examples

In this section we illustrate the theory developed so far using some commonly used examples.

We first consider a fixed channel, $Z = [.41, .82, .41]$. This channel is very widely used (see p. 414, [25] and p. 165, [21]). We use a two-tap linear equalizer. We plot the DD-LMS-LE, its ODE approximation and the Wiener filter for two values of noise variances $\sigma^2 = 0.01$, 1 in Figure 5.1. We see that the ODE approximation is quite accurate for all the coefficients. We can also see that the DD-LMS coefficients as well as their ODE approximations converge to the DD attractor for both the noise variances. The ODE approximation thus confirms that, with high probability the realizations of the DD-LMS trajectory (the DD-LMS trajectory in the figure being one such realization) converge to an attractor (of course, when initialized properly). One can see from this example that the DD-LMS attractors are close to the corresponding Wiener filters at high SNRs ($\sigma^2 = 0.01$) as is shown theoretically in Section 5.2, but are away from the same at low SNRs ($\sigma^2 = 1$).

We next consider two examples of a time varying channel equalized by a four-tap equalizer. We consider stable and marginally stable channels in Figure 5.2, 5.3 respectively. For both the examples, the mean channel is a constant multiple of a commonly used raised

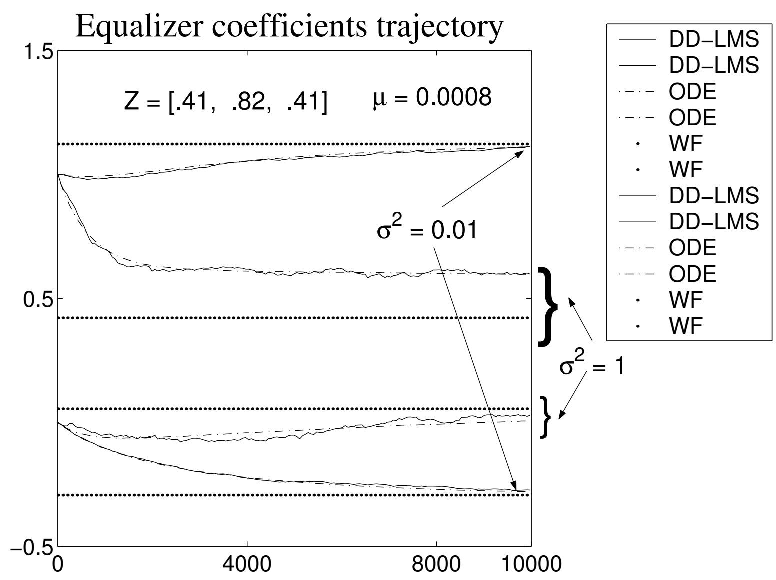

Figure 5.1: The fixed channel equalizer coefficients for three tap channel $Z = [.41, .8, .41]$

cosine channel, $[-0.4, 1, 0.6, -0.3, 0.1]$. The AR parameters, $d_1 = .497$, $d_2 = 0.5$ and $\mu = 0.0007$ are used for the stable channel, while the same parameters for the marginally stable channel are set at 1.99999, -1 and 0.000001 respectively (these parameter are chosen appropriately to obtain a suitable period for oscillations as from (2.2) of Chapter 2, the period of oscillation is controlled by $\sqrt{\frac{1-d_1-d_2}{\mu}}$). Both the examples are run under high SNR conditions (σ^2 equals 0.1 and 0.05 for marginally stable and stable channels respectively). In both the examples, the DD-LMS and the ODE are started with the initial value of the WF. One can see from both the figures that the ODE once again approximates the DD-LMS quite accurately. Also, the DD-LMS and the ODE track the instantaneous WF quite well. We also plot the instantaneous BER of the DD-LMS, the ODE and the WF in the last subfigure of the Figures 5.2, 5.3. One can see that the performance of the DD-LMS and ODE are quite close to that of the WFs throughout the time axis. The proximity of the ODE solution and the BER once again indicate that with high probability the realizations of DD-LMS track the instantaneous WFs.

Finally, in Figure 5.4, we consider a stable channel with d_2 close to -1 (from the figure, it actually looks like a marginally stable channel but its magnitude is reducing at a very small rate, $\frac{1+d_2}{\sqrt{\mu}}$, as d_2 is very close to -1). In this case, as is shown theoretically, a better

ODE approximation is obtained by a second order ODE. Here, the channel trajectory is approximated by an exponentially decaying cosine waveform. We considered the AR(2) process, which approximates the fading channel with band limited and U-shaped spectrum and received with $f_dT = 0.002$ and hence $d_1 = 1.999926$, $d_2 = -.9999789$, $\mu = 2.218332e-009$. The mean of the channel is constant multiple of $[.41, .8, -.4]$ while the noise variance is set at 0.03. Once again, as is seen from the Figure, the DD-LMS (and also the ODE) is tracking the WF well. Further, the performance (BER) of the DD-LMS and ODE is quite close to that of the WFs throughout the time axis.

Next we plot the DD-LMS, the ODE and the instantaneous WFs at two different noise variances in Figure 5.5 for a marginally stable channel. It is evident from the figure that the LMS-LE in DD mode, can track the channel variations at high SNR ($\sigma^2 = 0.05$), while it looses out at low SNRs ($\sigma^2 = 1$).

5.5 Conclusions

We obtain theoretical performance analysis of an LMS linear equalizer in decision directed mode. We first study a decision directed LMS-LE for a fixed channel. We approximate the trajectory of the DD-LMS-LE by an ODE. Next using this ODE, we obtain the existence of DD attractors in the vicinity of the WFs at high SNRs. We also illustrate this for some commonly used channel examples. We further show via some examples that, a DD attractor may be away from the Wiener filter at low SNRs. We thus conclude that at high SNRs, one can update the LMS-LE in decision directed mode to obtain the WFs, by initializing it with a 'good' enough (the initializer must be in the region of attraction of the desired attractor) training based estimate.

We next consider time varying channels. We model the time varying channel by an AR(2) process and obtain an ODE approximation for the tracking DD-LMS-LE. Using this ODE approximation, via some examples, we illustrated that LMS-LE in decision directed mode, can also be used to track the instantaneous WF at high SNRs. We also show that, at low SNRs the decision directed mode does not provide a good equalizer.

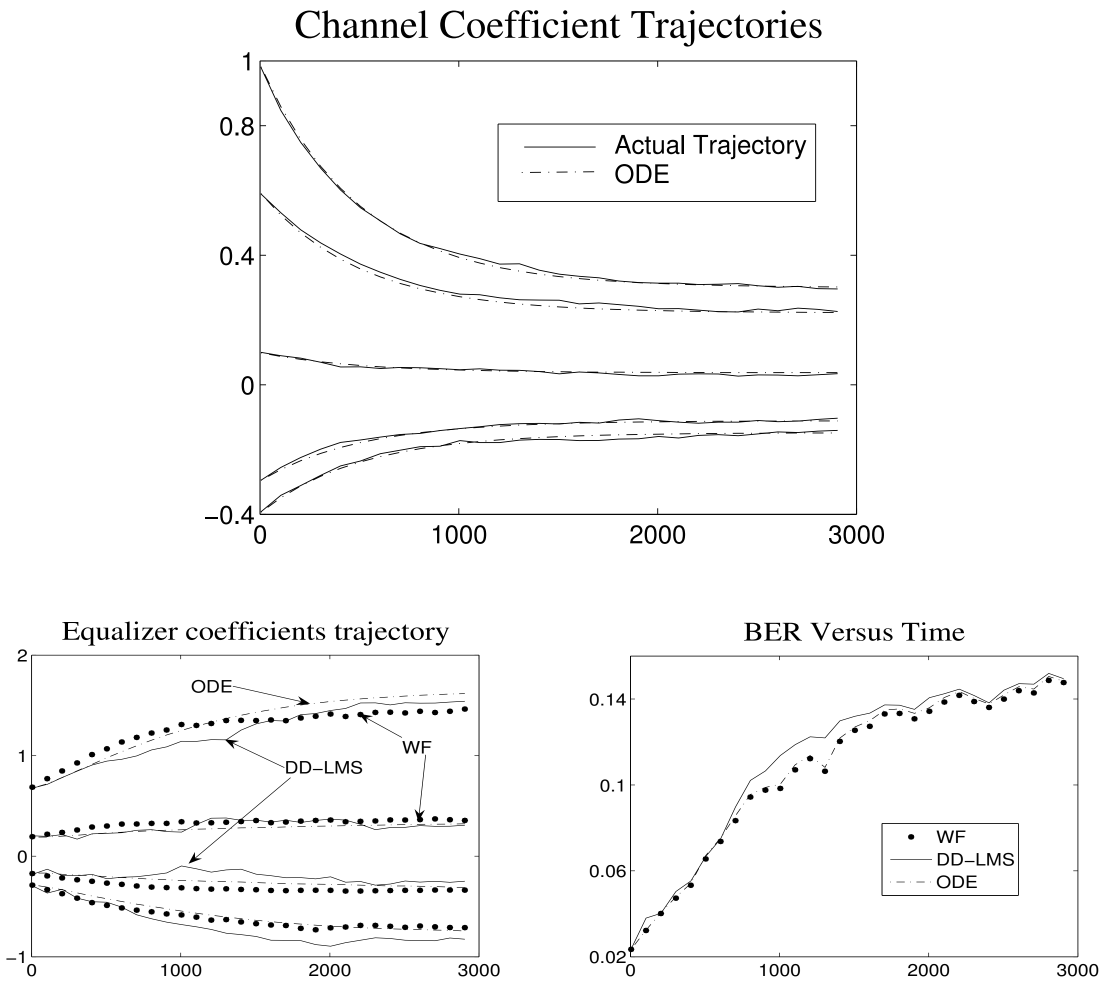

Figure 5.2: Trajectories of raised cosine AR(2) channel, the DD-LMS filter coefficients and the BER for a stable channel.

Appendix A

Proof of Theorem 5.2 : Defining $G_{k+1} \triangleq \left[U_k^T, S_k^T\right]^T$, one can rewrite the AR(2) process and the DD equalizer adaptation as,

$$\begin{aligned}
Z_{k+1} &= (1-d_2)Z_k + d_2 Z_{k-1} + \mu(W_k + \eta Z_k) \\
\theta_{k+1} &= \theta_k + \mu H_1(\theta_k, G_{k+1}) \\
H_1(\theta_k, G_{k+1}) &\triangleq -U_k(\theta_k^T U_k - \hat{s}_k) \\
&= -U_k(\theta_k^T U_k - Q(U_k^t \underline{\theta}_k)).
\end{aligned}$$

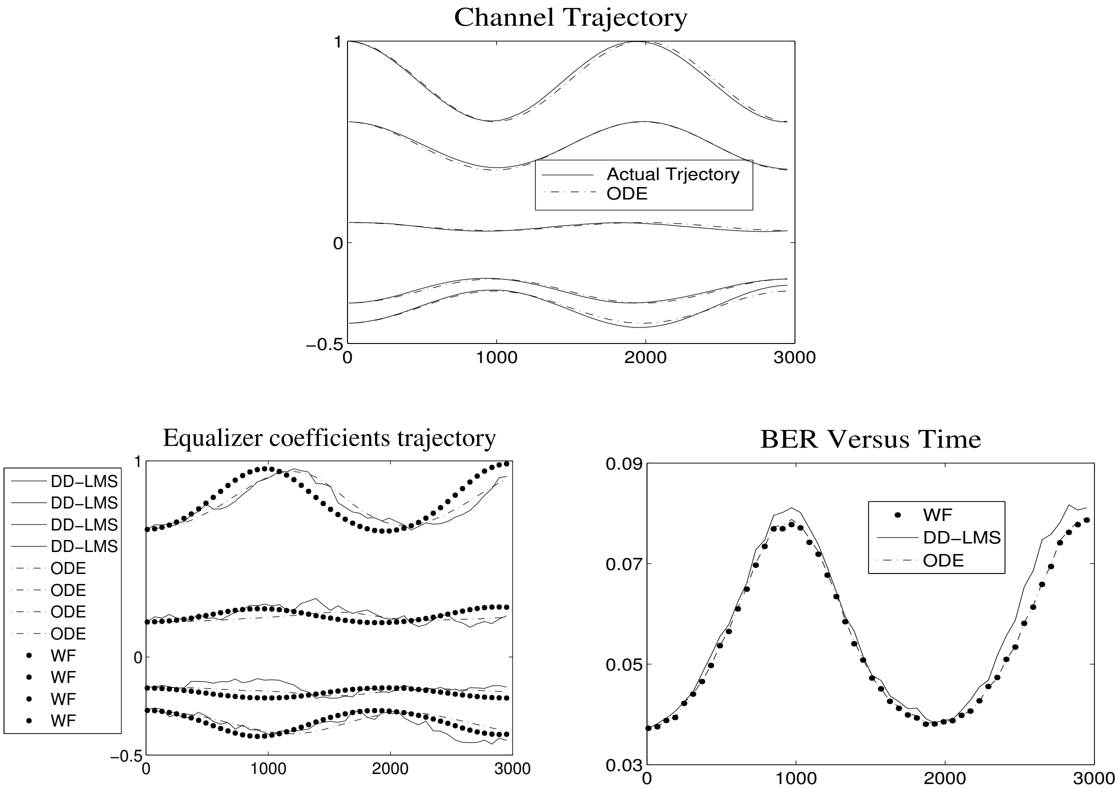

Figure 5.3: Trajectories of raised cosine AR(2) channel, the DD-LMS filter coefficients and the BER for a marginally stable channel.

This is similar to the general system (1), (2) of Appendix I (provided at the end of the thesis). Hence by Theorem A.1 of Appendix I, it suffices to show that our system satisfies the assumptions **A.**1-**A.**3 of Chapter 2, **B.**1-**B.**4 of the Appendix I and that the above system of ODEs has a bounded solution for any finite time.

The AR(2) process $\{\underline{Z}_k\}$ in (5.2) clearly satisfies the assumptions **A.**1 - **A.**3 as is shown in Chapter 2. Assumption **B.**2 is satisfied as for any compact set Q and for any $\theta \in Q$,

$$|H_1(\theta, G)| \leq 2 \left[max \left\{ 1, \sup_{\theta \in Q} |\theta| \right\} \right] (1 + |G|^2).$$

Fixing channel $Z_k = Z$ for all k, we obtain the transition kernel $\Pi_Z(.,.)$ for $\{G_k\}$ which is a function of Z alone. Thus condition **B.**1 is satisfied. It is easy to see that $\{G_k(Z)\}$

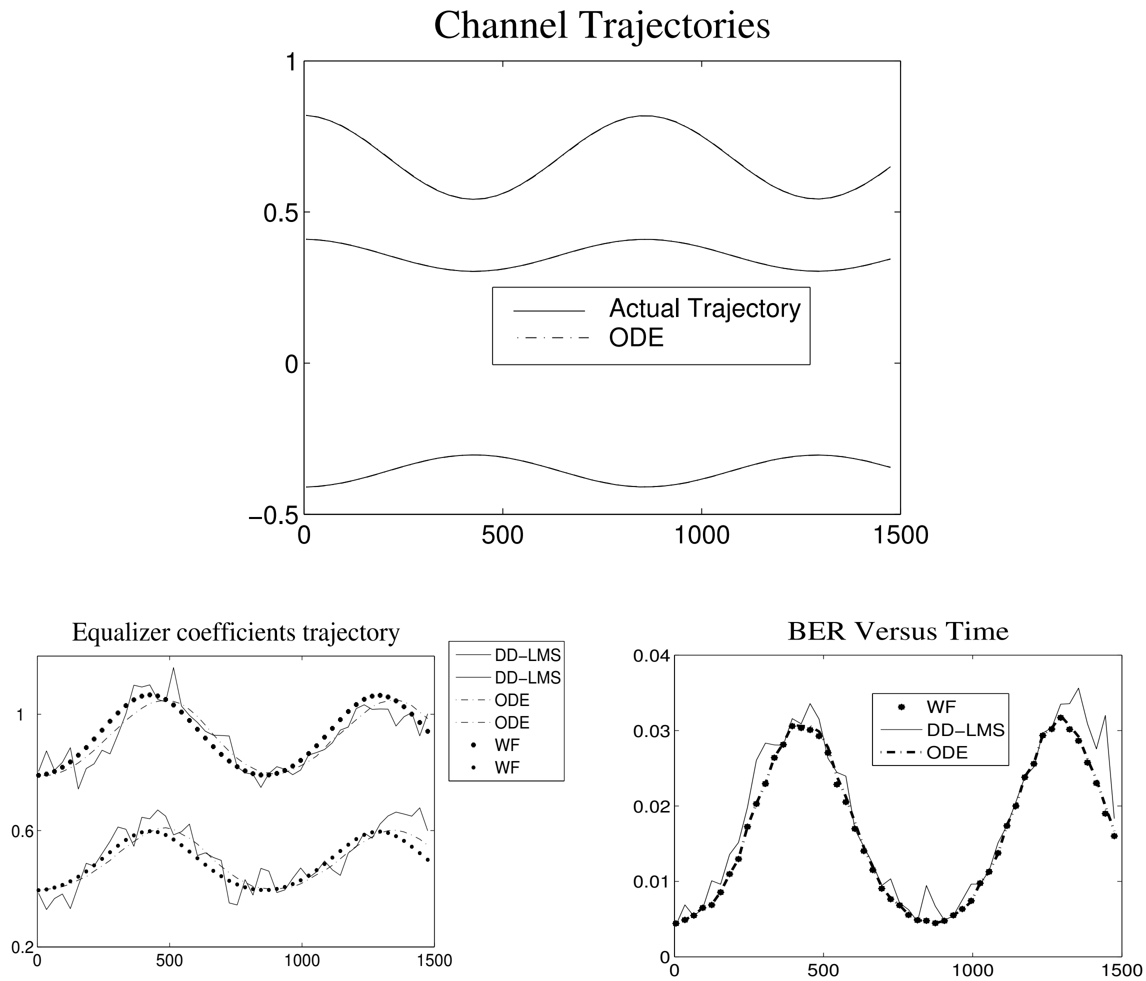

Figure 5.4: Trajectories of an AR(2) channel, the DD-LMS filter coefficients and the BER for a stable channel with $f_d T = 0.002$.

has a stationary distribution given by,

$$\Pi_Z(\mathbf{A_1} \times [s_1, s_2, \cdots s_n]) \;=\; Prob(S = [s_1, s_2, \cdots s_n]) Prob(N \in \mathbf{A_1} - \pi_Z [s_1, s_2, \cdots s_n]^T),$$

where π_Z is the $M \times M + L - 1$ length convolutional matrix formed from vector Z (as in the fixed channel case) and S, N are the input and noise vectors of length $M + L - 1$, M respectively. Define,

$$h_1(\underline{\theta}, \underline{Z}) \;\stackrel{\triangle}{=}\; E_Z(H_1(\theta, G(Z)) = -R_{uu}(Z)\theta + R_{u\hat{s}}(\underline{\theta}, \underline{Z}),$$

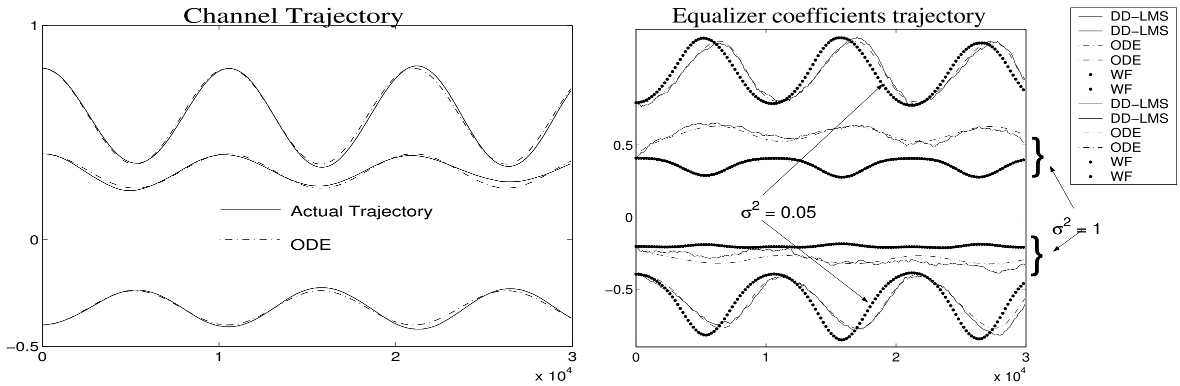

Figure 5.5: DD-LMS versus WFs at varying σ^2 in a time varying channel

$$\nu_{\underline{\theta},\underline{Z}}(G) \stackrel{\triangle}{=} \sum_{k\geq 0} \Pi_Z^k(H_1(\theta, G) - h_1(Z,\theta)).$$

By Lemma 5.2 of Appendix C, h_1 is locally Lipschitz. Thus conditions **B**.3 a, b are met.

We now prove condition **B**.3.c using Proposition 4.1 provided in Appendix A of Chapter 4 in similar lines as in Chapter 4. $\{G_k\}$ is a linear dynamic process depending upon the channel realization Z and it can be written as,

$$\begin{aligned}
G_{k+1} &= A(Z)G_k + B(Z)\mathcal{W}_{k+1}, \\
\text{where, } A(Z) &= \begin{bmatrix} \mathbf{J}_M & \mathbf{P} \\ \mathbf{0}_{L+M-1\times M} & \mathbf{J}_{L+M-1} \end{bmatrix}, \\
B(Z) &= \begin{bmatrix} Z_1 & 1 \\ \multicolumn{2}{c}{\mathbf{0_{M-1\times 2}}} \\ 1 & 0 \\ \multicolumn{2}{c}{\mathbf{0_{L+M-2\times 2}}} \end{bmatrix}, \\
\mathcal{W}_{k+1} &= [s_k, n_k].
\end{aligned}$$

In the above definitions, $\mathbf{0}_n$ is a $n \times n$ zero matrix, $\mathbf{J}_n$ is a $n \times n$ shift matrix given by,

$$\mathbf{J}_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and the matrix $\mathbf{P}$ is a $M \times L + M - 1$ matrix defined by,

$$\mathbf{P} = \begin{bmatrix} Z_2 \quad Z_3 \qquad\qquad \cdots \qquad\qquad Z_L \quad 0 \quad \cdots \quad 0 \\ \mathbf{0_{M-1 \times L+M-1}} \end{bmatrix}.$$

It is easy to see that, $A^n(Z) = 0$ for all $n \geq max\{L, L+M-1\}$ for all Z as it involves the powers of shift matrices $\mathbf{J}_L$, $\mathbf{J}_{L+M-1}$, which satisfy $\mathbf{J}_n^n = 0$. By Lemma 5.3, the function $P_{\underline{\theta},\underline{Z}} H(\theta, G)$ is $L_i(R^n)$. Now all other conditions of Proposition 4.1 are satisfied trivially (because $A(Z)$ and $B(Z)$ are linear in Z) and hence assumption **B**.3.c is satisfied for any $\lambda < 1$.

The condition **B**.4 is trivially met as for any $n > M + L - 1$, the expectation does not depend upon the initial condition G, but is bounded based on the compact set Q and because of the Gaussian random variable N and discrete random variable S.

Finally the theorem follows by Lemma 5.1, which gives the existence of unique bounded solution, for any finite time, for the DD-LMS ODE. ■

Appendix C

Lemma 5.1 *The ODE (5.4) has a unique solution, which satisfies,*

$$|\theta(t)| \leq c_0 + c_1 e^{-\sigma^2 t},$$

for appropriate positive constants c_0 and c_1.

Proof : For convenience, we reproduce the ODE (5.4),

$$\dot{\theta}\,(t) \;=\; -R_{uu}(Z(t))\theta(t) + R_{u\hat{s}}(\underline{\theta}(t), \underline{Z}(t)).$$

The matrix $R_{uu}(Z(t))$ is positive definite for all t, and it's minimum eigen value is greater than σ^2 for all t. Also, $|R_{u\hat{s}}(\underline{\theta}(t), \underline{Z}(t))| \leq C\,|Z(t)|$ for all t for some constant $C > 0$. Using the channel ODE solution curves (given in Chapter 2), $|R_{u\hat{s}}(\underline{\theta}(t), \underline{Z}(t))| \leq C(T)$ for all $t \leq T$ for any finite time T for some positive constant $C(T)$ depending only on T. Thus, for any vector θ, the inner product,

$$\begin{aligned}\left\langle \dot{\underline{\theta}}\,(t), \underline{\theta} \right\rangle &\leq -\sigma^2|\underline{\theta}|^2 + C(T)|\underline{\theta}| \\ &= \left[-\sigma^2|\underline{\theta}| + C(T)\right]|\underline{\theta}|.\end{aligned}$$

Therefore by Global existence theorem (pp 169 - 170 of [42]), the ODE (5.4), has a unique solution for any finite time and the solution is bounded by the solution of the following scalar ODE (after choosing the initial conditions properly),

$$\dot{k}\,(t) = -\sigma^2 k(t) + C(T),$$

whose solution is given by, $k(t) = c_1 e^{-\sigma^2 t} + C(T)$, where c_1 is an appropriate constant. ■

Lemma 5.2 *The function $R_{u\hat{s}}(\underline{\theta}, \underline{Z})$ is continuously differentiable in $(\underline{\theta}, \underline{Z})$, σ^2 and hence is locally Lipschitz.*

Proof : With $f_{\mathcal{N}}(\sigma^2, N)$ representing the zero mean M dimensional Gaussian density, with variance $\sigma^2 I$,

$$\begin{aligned}R_{u\hat{s}}(\underline{\theta}, \underline{Z}) &= E[(\pi_Z S + N)Q(\theta^t(\pi_Z S + N))] \\ &= \sum_S \int_{\{N:\theta^t(\pi_Z S+N)>0\}} (\pi_Z S + N) f_{\mathcal{N}}(\sigma^2, N)dN \\ &\qquad\qquad - \int_{\{N:\theta^t(\pi_Z S+N)<0\}} (\pi_Z S + N) f_{\mathcal{N}}(\sigma^2, N)dN.\end{aligned}$$

We make the following change of variable,

$$
\begin{aligned}
Y &= A(\theta)(\pi_Z S + N) \text{ where matrix} \\
A(\theta) &\triangleq \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_M \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\end{aligned}
$$

With $|B|$ representing the determinant of the matrix B,

$$
\begin{aligned}
R_{u\hat{s}}(\underline{\theta}, \underline{Z}) &= \sum_S \int_{\{Y \,:\, Y_1>0\}} A(\theta)^{-1} Y \left|A(\theta)^{-1}\right| f_{\mathcal{N}}(\sigma^2, A(\theta)^{-1}Y - \pi_Z S) dY. \\
&\quad - \int_{\{Y \,:\, Y_1<0\}} A(\theta)^{-1} Y \left|A(\theta)^{-1}\right| f_{\mathcal{N}}(\sigma^2, A(\theta)^{-1}Y - \pi_Z S) dY.
\end{aligned}
$$

Using dominated convergence theorem, we can show that the terms on the right hand side are continuously differentiable. ■

Lemma 5.3 *Let $P_{\underline{\theta},\underline{Z}}(.|.)$ represent the transition function of the Markov chain $\{G_k(\underline{\theta}, \underline{Z})\}$ (when the channel and equalizer are fixed at $(\underline{\theta}, \underline{Z})$). The function $P_{\underline{\theta},\underline{Z}} H_\theta(G)$ is locally Lipschitz.*

Proof : Note that,

$$
\begin{aligned}
P_{\underline{\theta},\underline{Z}} H_\theta(G_0) &= E\left[H_1(\theta, G_1) | G_0 = (U_0, S_0)\right] \\
&= E\left[H_1\left(\theta, (A(Z)G_0 + B(Z)\mathcal{W}_1)\right)\right].
\end{aligned}
$$

For all $(\underline{\theta}, \underline{Z})$ in a compact set, one can get a positive constant C_1 depending only upon the compact set Q such that,

$$
\begin{aligned}
&\left|P_{\underline{\theta},\underline{Z}} H_\theta(G_0) - P_{\underline{\theta}',\underline{Z}'} H_{\theta'}(G_0')\right| \\
&\qquad \leq E\left|(\theta^t U_1)U_1 - ({\theta'}^t U_1')U_1'\right| + 2E\left|U_1 - U_1'\right| + C_1 E\left|Q(\theta^t U_1) - Q({\theta'}^t U_1')\right|,
\end{aligned}
$$

where $U_1 \triangleq A(Z)G_0 + B(Z)\mathcal{W}_1$, $U_1' \triangleq A(Z')G_0' + B(Z')\mathcal{W}_1$. Thus it suffices to show the

Lipschitz continuity for the last term. Now,

$$E\left|Q(\theta^t U_1) - Q(\theta'^t U_1')\right| \;=\; 2P(Q(\theta^t U_1) \neq Q(\theta'^t U_1'))$$

The Lemma follows because, using the steps as in Lemma 5.2, we can show that the above term is continuously differentiable. ■

Appendix D

In this section we provide the statement of Theorem 13, p.278, [4] that is used in this chapter as well as the next chapter. We state the Theorem 13, [4], in our setup using our notations.

We consider the following dynamical process $\{V_k, U_k\}$:

$$\begin{aligned} V_k &= AV_{k-1} + Bs_k, \\ u_{k+N} &= CV_{k-1} + n_k, \\ U_k^T &= [u_{k+N}, \cdots, u_{k-N}]. \end{aligned}$$

For the quantizer Q we define a sequence of random variables $\hat{s}_k$, $\hat{S}_k$, $\underline{\theta}_n$ by:

$$\begin{aligned} \hat{S}_k^T &= [\hat{s}_k, \cdots, \hat{s}_{k-p+1}] \\ \hat{s}_k &= Q\left(\underline{\theta}_{f\,k-1}^{\,t} U_k + \underline{\theta}_{bk-1}^{\,t} \hat{S}_{k-1}\right) \\ \underline{\theta}_k &= \underline{\theta}_{k-1} + \mu_k H(\underline{\theta}_{k-1}, \xi_k), \quad \underline{\theta}_k = [\underline{\theta}_{f\,k}^{\,T}, \underline{\theta}_{bk}^{\,T}], \end{aligned} \tag{5.5}$$

where $\xi_k = [V_{k+N}^T, U_k^T, \hat{S}_k^T]$.

We make the following assumptions:

D.1 The matrix A satisfies :

$$|A^n| \leq K.\rho^n, \quad 0 < \rho < 1.$$

D.2 The sequence $\{(s_k, n_k)\}$ is a sequence of IID random vectors with values in $\mathcal{S} \times \mathcal{R}$, for some finite set $\mathcal{S}$ of $\mathcal{R}$. The $\{s_k\}$ are uniformly distributed over $\mathcal{S}$.

D.3 The decoder Q is assumed to satisfy

(a) There exist $s_1, \cdots, s_q \in \mathcal{R}$ such that $\{|s - s_i| > |s - s'|$ for all $i\} \Rightarrow Q(s) = Q(s')$.

(b) There exists an $\bar{\int} \in \mathcal{S}$, $\bar{s} \in \mathcal{R}$ such that $s > \bar{s} \Rightarrow Q(s) = \bar{\int}$.

D.4 H is of class $Li(K)$ (defined in Appendix A of Chapter 4) for any compact set K.

D.5 The random variable n_k has a density f satisfying,

$$f > 0 \text{ and for all } m \geq 0 : \sup_t (1 + |t|^m) f(t) < \infty.$$

D.6 For any integer r

$$\liminf_n \frac{\mu_{n+r}}{\mu_n} > 0.$$

D.7 $\sum_n \mu_n = \infty$.

We consider the ODE

$$\dot{\underline{\theta}}\ (t) = h(\underline{\theta}(t)), \tag{5.6}$$

where $h \stackrel{\triangle}{=} \lim_n P^n_{\underline{\theta}} H(\underline{\theta}, .)(\xi)$ (This limit is shown to exist in Theorem 13. Here, $P^n_{\underline{\theta}}$ represents the $n-$step transition function of the Markov chain $\{\xi_k\}$ obtained by fixing $\underline{\theta}_k = \underline{\theta}$ for all k) for $t \geq t_0$. Let $\underline{\theta}(t, t_0, a)$ denote the solution of this ODE for $\underline{\theta}(t_0) = a_0$.

Define $D = \{\underline{\theta}^T = (\underline{\theta}_f^T, \underline{\theta}_b^T) : |\underline{\theta}_f| > 0\}$. We choose $T > 0$ and two compact sets Q_1, Q_2 of D with $Q_1 \subset Q_2$, which we assume to satisfy:

$$d(\underline{\theta}(t, 0, a), Q_2^c) \geq \delta_0 > 0,$$

for all $a \in Q_1$ and all $t \leq T$. This condition can be satisfied by showing that a ODE has a bounded solution for any finite time (as in Chapter 2). Let $t(r) := \sum_{k=0}^{r} \mu_k$, $m(n,T) := \max_{r \geq n}\{t(r) - t(n) \leq T\}$.

Theorem 5.3 (Theorem 13, p.278, [4]) : *Under the above assumptions, the algorithm (5.5) and the ODE (5.6) satisfy the following:*

For any compact set $Q \subset D$ there exist constants B_5, L_2 and s, such that for all $k \geq 0$ with $\mu_k \leq 1$, all $a \in Q$, all ξ, all $\delta \leq \delta_0$, and for all $\lambda < 1$,

$$P_{n,\xi,a}\left\{\sup_{n \leq r \leq m(n,T)} |\underline{\theta}_r - \underline{\theta}(t_r, t_n, a)| \geq \delta\right\} \leq \frac{B_5}{\delta}(1 + \xi^s)(1 + T)exp(2L_2T).\sum_{k \geq n} \mu_k^{2\wedge(1+\lambda/2)}$$

where $P_{n,\xi,a}$ denotes the $n-$step transition function of the chain $\{\xi_k, \theta_k\}$ with initial condition ξ, a.

Chapter 6

LMS-DFE versus DFE-WF for a Fixed Channel

A Decision Feedback Equalizer (DFE) can significantly improve the performance of a system as compared to an LE (especially when there are deep nulls in the frequency response of the channel). A DFE feeds back the previous decisions of the transmitted symbols, to nullify the ISI due to them and makes a better decision about the current symbol. However, due to feedback, studying its performance is complicated and hence there is no known method to obtain an MSE optimal DFE (henceforth we will call it DFE-WF) even for a fixed channel ([10], [32], [43]). An 'optimal' DFE is commonly obtained by assuming error free decisions. We will call it IDFE. It is believed that the IDFE is close to the optimal DFE under high SNR. However it is not known how an IDFE really compares with the DFE-WF.

Another way to obtain an optimal DFE is to replace the feedback filter of the DFE with a precoder at the transmitter. The precoder requires that the channel state be known at the transmitter. This is possible for a wireline channel by estimating at the receiver via a training sequence and sending this information to the transmitter. But for a wireless channel, due to its time varying nature, this will be inefficient.

A common iterative algorithm to obtain MMSE filters is LMS (as we studied in the previous chapters). LMS has also been used to obtain a DFE ([43]). But the convergence

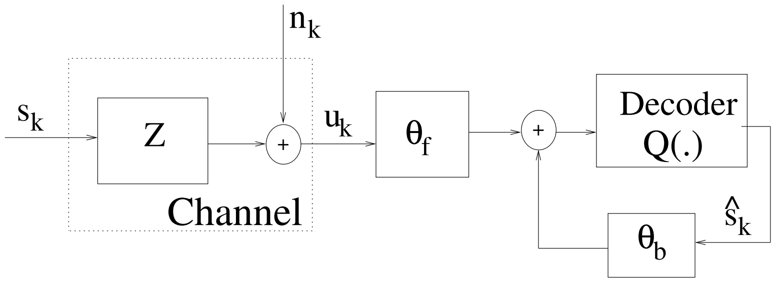

Figure 6.1: Block Diagram of Decision Feedback Equalizer (DFE)

behavior of an LMS for a DFE is not well understood. In particular, it is not known whether the LMS will converge to the DFE-WF even for a fixed channel.

In this chapter we theoretically study the LMS-DFE and show that under high SNR it converges to a limit close to the optimal DFE, the DFE-WF. We also show, via examples, that the IDFE (designed also using perfect channel estimate) may perform much worse than the DFE-WF and the limiting LMS-DFE (limiting LMS-DFE is close to the DFE-WF). In fact, the performance improvement is very significant even at high SNRs (up to 50%), where the IDFE is believed to be closer to the optimal one. To obtain these results we again approximate the LMS-DFE trajectory with an ODE and relate its attractors to the DFE-WF using implicit function theorem. We thus conclude that, via LMS one can obtain a computationally efficient way to obtain the true DFE Wiener filter under high SNR for a fixed channel. In the next chapter, we will study the tracking behavior of an LMS-DFE for a wireless channel.

This chapter is organized as follows. In Section 6.1 we define the model and specify our assumptions. In Section 6.2 we discuss the problems involved and the approach followed by us. In Section 6.3 we show that the trajectory of the LMS-DFE can be approximated by the solution of an ODE. We use this ODE for further analysis of the LMS algorithm. We study the differentiability of the stationary distribution of the system in Section 6.4. In Section 6.5 we show that the LMS attractors are close to that of the DFE Wiener filters at high SNRs. Section 6.6 provides some examples while Section 6.7 concludes the chapter. Proofs are contained in the appendices.

6.1 The model and notations

We consider the system shown in Figure 6. Inputs $\{s_k\}$ enter a time invariant finite impulse response channel $\{z_l\}_{l=0}^{L-1}$ and are corrupted by additive white Gaussian noise $\{n_k\}$ with variance σ^2. The channel output, u_k, at any time k, is given by,

$$u_k = \sum_{l=0}^{L-1} s_{k-l} z_l + n_k.$$

We use a DFE with a forward filter $\underline{\theta}_f$ and a feedback filter $\underline{\theta}_b$. The decisions are made by hard decoding. The performance of this system will depend upon the DFE filters $\underline{\theta}_f$ and $\underline{\theta}_b$. We are interested in obtaining the $(\underline{\theta}_f, \underline{\theta}_b)$ that minimizes the probability of error. Since it is not easy to optimize the probability of error, the common criterion used is to minimize $E\left[|s_k - \hat{s}_k|^2\right]$. We consider this problem in this chapter.

We make the following assumptions (some of these can be easily weakened) and use the following notations:

- We assume BPSK modulation, i.e., inputs $s_k \in \{+1, -1\}$.
- Sequences $\{s_k\}$ and $\{n_k\}$ are IID (independent, identically distributed) and independent of each other. The inputs $\{s_k\}$ are uniformly distributed.
- $f_{\mathcal{N}}(\underline{y})$ is the N dimensional standard IID Gaussian density, where N is the dimension of the vector $\underline{y}$, i.e.,

$$f_{\mathcal{N}}(\underline{y}) = \frac{1}{(2\pi)^{N/2}} exp^{-\frac{|\underline{y}|^2}{2}}.$$

- The equalizer forward filter is given by $\{\theta_{f_l}\}_{l=0}^{N_f-1}$, while the feedback filter is given by $\{\theta_{b_l}\}_{l=1}^{N_b}$. Also, $N_L \triangleq N_f + L - 1$.
- The decisions are obtained after hard decoding. Hence decision $\hat{s}_k$ is given by,

$$\hat{s}_k \quad = \quad Q\left(\sum_{l=0}^{N_f-1} \theta_{f_l} u_{k-l} + \sum_{l=1}^{N_b} \theta_{bl} \hat{s}_{k-l}\right) \text{ where}$$

$$Q(x) \;:=\; \begin{cases} +1 & if \quad x \geq 0, \\ -1 & if \quad x < 0. \end{cases} \tag{6.1}$$

- The following vector notations are used throughout:

$$\begin{aligned}
\underline{S}_k &\triangleq [\ s_k \quad s_{k-1} \quad . \quad . \quad s_{k-N_L+1}\ ]^T, \\
\underline{N}_k &\triangleq [\ n_k \quad n_{k-1} \quad . \quad . \quad n_{k-N_f+1}\ ]^T, \\
\underline{U}_k &\triangleq [\ u_k \quad u_{k-1} \quad . \quad . \quad u_{k-N_f+1}\ ]^T, \\
\underline{\hat{S}}_k &\triangleq [\ \hat{s}_k \quad \hat{s}_{k-1} \quad . \quad . \quad \hat{s}_{k-N_b+1}\ ]^T, \\
\underline{X}_k &\triangleq [\ \underline{U}_k^T \quad \underline{\hat{S}}_{k-1}^T\ ]^T, \\
\underline{G}_k &\triangleq [\ \underline{S}_k^T \quad \underline{\hat{S}}_{k-1}^T \quad \underline{N}_k^T\ ]^T, \\
\underline{\theta}_f &\triangleq [\ \theta_{f_0} \quad \theta_{f_1} \quad . \quad . \quad \theta_{f_{N_f-1}}\ ]^T, \\
\underline{\theta}_b &\triangleq [\ \theta_{b1} \quad \theta_{b2} \quad . \quad . \quad \theta_{bN_b}\ ]^T, \\
\underline{\theta} &\triangleq [\ \underline{\theta_f}^T \quad \underline{\theta_b}^T\ ]^T.
\end{aligned}$$

- With $\mathcal{S} := \{+1, -1\}$, for a fixed $\underline{\theta}$, $\{G_k\}$ is a Markov chain taking values in $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b} \times \mathcal{R}^{N_f}$, where $\mathcal{R}$ is the set of real numbers. We represent throughout this chapter the current and previous state values of this Markov chain by the ordered pairs $(i, \underline{y})$, $(j, \underline{y}')$ respectively. Here i, j take values in the discrete part of the state space, $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b}$, while $\underline{y}, \underline{y}'$ take values in $\mathcal{R}^{N_f}$.

- For any vector, $\underline{x}$, we use x_l to represent its l^{th} component and $\underline{x}_l^k$, $l \leq k$, represents the vector $\begin{bmatrix} x_l & x_{l+1} & \cdots & x_k \end{bmatrix}^T$.

- $Z\underline{\theta} = \{z\theta_l\}_{l=0}^{N_L-1}$ represents the convolution of the channel $\{z_l\}$and the forward filter $\underline{\theta}_f$.

- The input to the hard decoder for a given state of the Markov chain is represented by,

$$e_{\underline{\theta}}(i,\underline{y}) := \sum_{l=0}^{N_L-1} z\theta_l s_{k-l} + \sum_{l=0}^{N_f} \theta_{f_l} n_{k-l} + \sum_{l=1}^{N_b} \theta_{bl}\hat{s}_{k-l}.$$

Note that $\hat{s}_{k-1} = Q(e_{\underline{\theta}}(j,\underline{y}'))$.

- $B(\underline{\theta},\delta)$, $\bar{B}(\underline{\theta},\delta)$ are the open and closed balls respectively with center $\underline{\theta}$ and radius δ.
- Throughout the chapter, unless otherwise mentioned, integrability is always with respect to the measure $f_{\mathcal{N}}(\underline{y})d\underline{y}$.
- The equalizer output without noise, $e_{\underline{\theta}}(i,0) \neq 0$ for all values of i at the LMS attractor. Without this assumption the LMS algorithm makes more errors than the correct decisions.

One can easily extend the theory in this chapter to any finite alphabet input source with any arbitrary distribution as in the next chapter. Also, the theory to follow, considers an optimal equalizer for delay 0. However, the entire theory will go through for any arbitrary delay. Indeed in Section 6.6, an example with an optimal equalizer for delay 1, is presented.

6.2 The issues and our approach

A DFE-WF is given by,

$$\underline{\theta}^* = \arg\min_{\underline{\theta}} E\left[\underline{\theta}^t \underline{X}_k - s_k\right]^2 \tag{6.2}$$

where the expectation on the right hand side is under stationarity for a given $\underline{\theta}$ and is shown to exist in Theorem 6.1 below.

For a DFE, vector $\underline{X}_k = \left[U_k^T,\ \hat{S}_{k-1}^T\right]^T$, includes previous decisions $\hat{S}_{k-1}$ along with channel outputs U_k and hence its distribution depends upon the parameter $\underline{\theta}$. Thus, there is no known way to compute a Wiener filter. We address this problem directly via the iterative algorithm, LMS, given by the following iteration:

$$\underline{\theta}_{k+1} = \underline{\theta}_k - \mu_k \underline{X}_k \left(\underline{X}_k^t \underline{\theta}_k - s_k \right), \tag{6.3}$$

where $\{\mu_k\}$ is a decreasing sequence.

In practical systems, a DFE Wiener filter is commonly designed by assuming perfect decisions (i.e. $\hat{\underline{S}}_k = \underline{S}_{k-N_b+1}^k$), which we called IDFE. It is easy to see that the IDFE is given by,

$$\underline{\theta}_{IDFE} = \left(E\left[\underline{\mathcal{X}}_k \underline{\mathcal{X}}_k^t \right] \right)^{-1} E\left[\underline{\mathcal{X}}_k s_k \right],$$

where $\underline{\mathcal{X}}_k^t := \left[\begin{array}{cc} \underline{U}_k^t & {\underline{S}_{k-N_b+1}^{k-1}}^t \end{array} \right]$. This computation may be expensive because of matrix inversion and LMS (6.3) is actually used as an alternative. Our claim is that in case of a DFE, apart from being computationally efficient the LMS algorithm also outperforms the popularly used Wiener filter, IDFE, $\underline{\theta}_{IDFE}$ and can indeed be close to DFE-WF. We achieve this goal by showing that the LMS-DFE attractors are close to that of the DFE-WF at least for high SNRs (later in Section 6.6 we show via examples that this covers the practically used SNRs).

Towards this, we first show the existence of a unique stationary distribution for $\{\underline{X}_k\}$ for every $\underline{\theta}$ (one can now define the optimum in (6.2) under stationarity). Using this along with the ODE approximation in [4], we obtain a more tractable ODE,

$$\dot{\underline{\theta}}(t) = -\frac{1}{2} E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s \right]^2 \right] = -E_{\underline{\theta}} \left[\underline{X} \left(\underline{\theta}^t \underline{X} - s \right) \right]. \tag{6.4}$$

The attractors of the LMS-DFE will be the zeros of the RHS of the above ODE, while the DFE Wiener filter will be a zero of the gradient (if it exists) of the RHS of equation (6.2). Under certain conditions these two terms can be related by,

$$\nabla_{\underline{\theta}} E_{\underline{\theta}} \left[\left[\underline{\theta}^t \underline{X} - s \right]^2 \right] = E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s \right]^2 \right] + E\left[\left[\underline{\theta}^t \underline{X} - s \right]^2 \nabla_{\underline{\theta}} \pi_{\underline{\theta}} \right], \tag{6.5}$$

where $\pi_{\underline{\theta}}$ is the stationary density of $\{\underline{G}_k\}$ wrt the Lebesgue measure (we will show that it exists), when the DFE $\underline{\theta}$ is used. One can expect the LMS-DFE attractors to be close to the DFE Wiener filter, if the second term in the RHS of (6.5) is close to zero. We show in this chapter that, the above is indeed true at high SNRs, for a decoder that is slightly perturbed from the original one. The perturbation of the decoder is required because with the original decoder $\pi_{\underline{\theta}}$ may not be differentiable. We show that the DFE Wiener filter and an LMS-DFE attractor of this perturbed decoder converge to that of the original hard decoder as the level of perturbation tends to zero. Then we analyze this perturbed decoder and show that the LMS-DFE attractors of this decoder are close to the DFE Wiener filters of the same at high SNR. This suggests that at high SNR an LMS attractor for the original decoder is close to its DFE-WF.

6.3 ODE Approximation

We first show that the Markov chain $\{G_k\}$ has a stationary distribution for any given DFE, $\underline{\theta}$. This result was not known so far and is of obvious interest. Later we use this result to obtain an LMS-DFE ODE approximation with a more tractable ODE.

Theorem 6.1 *For every $\underline{\theta}$, Markov chain $\{G_k\}$ has a unique stationary distribution $\Pi_{\underline{\theta}}$. Starting from any initial conditions, the Markov chain converges geometrically to the stationary distribution. The continuous part of the stationary distribution has a density, $\pi_{\underline{\theta}}$ that is continuous with respect to $\underline{\theta}$ in L_1 norm. Also the MSE under stationarity is continuous in $\underline{\theta}$.*

Proof : Please refer to Appendix A.

Because of the continuity of the MSE as a function of $\underline{\theta}$, by confining our search to an approximately compact region, *we obtain the existence of the DFE Wiener filter.* Next we consider the LMS attractors. For this we first approximate the trajectory of the LMS-DFE with an ODE and then study the ODE's attractors.

DFE with a hard decoder has been approximated by an ODE in [4] (given by Theorem 13, p.278 . For convenience, the Theorem 13, p.278, [4] and its assumptions are reproduced in Appendix D of Chapter 5 as Theorem 5.3). We start our LMS-DFE analysis with this ODE. Towards this goal, as a first step the LMS-DFE algorithm (6.3) is rewritten to fit in the setup of Appendix D (i.e., as an example of system (5.5))

$$\begin{aligned}
\underline{\xi}_k &:= \left[\begin{array}{cc} \underline{S}_k^t & \underline{X}_k^t \end{array} \right]^t, \\
H(\underline{\theta}, \xi) &:= \underline{X}^t \left(\underline{\theta}^t \underline{X} - s \right), \\
\underline{\theta}_k &= \underline{\theta}_{k-1} - \mu_{k-1} H(\underline{\theta}_{k-1}, \xi_k).
\end{aligned}$$

Let $\underline{\theta}(t, t_0, a)$ denote the solution of the following ODE with initial condition $\underline{\theta}(t_0) = a$,

$$\begin{aligned}
\dot{\underline{\theta}}\,(t) &= -h(\underline{\theta}) \\
\text{where } h(\underline{\theta}) &= \lim_{n \to \infty} P_{\underline{\theta}}^n H_{\underline{\theta}}(j, \underline{y'}),
\end{aligned} \tag{6.6}$$

and $P_{\underline{\theta}}^n$ is the n-step transition function of the Markov chain $\{G_k\}$ with DFE $\underline{\theta}$, and $P_{\underline{\theta}}^n H_{\underline{\theta}}(j, \underline{y'})$ is the expectation of the function $H(\underline{\theta}, \xi)$ using the conditional measure $P_{\underline{\theta}}^n(.|j, \underline{y'})$ (note that ξ_k is a fixed function of G_k). The existence of the limit in the right hand side of (6.6) is established in the next para. We will show later on that, this limit will be independent of the initial condition $(j, \underline{y'})$ and will equal $E_{\underline{\theta}} H(\underline{\theta}, \xi)$.

We will now show that our algorithm satisfies the assumptions D.1-6 of Appendix D of Chapter 5. The LMS algorithm satisfies assumption D.1 as the matrix A of D.1 is a shift matrix now and hence for any $n > N_L$, $|A^n| = 0$. Assumption D.3 is satisfied for a BPSK source (for a fixed channel DFE, we obtain the ODE approximation using BPSK source. However we relax this condition in the next chapter and obtain an ODE approximation for tracking DFE for any finite alphabet source. The fixed channel DFE is a special case of the tracking DFE and hence can be approximated for any finite alphabet source) and the hard decoder Q with $s_1 = 0$, $\bar{\int} = 1$ and $\bar{s} = 0$ (p.277, [4]). Assumptions D.4 and D.5 are easily satisfied by the function H and the noise process $\{n_k\}$ respectively. As in previous chapters, one can see that the above ODE will have unique bounded solution for any finite time (this will be a special case of Lemma 7.1 of next chapter). Hence the LMS algorithm satisfies all the required hypothesis of Theorem 5.3 and we obtain :

- The existence of the limit in the right hand side of (6.6).
- For finite time T, with $t(r) := \sum_{k=0}^{r} \mu_k$, $m(n,T) := \max_{r \geq n}\{t(r) - t(n) \leq T\}$, and for any initial condition $\underline{\theta}_0$,

$$\sup_{n \leq r \leq m(n,T)} |\underline{\theta}_r - \underline{\theta}\,(t(r),\ t(n),\ \underline{\theta}_0)| \xrightarrow{p} 0 \text{ as } n \to \infty$$

whenever,

  - $\sum_k \mu_k = \infty$,
  - $\sum_k \mu_k^{1+\delta} < \infty$ for some $0 < \delta < 0.5$,
  - $0 < \mu_k \leq 1$ for all k and
  - $\liminf_k \frac{\mu_{k+r}}{k} > 0$ for every integer r.

We will show below that the RHS of the ODE (6.6) is same as that of the ODE (6.4) and hence equate the ODE (6.6) with a more tractable ODE (6.4). For each $\underline{\theta}$, $\{G_k\}$ is a Markov chain taking values in a general state space. Its transition function is given by,

$$P_{\underline{\theta}}(i, \underline{y} \in B | j, \underline{y'}) \;=\; \tilde{\delta}(i,j)\bar{\delta}(\underline{y}, \underline{y'})P(i_1)P(\underline{y}_1 \in B_{\underline{y'}})P_{\underline{\theta}}(i_{N_L+1}|j, \underline{y'}), \tag{6.7}$$

where $\bar{\delta}(\underline{y}, \underline{y'})$ equals 1 only when the vector formed from all but the last component of the vector $\underline{y'}$ equals the vector formed from all but the first component of the vector $\underline{y}$ and $\tilde{\delta}(i,j) = \bar{\delta}\left(i_1^{N_L}, j_1^{N_L}\right)\bar{\delta}\left(i_{N_L+1}^{N_b+N_L}, j_{N_L+1}^{N_b+N_L}\right)$ (note that the first component, $i_1^{N_L}$, represents the sample value of $\underline{S}_k$, while the second one, $i_{N_L+1}^{N_b+N_L}$, represents the sample value of $\underline{\hat{S}}_{k-1}$). Note here that, i_1 represents the current input s_k while, i_{N_L+1} represents the most recent decision $\hat{s}_{k-1}$. The only component of the transition function (6.7) that depends upon $\underline{\theta}$ is, $P_{\underline{\theta}}(i_{N_L+1}|j, \underline{y'})$ given by,

$$P_{\underline{\theta}}(i_{N_L+1} = 1 | j, \underline{y'}) = 1_{\left\{e_{\underline{\theta}}(j, \underline{y'}) > 0\right\}}.$$

By IID nature of the input s_k and noise n_k one can choose n_0 large enough such that the continuous part of the n step transition function $P^n_{\underline{\theta}}(i, \underline{y} \in B | j, \underline{y'})$ is absolutely continuous with respect to $f_{\mathcal{N}}(\underline{y})d\underline{y}$ for all $n \geq n_0$. Further, n_0 is chosen larger than N_L to ensure $\underline{S}_k$, $\underline{S}_{k-n_0}$ are independent. Fix one such n. The corresponding density (Radon-Nikodym derivative)

$$p^n_{\underline{\theta}}\left(i, \underline{y}|j, \underline{y}'\right) = \sum_l \int_{\underline{v}} P(\underline{S}^{k+n}_{k+1})\Pi^n_{q=1} P_{\underline{\theta}}(\hat{s}_{k+n-q}|\underline{x}(q)) f_{\mathcal{N}}(\underline{v})\ d\underline{v}, \tag{6.8}$$

where (the following notations are used throughout),

$$\begin{aligned} l &:= (\underline{S}^{k+n-N_L}_{k+1}, \underline{\hat{S}}^{k+n-1-N_b}_{k}), \\ \underline{v} &:= \underline{N}^{k+n-N_f}_{k+1}, \\ \underline{x}(q) &:= \left(\underline{S}^{k+n-q}_{k+n-q-N_L+1}, \underline{\hat{S}}^{k+n-q-1}_{k+n-q-N_b}, \sigma \underline{N}^{k+n-q}_{k+n-q-N_f+1}\right). \end{aligned}$$

It is easy to see that the density of the $n-$step transition function $p^n_{\underline{\theta}}(i, \underline{y}|j, \underline{y}') \leq 1$, for all values of $i, \underline{y}$, $j, \underline{y}'$. Also, we have by Theorem 6.1, $\left|p^n_{\underline{\theta}}(.|j, \underline{y}') - \pi_{\underline{\theta}}\right| \to 0$ for every value of $j, \underline{y}'$ as $n \to \infty$ in L_1 norm. The function $H()$ can be bounded uniformly by,

$$|H(\underline{\theta}, \xi_k)| \leq C_1 \left|\underline{X}_k\right|^2 + C_2 \left|\underline{X}_k\right|$$

for all $\underline{\theta}$ in a small neighborhood, for some appropriate constants C_1, C_2. The above bound is square integrable and depends only on $\{G_k\}$. Hence by Lemma 6.1 of Appendix E,

$$\begin{aligned} h(\underline{\theta}) &= \lim_{n\to\infty} P^n_{\underline{\theta}} H_{\underline{\theta}}(j, \underline{y}') = \Pi_{\underline{\theta}} H(\underline{\theta}, \xi), \\ &= \frac{1}{2} E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s\right]^2\right]. \end{aligned}$$

In the above, $\Pi_{\underline{\theta}} H(\underline{\theta}, \xi)$ represents the expectation of the function $H(\underline{\theta}, .)$ with respect to the stationary measure $\Pi_{\underline{\theta}}$. Thus the trajectory of the LMS-DFE can be approximated by the solution of the ODE (6.4), reproduced here for convenience,

$$\dot{\underline{\theta}}\ (t) = -\frac{1}{2} E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s\right]^2\right]$$

for any $t \leq T$, where T is any finite constant.

One expects that the attractors of the above ODE will be LMS-DFE attractors. The ODE attractors will be zeros of the RHS of the above equation. While the DFE Wiener filter will be the zero of the gradient (if it exists) of the MSE (the cost in the RHS of (6.2)). Under certain conditions (which will be discussed in Section 6.5) we get,

$$\nabla_{\underline{\theta}} E_{\underline{\theta}} \left[\left[\underline{\theta}^t \underline{X} - s \right]^2 \right] = E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s \right]^2 \right] + \sum_{\underline{S}, \hat{\underline{S}}} E_{f_{\mathcal{N}}} \left[\left[\underline{\theta}^t \underline{X} - s \right]^2 \nabla_{\underline{\theta}} \pi_{\underline{\theta}} \right],$$

where $\nabla_{\underline{\theta}} \pi_{\underline{\theta}}$ represents the gradient of the stationary density. One can easily see that an LMS-DFE attractor will be exactly (close to) the DFE Wiener filter if in addition the gradient of the stationary density equals zero (is close to zero). This shows that to get the connection between an LMS-DFE attractor and the DFE Wiener filter one needs to look at the differentiability of the stationary density.

6.4 Differentiability of the Stationary density.

One can see from equation (6.8) that it is very difficult to comment on differentiability of the n-step transition density itself. Thus, it is even more difficult to discuss the differentiability of the stationary density. To proceed further with the analysis, we perturb the hard decoder Q such that the n-step transition density and the stationary density become differentiable. Next we show that the LMS attractors and the DFE Wiener filter of this perturbed decoder converge to that of the original decoder as the level of perturbation tends to zero. Finally we study the DFE using these perturbed decoders in Section 6.5.

We alter the decoder function $Q(x)$ to,

$$Q_{\epsilon_0}(x) = \begin{cases} 1, & with \quad prob \quad 1, \quad x > \epsilon_0, \\ -1, & with \quad prob \quad 1, \quad x < -\epsilon_0, \\ 1, & with \quad prob \quad \frac{1}{2}\left[cos\left(\frac{(x-\epsilon_0)\pi}{2\epsilon_0} \right) + 1 \right], \quad |x| \leq \epsilon_0, \end{cases} \tag{6.9}$$

where ϵ_0 is a small constant. Observe that the perturbed decoder is also a hard decoder.

With the perturbed decoder $Q_{\epsilon_0}(x)$, the $\underline{\theta}$ dependent component of the transition function is given by,

$$P_{\underline{\theta}}^{\epsilon_0}(i_{N_L+1} = 1|j, \underline{y}') = \frac{1_{\{|e_{\underline{\theta}}(j,\underline{y}')| \leq \epsilon_0\}}}{2} \left[cos \left(\frac{(e_{\underline{\theta}}(j, \underline{y}') - \epsilon_0)\pi}{2\epsilon_0} \right) + 1 \right] + 1_{\{e_{\underline{\theta}}(j,\underline{y}') \geq \epsilon_0\}}.$$

The partial derivative, $\frac{\partial P_{\underline{\theta}}^{\epsilon_0}(i_{N_L+1}=1|j,\underline{y}')}{\partial \underline{\theta}}$ exists everywhere and equals,

$$\frac{-1_{\{|e_{\underline{\theta}}(j,\underline{y}')| \leq \epsilon_0\}}\pi}{4\epsilon_0} sin \left(\frac{(e_{\underline{\theta}}(j, \underline{y}') - \epsilon_0)\pi}{2\epsilon_0} \right) \frac{\partial e_{\underline{\theta}}(j, \underline{y}')}{\partial \underline{\theta}}. \tag{6.10}$$

By the uniform upper bound on the derivative (6.10) and by the bounded convergence theorem one can see that the n-step transition density (6.8) (with $n \geq n_0$) becomes differentiable (more details are in Appendix B) and equals (using the notations of equation (6.8)),

$$\begin{aligned} \frac{\partial p_{\underline{\theta}}^{\epsilon_0,n}}{\partial \underline{\theta}}(i, \underline{y}|j, \underline{y}') &= \sum_l \int_{\underline{v}} \sum_{m=1}^{n} \Pi_{\substack{q=1 \\ q \neq m}}^{n} P_{\underline{\theta}}^{\epsilon_0}(\hat{s}_{k+n-q}|\underline{x}(q)) \frac{\partial P_{\underline{\theta}}^{\epsilon_0}(\hat{s}_{k+n-m}|\underline{x}(m))}{\partial \underline{\theta}} \\ &\qquad P(\underline{S}_{k+1}^{k+n}) f_{\mathcal{N}}(\underline{v}) \ d\underline{v}. \end{aligned} \tag{6.11}$$

For these perturbed decoders, we show that the stationary density (with respect to N_f dimensional Gaussian IID vector) also becomes differentiable. Furthermore, using an Implicit function theorem, we get a bound on the norm of this gradient.

Theorem 6.2 *For every $\epsilon_0 > 0$, for every $\underline{\theta}_0$, the Markov chain $\{G_k\}$ has a unique stationary distribution, $\Pi_{\underline{\theta}}^{\epsilon_0}$. It's continuous part has a density, $\pi_{\underline{\theta}}^{\epsilon_0}$, that is continuously differentiable with respect to $\underline{\theta}$ in L_2 norm.*

Proof : Please refer to Appendix B.

In the proof of the above theorem, we also get the following upper bound. For every $\delta > 0$ and $\sigma_0^2 > 0$ there exists a constant $C < \infty$ such that for all $\underline{\theta} \in B(\underline{\theta}_0, \delta)$, $\sigma^2 \leq \sigma_0^2$,

$$\left| \nabla_{\underline{\theta}} \pi_{\underline{\theta}}^{\epsilon_0} \right|^2 \ \leq \ C \left(\sum_i P \left(\left| \underline{S}_k^t \underline{Z\theta} + \underline{\theta}_b^t \underline{\hat{S}}_{k-1} + \underline{\theta}_f^t N_k \right| \leq \epsilon_0 \right) + \sigma^2 \right). \tag{6.12}$$

We conclude this section by showing that the DFE Wiener filters and the LMS-DFE attractors of the perturbed decoder converge to that of the original decoder. Let $\underline{\theta}_n^*$ and $\underline{\theta}_n^{LMS}$ denote the DFE Wiener filter and an LMS-DFE attractor for perturbation ϵ_{0n}.

Theorem 6.3 *There exists a sequence $\epsilon_{0n} \to 0$, $\underline{\theta}_*$, a DFE Wiener filter of the original decoder, and $\underline{\theta}^{LMS}$ an LMS-DFE attractor of the original decoder, such that,*

$$\begin{aligned} \underline{\theta}_n^* &\to \underline{\theta}^*, \\ \underline{\theta}_n^{LMS} &\to \underline{\theta}^{LMS}, \end{aligned}$$

for any fixed noise variance, σ^2.

Proof : Please refer to Appendix C.

Thus we can always take the perturbation ϵ_0 in (6.9) small enough such that the LMS attractors and the DFE Wiener filters for the perturbed decoder are close enough to the corresponding equalizers for the original decoder. Henceforth, we analyze these perturbed decoders to draw important conclusions.

6.5 LMS attractors versus Wiener filter at high SNRs

In this section we would like to understand the connection between an LMS attractor and a DFE Wiener filter for a perturbed decoder. Since the former is a zero of the RHS of equation (6.4) and the later is the zero of the gradient of the MSE (the cost in the RHS of equation (6.2)), we study the connection between the two RHSs.

Fix an $\epsilon_0 > 0$. With the error defined by, $err_{\underline{\theta}}(G_k) := (s_k - e_{\underline{\theta}}(G_k))$ (note that i defined in the notations in Section 6.1 represents, $(\underline{S}_k, \hat{\underline{S}}_{k-1})$, the discrete part of the Markov chain, $\{G_k\}$),

$$\nabla_{\underline{\theta}} E_{G_k(\underline{\theta})}\left[err_{\underline{\theta}}(G_k)^2\right] \overset{a}{=} \sum_i \nabla_{\underline{\theta}} E_{f_{\mathcal{N}}}\left[err_{\underline{\theta}}(G_k)^2 \pi_{\underline{\theta}}^{\epsilon_0}(G_k)\right]$$

$$
\begin{aligned}
&\stackrel{b}{=} \sum_i E_{f_{\mathcal{N}}} \nabla_{\underline{\theta}} \left[err_{\underline{\theta}}(G_k)^2 \pi_{\underline{\theta}}^{\epsilon_0}(G_k) \right] \\
&= \sum_i E_{f_{\mathcal{N}}} \left[\nabla_{\underline{\theta}} \left(err_{\underline{\theta}}(G_k)^2 \right) \pi_{\underline{\theta}}^{\epsilon_0}(G_k) \right] + \sum_i E_{f_{\mathcal{N}}} \left[err_{\underline{\theta}}(G_k)^2 \nabla_{\underline{\theta}} \pi_{\underline{\theta}}^{\epsilon_0}(G_k) \right] \\
&= E_{G_k(\underline{\theta})} \left[\nabla_{\underline{\theta}} \left(err_{\underline{\theta}}(G_k)^2 \right) \right] + \sum_i E_{f_{\mathcal{N}}} \left[err_{\underline{\theta}}(G_k)^2 \nabla_{\underline{\theta}} \pi_{\underline{\theta}}^{\epsilon_0}(G_k) \right].
\end{aligned} \tag{6.13}
$$

Here equality a follows by the existence of the stationary density $\pi_{\underline{\theta}}^{\epsilon_0}$ with respect to the Gaussian measure $f_{\mathcal{N}}(\underline{y})d\underline{y}$. Equality b is given by Lemma 6.2 of Appendix E. The above equality (6.13) is true for any $\epsilon_0 > 0$ and for any σ^2.

We will show that the DFE Wiener filter will be close to the limiting LMS-DFE if the second term on the right hand side of (6.13) is small.

We have assumed that $\underline{S_k^t Z\theta} + \underline{\theta_b^t \hat{S}_{k-1}} \neq 0$ at an LMS attractor. By continuity, we choose an ϵ_1 small enough such that

$$0 \notin [\underline{S_k^t Z\theta} + \underline{\theta_b^t \hat{S}_{k-1}} - \epsilon_1, \underline{S_k^t Z\theta} + \underline{\theta_b^t \hat{S}_{k-1}} + \epsilon_1],$$

for all $(\underline{S}_k, \underline{\hat{S}}_{k-1})$ and for all $\underline{\theta}$ in a small neighborhood of an LMS attractor. Choose $\epsilon_0 \leq \epsilon_1$. By Chebyshev's inequality, if $0 \notin [c - \epsilon_0, c + \epsilon_0]$ and if n is a Gaussian random variable with mean zero and variance σ^2,

$$P(|c + n| \leq \epsilon_0) \leq P(|n| \geq \min\{|c - \epsilon_0|, |c + \epsilon_0|\}) \to 0$$

as $\sigma^2 \to 0$. Thus, from the upper bound 6.12 of Theorem 6.2, for any fixed $\epsilon_0 \leq \epsilon_1$,

$$|\nabla_{\underline{\theta}} \pi_{\underline{\theta}}| \to 0 \text{ as } \sigma^2 \to 0.$$

Thus by Cauchy Schwartz inequality as $\sigma^2 \to 0$,

$$
\begin{aligned}
&\left| \nabla_{\underline{\theta}} E_{\underline{\theta}} \left[err_{\underline{\theta}}(G_k)^2 \right] - E_{\underline{\theta}} \left[\nabla_{\underline{\theta}} \left(err_{\underline{\theta}}(G_k)^2 \right) \right] \right| \\
&\qquad = \left| \sum_i E_{f_{\mathcal{N}}} \left[err_{\underline{\theta}}(G_k)^2 \nabla_{\underline{\theta}} \pi_{\underline{\theta}}(G_k) \right] \right|
\end{aligned}
$$

$$\leq \quad \sum_i \left(E_{f_{\mathcal{N}}}\left[err_{\underline{\theta}}(G_k)^4\right]\right)^{1/2} |\nabla_{\underline{\theta}}\pi_{\underline{\theta}}| \quad \to 0.$$

Now we show that this implies that the LMS-DFE attractors will be close to the DFE Wiener filters. In general two functions f_1, f_2 can be close to each other at every point, but their zeros may be far apart, i.e., if x_1 is a zero of f_1 then $f_2(x_1)$ will be close to zero but the zero of f_2 closest to x_1 may still be far away. It is useful to rule out this possibility in our scenario. We show this using the following theorem. Define,

$$\begin{aligned} s(\underline{\theta}, \sigma^2) &:= E_{G_k(\underline{\theta})}\left[\nabla_{\underline{\theta}}\left(err_{\underline{\theta}}(G_k)^2\right)\right] \text{ and} \\ w(\underline{\theta}, \sigma^2, \eta) &:= s(\underline{\theta}, \sigma^2) + \eta. \end{aligned}$$

Theorem 6.4 *There exists an ϵ_2 with $0 < \epsilon_2 \leq \epsilon_1$ such that for any $\epsilon_0 \leq \epsilon_2$, there exists a continuous function*
$q : B(0, \delta) \subset \mathcal{R} \times \mathcal{R} \mapsto \mathcal{R}^{N_f}$, *such that,*

$$w\left(q\left(\sigma^2, \eta\right), \sigma^2, \eta\right) = 0.$$

Proof : Please refer to Appendix D.

For any fixed $\epsilon_0 \leq \epsilon_2$, the gradient of the stationary density, near an LMS attractor, is tending to zero as $\sigma^2 \to 0$. Thus there exists a small enough σ_0^2 such that for all $\sigma^2 \leq \sigma_0^2$,

$$\left|\left(\sigma^2, \;\; \nabla_{\underline{\theta}} E_{\underline{\theta}} err_{\underline{\theta}}(G_k)^2 - E_{\underline{\theta}} \nabla_{\underline{\theta}} err_{\underline{\theta}}(G_k)^2\right)\right| \;\; \leq \;\; \delta.$$

For these $\sigma^2 \leq \sigma_0^2$, the LMS attractors, $q(\sigma^2, 0)$, will be close to that of the Wiener filters, $q\left(\sigma^2, \;\; (\nabla_{\underline{\theta}} E_{\underline{\theta}} err_{\underline{\theta}}(G_k)^2 - E_{\underline{\theta}} \nabla_{\underline{\theta}} err_{\underline{\theta}}(G_k)^2)\right)$.

It is clear from the above Theorem that at high SNRs, for very small ϵ_0 (close to the practical decoder), the LMS attractor is close to the DFE Wiener filter. Since, IDFE $\underline{\theta}_{IDFE}$, is designed with an improper assumption (like perfect decisions), there is a good chance of these filters to be inefficient in comparison to the LMS attractors. We will see this in the examples of the next section.

We conclude this section by pointing out another useful consequence of the Theorem 6.4. This theorem also establishes the existence of the LMS attractors at high SNRs for perturbed decoders with perturbation level ϵ_0 small. A Remark at the end of Appendix D establishes this point.

6.6 Examples

In this section we reinforce the theory developed so far using some examples. We take a few examples of channels obtained from previous studies and show the proximity of the DFE Wiener filter and the LMS attractor for practical values of SNRs. We also show that in many cases, the IDFE performs much worse than the DFE Wiener filter and an LMS attractor close to the DFE-WF. Probability of Error (P_e) and the MSE are used to compare the various equalizers. We used Monte-Carlo simulations to estimate P_e using one million samples of data.

DFE Wiener filter, $\underline{\theta}^*$ for these examples was obtained by running a gradient descent type of algorithm on the cost function (6.2) itself, where the gradient was approximated at each point by finite difference approximation,

$$\frac{1}{|\Delta|}\left[E_{\underline{\theta}+\Delta}\left(X(\underline{\theta}+\Delta)^t(\underline{\theta}+\Delta)-s\right)^2 - E_{\underline{\theta}}\left(X(\underline{\theta})^t(\underline{\theta})-s\right)^2\right].$$

Here the expectation $E_{\underline{\theta}}\left(X(\underline{\theta})^t(\underline{\theta})-s\right)^2$ is obtained approximately by the sample path averages,

$$\frac{1}{N}\sum_{i=1}^{N}\left(X_i^t(\underline{\theta})\underline{\theta}-s_i\right)^2$$

using a large number of samples, N. Vector sequence $\{X_i(\underline{\theta})\}_{i=1}^N$ is obtained by running the DFE with fixed coefficients $\underline{\theta}$. Thus $\underline{\theta}^*$ is estimated by the following algorithm,

$$\underline{\theta}_{k+1} = \underline{\theta}_k - \frac{\mu_k}{N\,|\Delta_k|}\left[\sum_{i=1}^{N}\left(X_{k,i}^t(\underline{\theta}_k)\underline{\theta}_k - s_{k,i}\right)^2 - \sum_{i=1}^{N}\left(X_{k,i}^t(\underline{\theta}_k+\Delta_k)\left(\underline{\theta}_k+\Delta_k\right) - s_{k,i}\right)^2\right].$$

Here $s_{k,i}$ are IID with the distribution of the inputs, s_k. Vector sequence $\{X_{k,i}(\underline{\theta})\}_{i=1}^N$ is obtained by running the DFE with fixed coefficients $\underline{\theta}$. Sequences $\{\Delta_k\}$, $\{\mu_k\}$ are chosen appropriately to reduce to zero. In our simulations we used $\mu_k = \frac{0.07}{k^{0.6}}$, $\Delta_k = 5\mu_k$. We obtained the estimates with $N = 4 \times 10^5$ samples.

In Table 6.1, we have used an interesting example (significant part of the raised cosine channel of [21], p. 199) to show that the LMS attractors will be very close to the Wiener filters at practical SNRs. Its coefficients are provided in the table. We also provide P_e in this table. One can see an improvement up to 25% in P_e in LMS in comparison with the IDFE. In fact this improvement is more at high SNRs (where the IDFE is assumed to have lesser problem because of error propagation). We can also see that the P_e of the DFE Wiener filter is very close to that of the LMS attractor.

We have developed the entire theory for an equalizer with delay zero. One can easily extend these results to the equalizer with any arbitrary delay. In fact, the channel in Table 6.2 is one such example. Here the equalizer with delay 1 will be the best one. The channel of Table 6.2 is very widely used (see p. 414 of [25], p. 165 of [21]). We can see once again a huge improvement (up to 50%) in P_e of LMS with respect to $\underline{\theta}_{IDFE}$. We can also see that the LMS attractors are very close to the DFE Wiener filter, $\underline{\theta}^*$ for all practical SNRs.

6.7 Conclusions

Obtaining MSE optimal filter for DFE is a long-standing problem. Precoding provides one practical solution but may not be feasible with wireless channels. Otherwise one commonly uses the optimal Wiener filter obtained assuming perfect past decisions.

In this chapter we show via ODE analysis, that LMS itself can provide the optimal

Table 6.1: Comparison of DFEs for raised cosine channel with $N_f = 5$, $N_b = 10$ and Channel = $\left[\begin{array}{cccccccccc} 0.45 & 0.59 & 0.43 & 0.11 & -0.22 & -0.32 & -0.27 & 0 & 0.11 & 0.11 \end{array}\right]$

SNR	$\underline{\theta}^*$		$\underline{\theta}_{IDFE}$			$\underline{\theta}_{LMS}$		
	MSE	P_e	Dist from $\underline{\theta}^*$	MSE	P_e	Dist from $\underline{\theta}^*$	MSE	P_e
16.7	.21	.024	1.1	.26	.027	.035	.21	.024
14.5	.38	.089	1.7	.64	.10	.037	.38	.091
12.5	.49	.150	1.6	.94	.184	.027	.49	.151
11.5	.54	.176	1.5	1.0	.215	.023	.54	.177
4.5	.81	.311	.64	.94	.33	.021	.80	.311
1.5	.87	.353	.37	.93	.364	.023	.87	.353

Table 6.2: Comparison of DFEs with $N_f = N_b = 2$, Channel = $[\,0.41\ \ .82\ \ 0.41]$

SNR	$\underline{\theta}^*$		$\underline{\theta}_{IDFE}$			$\underline{\theta}_{LMS}$		
	MSE	P_e	Dist from $\underline{\theta}^*$	MSE	P_e	Dist from $\underline{\theta}^*$	MSE	P_e
16.7	.11	.0027	.13	.12	.0035	.014	.11	.0028
14.5	.16	.01	.26	.18	.015	.021	.16	.011
12.5	.23	.03	.35	.26	.037	.025	.23	.032
11.5	.27	.047	.40	.30	.055	.031	.28	.050
4.5	.54	.184	.43	.59	.2	.009	.54	.184
1.5	.65	.235	.32	.69	.25	.008	.65	.235

Wiener filter. We show it by proving that the attractors of the LMS are close to that of the optimal DFE at high SNRs. Proofs become nontrivial partly because of the non-differentiability of the hard decoder. Next, we show by examples that the SNRs need not be very high, i.e., in fact practically used SNRs can be sufficient. We also show that the BER of the commonly used Wiener filter, designed assuming perfect past decisions (also designed using perfect channel estimate), can be up to 50% higher than the optimal Wiener filter even at high SNRs (where the former is believed to be closer to the later).

Of course, one advantage of LMS over Precoding is that it can be used at the receiver and has tracking capability when the channel is time varying. In the next chapter we will study the tracking performance of LMS-DFE.

Appendix A

Proof of Theorem 6.1 :

We prove the existence and continuity of the stationary distribution of the Markov chain, $\{G_k\}$, using the results on Markov chains and Stochastic stability ([39]).

For any $\underline{\theta}_0$ and for any $\epsilon > 0$,

$$M_1 \stackrel{\triangle}{=} \min_{\underline{S}_k, \hat{\underline{S}}_{k-1}, \underline{\theta} \in \bar{B}(\underline{\theta}_0, \epsilon)} \underline{Z\theta^t} \underline{S}_k + \underline{\theta}_b^t \hat{\underline{S}}_{k-1}. \tag{6.14}$$

Continuity of the map considered in the bound and compactness of the closed ball $\bar{B}(\underline{\theta}_0, \epsilon)$ ensures $|M_1| < \infty$.

By continuity of the map $(\underline{\theta}, \underline{N}_k) \mapsto \underline{\theta}_f{}^t \underline{N}_k$, its inverse image of the open set, $\{x > -M_n\}$, is open and hence it is possible to get a open set C and a $\delta \leq \epsilon$ such that,

$$\left\{(\underline{N}_k, \underline{\theta}) : \underline{\theta}_f^t \underline{N}_k > -M_1\right\} \supset C \times \bar{B}(\underline{\theta}_0, \delta). \tag{6.15}$$

Thus decoder (6.1) outputs 1 (irrespective of the inputs/past decisions) once the noise vector is in C whenever $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \epsilon)$. Hence,

$$\begin{aligned} P(\hat{\underline{S}}_k = [\ 1 \quad .. \quad 1\ ]) &= P(\cap_{l=k-N_b-1}^{k} \{\hat{s}_l = 1\}) \\ &\geq P\left(\cap_{l=k-N_b-1}^{k} \{\underline{N}_l \in C\}\right) \\ &\geq P(\underline{N}_{k-N_b-N_f+1}^{k} \in C_1 \times C_2), \end{aligned}$$

where sets $C_1 \in \mathcal{R}^{N_b}$, $C_2 \in \mathcal{R}^{N_f}$ are selected such that their respective Lebesgue measures are not equal to zero and such that the open set,

$$\cap_{l=k-N_b-1}^{k} \{\underline{N}_l \in C\} \supset C_1 \times C_2.$$

Define $\mathcal{G} := [\ 1 \quad .. \quad 1\ ] \times [\ 1 \quad .. \quad 1\ ] \times \mathcal{R}^{N_f}$. For any $n_0 > max\{N_b + N_f + 1, N_L\}$, for any initial condition G_{k-n_0}, for any measurable set B_N, and for any $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \delta)$,

$$\begin{aligned} P(G_k \in \{[\ 1 \ \ .. \ \ 1\ ] \times [\ 1 \ \ .. \ \ 1\ ] \times B_N\} | G_{k-n_0}) & \\ &= P(\underline{S}_k = [\ 1 \ \ .. \ \ 1\ ], \underline{\hat{S}}_{k-1} = [\ 1 \ \ .. \ \ 1\ ], \underline{N}_k \in B_N | G_{k-n_0}) \\ &\geq P(\underline{S}_k = [\ 1 \ \ .. \ \ 1\ ], \underline{N}^k_{k-N_b-N_f+1} \in C_1 \times C_2, \underline{N}_k \in B_N) \\ &\geq \alpha P(\underline{N}_k \in B_N \cap C_2) \end{aligned}$$

where $\alpha := P(\underline{S}_k = [\ 1 \ \ .. \ \ 1\ ])P(\underline{N}^{k-N_f}_{k-N_b+N_f} \in C_1)$. Thus for any $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \delta)$, and for any initial condition G_{k-n_0}, the n_0-step conditional measure is majorized :

$$P_{\underline{\theta}}(G_k \in E | G_{k-n_0}) \ \ \geq \ \ \nu_{n_0}(E \cap \mathcal{G}),$$

where the measure $\nu_{n_0}()$ is defined by,

$$\nu_{n_0}([\ 1 \ \ .. \ \ 1\ ] \times [\ 1 \ \ .. \ \ 1\ ] \times B_E) \ \ := \ \ \alpha P(\underline{N}_k \in B_E \cap C_2).$$

Thus the entire state space $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b} \times \mathcal{R}^{N_f}$ is $\nu_{n_0}-$small (hence also a petite set) for all the Markov chains, $\{G_k\}$, parameterized by $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \delta)$. Thus using Proposition 9.1.7, p. 206 and Theorem 10.01, p. 230, [39] one obtains the existence and uniqueness of the probability stationary distribution, $\Pi_{\underline{\theta}}$ for each $\underline{\theta}$.

Define $\rho = 1 - \nu_{n_0}(\mathcal{G})$. Further, by Theorem 16.2.4 in page 392 of [39], for all $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \delta)$, we get the following uniform convergence in total variation norm,

$$\left| P^n_{\underline{\theta}}(.|j, \underline{y}') - \Pi_{\underline{\theta}} \right| \leq \rho^{\frac{n}{n_0}} \text{ for all initial conditions } (j, \underline{y}').$$

This along with the continuity of the transition function, establishes the continuity of the stationary distribution $\Pi_{\underline{\theta}}$ under total variation norm at $\underline{\theta}_0$. This is because, for any $\underline{\theta} \in \bar{B}(\underline{\theta}_0, \delta)$

$$\lim_{\underline{\theta} \to \underline{\theta}_0} \left| \Pi_{\underline{\theta}} - \Pi_{\underline{\theta}_0} \right| \ \ \leq \ \ \lim_{\underline{\theta} \to \underline{\theta}_0} \left[\left| \Pi_{\underline{\theta}} - P^n_{\underline{\theta}}(.|j, \underline{y}') \right| + \left| \Pi_{\underline{\theta}_0} - P^n_{\underline{\theta}_0}(.|j, \underline{y}') \right| + \left| P^n_{\underline{\theta}_0}(.|j, \underline{y}') - P^n_{\underline{\theta}}(.|j, \underline{y}') \right| \right]$$

$$\begin{aligned} &\leq \quad 2\rho^{\frac{n}{n_0}} + \lim_{\underline{\theta}\to\underline{\theta}_0} \left| P^n_{\underline{\theta}_0}(x,.) - P^n_{\underline{\theta}}(x,.) \right| \\ &\stackrel{a}{=} \quad 2\rho^{\frac{n}{n_0}} \qquad \text{for all } n \geq 1. \end{aligned}$$

Equality a follows by continuity of the transition function with respect to $\underline{\theta}$. By letting $n \to \infty$ we get,

$$\lim_{\underline{\theta}\to\underline{\theta}_0} \left| \Pi_{\underline{\theta}} - \Pi_{\underline{\theta}_0} \right| = 0.$$

The stationary distribution, $\Pi_{\underline{\theta}}$, has discrete and continuous components. The continuous component of $\Pi_{\underline{\theta}}$, is absolutely continuous with respect to the measure $f_{\mathcal{N}}(\underline{y})d\underline{y}$ for every $\underline{\theta}$. Hence the stationary density, $\pi_{\underline{\theta}}$ for $\{G_k\}$ exists. Continuity in total variation norm of the stationary distribution implies the continuity of the stationary densities in L_1 norm (Theorem 8.2, p. 110, [55]). It is also easy to see that the stationary density $\pi_{\underline{\theta}}(i,\underline{y}) \leq 1$ for all $(i,\underline{y})$.

MSE, the cost in RHS of equation (6.2), can be rewritten as,

$$E_{\underline{\theta}} \left[\underline{\theta}^t \underline{X} - s \right]^2 = \sum_{\underline{S},\hat{\underline{S}}} E_{f_{\mathcal{N}}} \left[\left(\underline{\theta}^t \underline{X} - s \right)^2 \pi_{\underline{\theta}} \right].$$

Lemma 6.1 in Appendix E, now gives the continuity of the MSE with respect to $\underline{\theta}$. ■

Appendix B

Proof of Theorem 6.2 :

The existence and continuity of the stationary density $\pi^{\epsilon_0}_{\underline{\theta}}$ for every ϵ_0 is achieved in a way similar to the proof of the Theorem 6.1. The only difference being, ϵ_0 must be added to $-M_1$ in the definition of the set (6.15). We leave superscript ϵ_0 to simplify the notation in the rest of the proof.

We use an implicit function theorem to prove differentiability. For that, we will need to prove the following results. Consider the Banach spaces :

- $X = \mathcal{R}^{N_f+N_b}$ with Euclidean norm.

- $Y = \{g : \mathcal{S}^{N_L+N_b} \times X \to \mathcal{R}; |g| < \infty\}$ with L_2 norm, $|.|$, defined by,

$$|g| := \frac{1}{|\mathcal{S}|} \sum_i \left(\int_{\underline{y}} |g(i, \underline{y})|^2 f_{\mathcal{N}}(\underline{y}) d\underline{y} \right)^{1/2},$$

where $|\mathcal{S}|$ represents the cardinality of set $\mathcal{S}^{N_L+N_b}$.

Fix $n_0 > max\{N_f + N_b, N_L\}$. We consider the following continuous map $f : X \times Y \mapsto Y$,

$$f(\underline{\theta}, \pi) = g(\underline{\theta}, \pi) - \pi + \left(\sum_j \int_{\underline{y}'} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' - 1 \right),$$

where,

$$g(\underline{\theta}, \pi)(i, \underline{y}) \quad := \quad \sum_j \int_{\underline{y}'} p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y}') \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}'.$$

Observe that $(\underline{\theta}, \pi_{\underline{\theta}})$ is a zero of f.

By Lemma B.1, Lemma B.2 the function f is differentiable with respect to π and $\underline{\theta}$ respectively and further the derivative $\frac{\partial f}{\partial \pi}$ is a homeomorphism. Also, $\left| \left(\frac{\partial f}{\partial \pi} \right)^{-1} \right|$ and $\left| \frac{\partial f}{\partial \underline{\theta}} \right|$ are upper bounded locally by the RHS of (6.16) and (6.19) respectively.

Using similar logic one can easily show that both the partial derivatives of f are continuous in $(\underline{\theta}, \pi)$. Hence by Implicit function theorem on Banach spaces, (Theorem 3.1.10 and corollary 3.1.11, p. 115, [5]), the map $\underline{\theta} \mapsto \pi_{\underline{\theta}}$ is continuously differentiable and the derivative is given by,

$$\nabla_{\underline{\theta}} \pi_{\underline{\theta}} = - \left[\left. \frac{\partial f}{\partial \pi} \right|_{(\underline{\theta}, \pi_{\underline{\theta}})} \right]^{-1} \left. \frac{\partial f}{\partial \underline{\theta}} \right|_{(\underline{\theta}, \pi_{\underline{\theta}})}.$$

Upper bound 6.12 is obtained by bounding the above gradient using the upper bounds (6.16), (6.19). ■

Lemma B.1 *f is differentiable with respect to π and the derivative is a homeomorphism.*

Also for any $\delta > 0, \sigma_0^2 > 0$ there exists a constant $C_0 < \infty$ such that,

$$\left| \left[\left. \frac{\partial f}{\partial \pi} \right|_{(\underline{\theta}, \pi_{\underline{\theta}})} \right]^{-1} \right| \leq C_0 \tag{6.16}$$

for all $\underline{\theta} \in B(\underline{\theta}_0, \delta)$, $\sigma^2 \leq \sigma_0^2$.

Proof : The function f is affine linear in the second variable $\pi \in Y$. Thus,

$$\left. \frac{\partial f}{\partial \pi} \right|_{(\underline{\theta}, \hat{\pi})} (\pi) = g(\underline{\theta}, \pi) - \pi + \left(\sum_j \int_{\underline{y}'} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' \right). \tag{6.17}$$

We will show below that this map is one-one through contradiction. It is easy to see that $g(\underline{\theta}, \pi) - \pi$ is in the set,

$$\mathcal{H} := \left\{ \pi : \sum_j \int_{\underline{y}'} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' = 0 \right\} \subset Y.$$

Operator $\pi \mapsto \left(\sum_j \int_{\underline{y}'} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' \right)$ has one-dimensional range which lies inside $\mathcal{H}^c$. We can show that the partial derivative (6.17) is one-one, if we show that there is no common non-zero vector in the null space of both the operators. Say there exists a vector $\pi \neq 0$ in the null space of both the operators. Let,

$$\begin{aligned} D &:= \left\{ (i, \underline{y}) : \pi(i, \underline{y}) \geq 0 \right\}, \\ \alpha &:= \sum_j \int_{\{\underline{y}' : (j, \underline{y}') \in D\}} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}', \\ |\pi|_1 &:= \sum_j \int_{\underline{y}'} \left| \pi(j, \underline{y}') \right| f_{\mathcal{N}}(\underline{y}') d\underline{y}'. \end{aligned}$$

As π is in the null space of the operator, $\pi \mapsto \sum_j \int_{\underline{y}'} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}'$,

$$\sum_j \int_{\{\underline{y}' : (j, \underline{y}') \in D^c\}} \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' = -\alpha.$$

Hence $|\pi|_1 = 2\alpha$. Also, because $g(\underline{\theta}, \pi) = \pi$,

$$g(\underline{\theta}, \pi)(i, \underline{y}) \geq 0 \quad \text{for all } i, \underline{y} \in D \quad \text{and}$$
$$g(\underline{\theta}, \pi)(i, \underline{y}) < 0 \quad \text{for all } i, \underline{y} \in D^c.$$

Then,

$$\begin{aligned}
&|g(\underline{\theta}, \pi)|_1 \\
&= \sum_i \int_{\underline{y}:(i,\underline{y}) \in D} g(\underline{\theta}, \pi)(i, \underline{y}) f_{\mathcal{N}}(\underline{y}) d\underline{y} - \sum_i \int_{\underline{y}:(i,\underline{y}) \in D^c} g(\underline{\theta}, \pi)(i, \underline{y}) f_{\mathcal{N}}(\underline{y}) d\underline{y} \\
&= \sum_i \int_{\underline{y}:(i,\underline{y}) \in D} \sum_j \int_{\underline{y'}} p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y'}) \pi(j, \underline{y'}) f_N(\underline{y'}) d\underline{y'} f_{\mathcal{N}}(\underline{y}) d\underline{y} \\
&\quad - \sum_i \int_{\underline{y}:(i,\underline{y}) \in D^c} \sum_j \int_{\underline{y'}} p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y'}) \pi(j, \underline{y'}) f_N(\underline{y'}) d\underline{y'} f_N(\underline{y}) d\underline{y} \\
&\overset{Fubini}{=} \sum_j \int_{\underline{y'}} \left(\sum_i \int_{\underline{y}:(i,\underline{y}) \in D} p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y'}) f_{\mathcal{N}}(\underline{y}) d\underline{y} \right) \pi(j, \underline{y'}) f_{\mathcal{N}}(\underline{y'}) d\underline{y'} \\
&\quad - \sum_j \int_{\underline{y'}} \left(\sum_i \int_{\underline{y}:(i,\underline{y}) \in D^c} p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y'}) f_{\mathcal{N}}(\underline{y}) d\underline{y} \right) \pi(j, \underline{y'}) f_{\mathcal{N}}(\underline{y'}) d\underline{y'} \\
&= \sum_j \int_{\underline{y'}} P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D | j, \underline{y'}) \;\; \pi(j, \underline{y'}) f_{\mathcal{N}}(\underline{y'}) d\underline{y'} \\
&\quad - \sum_j \int_{\underline{y'}} P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D^c | j, \underline{y'}) \;\; \pi(j, \underline{y'}) f_{\mathcal{N}}(\underline{y'}) d\underline{y'} \\
&= \sum_j \int_{\underline{y'}} \left(P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D | j, \underline{y'}) - P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D^c | j, \underline{y'}) \right) \pi(j, \underline{y'}) f_{\mathcal{N}}(\underline{y'}) d\underline{y'}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \int_{\underline{y}':j,\underline{y}' \in D} \left(1 - 2P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D^c | j, \underline{y}')\right) \left|\pi(j, \underline{y}')\right| f_{\mathcal{N}}(\underline{y}') d\underline{y}' \\
&\quad + \sum_j \int_{\underline{y}':j,\underline{y}' \in D^c} \left(1 - 2P_{\underline{\theta}}^{n_0}(i, \underline{y} \in D | j, \underline{y}')\right) \left|\pi(j, \underline{y}')\right| f_{\mathcal{N}}(\underline{y}') d\underline{y}' \\
&\leq \sum_j \int_{\underline{y}':j,\underline{y}' \in D} (1 - 2\nu_{n_0}(D^c)) \left|\pi(j, \underline{y}')\right| f_{\mathcal{N}}(\underline{y}') d\underline{y}' \\
&\quad + \sum_j \int_{\underline{y}':j,\underline{y}' \in D^c} (1 - 2\nu_{n_0}(D)) \left|\pi(j, \underline{y}')\right| f_{\mathcal{N}}(\underline{y}') d\underline{y}' \\
&= \frac{|\pi|_1}{2} (2 - 2\nu_{n_0}(D^c) - 2\nu_{n_0}(D)) \\
&= |\pi|_1 (1 - \nu_{n_0}(Y)) < |\pi|_1 .
\end{aligned}
$$

This provides a contradiction as $0 < \nu_{n_0}(Y) < 1$ and hence $|\pi|_1 = |g(\underline{\theta}, \pi)|_1 < |\pi|_1$. This proves that the partial derivative (6.17) is one-one. The inequality is obtained by using the majorizing measure, $\nu_{n_0}(.)$, defined in the proof of continuity of stationary distribution.

The map $g(\underline{\theta}, \pi)$ is compact integral operator (example 2, p. 277, [64]). The last map of the partial derivative has one-dimensional range and hence is compact. Therefore, the partial derivative equals $T - I$, where T is a compact operator. Then by Riesz-Schauder Theory (Theorem 1, p. 283, [64]), the fact that $\frac{\partial f}{\partial \pi}$ is one-one implies that it is onto and also further that the inverse is bounded. Hence $\frac{\partial f}{\partial \pi}$ is a linear homeomorphism.

Furthermore, the mapping $(\sigma^2, \underline{\theta}) \mapsto \left| \left[\frac{\partial f}{\partial \pi}\Big|_{(\underline{\theta}, \pi_{\underline{\theta}})} \right]^{-1} \right|$ is continuous. This continuity follows by the joint continuity of the n_0-step transition function, $p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y}')$ with respect to $(\sigma^2, \underline{\theta})$ and then by bounded convergence theorem (as $p_{\underline{\theta}}^{n_0}(i, \underline{y} | j, \underline{y}') + 1$ is uniformly bounded) and finally by the continuity of the map $x \mapsto x^{-1}$ (p. 135, [52]). Hence the lemma follows for some $C_0 < \infty$, $\delta > 0, \sigma_0^2 > 0$. ■

Lemma B.2 *f is differentiable with respect to $\underline{\theta}$. The partial derivative $\left.\frac{\partial f}{\partial \underline{\theta}}\right|_{(\underline{\theta},\pi_{\underline{\theta}})}$ is upper bounded by bound (6.19).*

Proof : We reintroduce the notations that will be used here (notation of equation (6.8)).

- $i = \left[\underline{S}_{k+n_0-(N_f+L-2)}^{k+n_0}, \underline{\hat{S}}_{k+n_0-1-(N_b-1)}^{k+n_0-1}\right]$, $\underline{y} = \underline{N}_{k+n_0-(N_f-1)}^{k+n_0}$, represent the current state of the Markov Chain, at $k+n_0$.
- $j = \left[\underline{S}_{k-(N_f+L-2)}^{k}, \underline{\hat{S}}_{k-(N_b-1)}^{k-1}\right]$, $\underline{y}' = \underline{N}_{k-(N_f-1)}^{k}$ represent the initial condition for n_0-step transition function, which is the state of the Markov chain at k.
- $l = \left[\underline{S}_{k+1}^{k+n_0-(N_f+L-3)}, \underline{\hat{S}}_{k}^{k+n_0-1-(N_b-2)}\right]$, $\underline{v} = \underline{N}_{k+1}^{k+n_0-(N_f-2)}$ represent the intermediate input, decision and noise vectors.
- $\underline{x}(q) := \left[\underline{S}_{k+n_0-q-(N_f+L-1)}^{k+n_0-q}, \underline{\hat{S}}_{k+n_0-q-N_b}^{k+n_0-q-1}, \underline{N}_{k+n_0-q-N_f}^{k+n_0-q}\right]$ represent the intermediate state of the Markov chain at $k+n_0-q$.

To begin with, we will show component wise differentiability of the function f, i.e., differentiability of $f(\underline{\theta},\pi)(i,\underline{y})$ for every $(i,\underline{y})$. We will show the differentiability of the n_0-step transition function, $p_{\underline{\theta}}^{n_0}\left(i,\underline{y}|j,\underline{y}'\right)$ along with that. Positive and finite constants are introduced in the derivations as and when required. While obtaining upper bounds we have taken advantage of the finite alphabet nature of the set $\mathcal{S}$. By simple computations, one can see that the density with respect to the Gaussian measure is,

$$p_{\underline{\theta}}^{n_0}\left(i,\underline{y}|j,\underline{y}'\right) = \sum_l \int_{\underline{v}} \Pi_{q=1}^{n_0} P_{\underline{\theta}}(\hat{s}_{k+n_0-q}|\underline{x}(q)) \;\; P(\underline{S}_{k+1}^{k+n_0}) \;\; f_{\mathcal{N}}(\underline{v}) \;\; d\underline{v}. \tag{6.18}$$

Hence,

$$\begin{aligned} f(\underline{\theta},\pi)(i,\underline{y}) = \; & \textstyle\sum_{l,j} \int_{\underline{v},\underline{y}'} \Pi_{q=1}^{n_0} P_{\underline{\theta}}(\hat{s}_{k+n_0-q}|\underline{x}(q)) P(\underline{S}_{k+1}^{k+n_0}) \pi(j,\underline{y}') f_{\mathcal{N}}(\underline{v}) f_{\mathcal{N}}(\underline{y}') d\underline{v} d\underline{y} \\ & -\pi(i,\underline{y}) + \left(\textstyle\sum_j \int_{\underline{y}'} \pi(j,\underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}' - 1\right). \end{aligned}$$

The only component of the above functions which depend upon $\underline{\theta}$ is $P_{\underline{\theta}}(\hat{s}_{k+n_0-q}|j,\underline{y}')$.

By (6.10),

$$\left| \frac{\partial P_{\underline{\theta}}}{\partial \underline{\theta}} \left(\hat{s}_{k+n_0-q} | \underline{x}(q) \right) \right| \leq c \left| \underline{N}_{k+n_0-q-N_f}^{k+n_0-q} \right|,$$

uniformly in $(i, \underline{y})$, for every $\underline{\theta}$, $(j, \underline{y}')$ and for every q. Thus for any $\underline{\theta}_h$ in a small neighborhood of 0 and for any $i, \underline{y}$, $j, \underline{y}'$, $\underline{\theta}$ and q (by mean value theorem, Theorem X.4.5, p.312, [7]),

$$\begin{aligned}
&\left| P_{\underline{\theta}}(\hat{s}_{k+n_0-q} | \underline{x}(q)) - P_{\underline{\theta}+\underline{\theta}_h}(\hat{s}_{k+n_0-q} | \underline{x}(q)) - \underline{\theta}_h^t \frac{\partial P_{\underline{\theta}}}{\partial \underline{\theta}} (\hat{s}_{k+n_0-q} | \underline{x}(q)) \right| \\
&\leq \left| P_{\underline{\theta}}(\hat{s}_{k+n_0-q} | j, \underline{y}') - P_{\underline{\theta}+\underline{\theta}_h}(\hat{s}_{k+n_0-q} | j, \underline{y}') \right| + \left| \underline{\theta}_h^t \frac{\partial P_{\underline{\theta}}}{\partial \underline{\theta}} \left(\hat{s}_{k+n_0-q} | j, \underline{y}' \right) \right| \\
&\leq 2c \left| \underline{\theta}_h \right| \left| \underline{N}_{k+n_0-q-N_f}^{k+n_0-q} \right|.
\end{aligned}$$

Finally by dominated convergence theorem, we obtain the existence of the following partial derivatives,

$$\begin{aligned}
\frac{\partial p_{\underline{\theta}}^{n_0}}{\partial \underline{\theta}} \left(i, \underline{y} | j, \underline{y}' \right) &= \sum_l \int_{\underline{v}} \frac{\partial \left(\Pi_{q=1}^{n} P_{\underline{\theta}}(\hat{s}_{k+n_0-q} | \underline{x}(q)) \right)}{\partial \underline{\theta}} \; P(\underline{S}_{k+1}^{k+n_0}) \;\; f_{\mathcal{N}}(\underline{v}) \;\; d\underline{v}, \\
\left. \frac{\partial f}{\partial \theta} \right|_{\theta,\pi} (i, \underline{y}) &= \sum_j \int_{\underline{y}'} \frac{\partial p_{\underline{\theta}}^{n_0}}{\partial \underline{\theta}} \left(i, \underline{y} | j, \underline{y}' \right) \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{y}') d\underline{y}'.
\end{aligned}$$

For obtaining the second partial derivative, the function inside each of the integral (one each for different values of i, j, l, r),

$$\int_{\underline{v},\underline{y}'} \left(P_{\underline{\theta}}(\hat{s}_{k+n_0-r} | \underline{x}(r)) - P_{\underline{\theta}+\underline{\theta}_h}(\hat{s}_{k+n_0-r} | \underline{x}(r)) - \underline{\theta}_h^t \frac{\partial P_{\underline{\theta}}}{\partial \underline{\theta}} (\hat{s}_{k+n_0-r} | \underline{x}(r)) \right)$$
$$\Pi_{q=1,q\neq r}^{n_0} P_{\underline{\theta}}(\hat{s}_{k+n_0-q} | \underline{x}(q)) \pi(j, \underline{y}') f_{\mathcal{N}}(\underline{v}) f_{\mathcal{N}}(\underline{y}') d\underline{v} d\underline{y}$$

(for which the limit $\lim_{|\underline{\theta}_h| \to 0}$ is taken inside the integral by dominated convergence theorem), is dominated by some constant multiple of the integrable function,

$$\left| \underline{N}_{k+n_0-r-N_f}^{k+n_0-r} \right| \left| \pi(j, \underline{y}') \right|.$$

The above function is integrable by Cauchy Schwartz inequality.

The component wise partial derivative, $\frac{\partial f}{\partial \theta}\big|_{\theta,\pi}(i,\underline{y})$, is uniformly upper bounded by,

$$\left| \frac{\partial f}{\partial \theta}\bigg|_{\theta,\pi}(i,\underline{y}) \right| \leq \frac{c''}{\epsilon_0}\left[|\underline{y}|+1\right] \text{ for all } (i,\underline{y}), \text{ all } \underline{\theta},$$

where the constant c'' depends on $|\pi|$.

One can now prove the existence of the overall partial derivative $\frac{\partial f}{\partial \underline{\theta}}$ at every $(\underline{\theta},\pi)$ using the above upper bound and the dominated convergence theorem (in L_2 norm). Consider the limit,

$$\begin{aligned}
&\lim_{|\underline{\theta}_h|\to 0} \frac{1}{|\underline{\theta}_h|} \sum_i \int_{\underline{y}} \left| f(\underline{\theta},\pi_{\underline{\theta}})(i,\underline{y}) - f(\underline{\theta}+\underline{\theta}_h,\pi_{\underline{\theta}})(i,\underline{y}) - \underline{\theta}_h^t \frac{\partial f}{\partial \theta}\bigg|_{\theta,\pi_{\underline{\theta}}}(i,\underline{y}) \right|^2 f_{\mathcal{N}}(\underline{y})d\underline{y} \\
&= \sum_i \int_{\underline{y}} \lim_{|\underline{\theta}_h|\to 0} \frac{1}{|\underline{\theta}_h|} \left| f(\underline{\theta},\pi_{\underline{\theta}})(i,\underline{y}) - f(\underline{\theta}+\underline{\theta}_h,\pi_{\underline{\theta}})(i,\underline{y}) - \underline{\theta}_h^t \frac{\partial f}{\partial \theta}\bigg|_{\theta,\pi_{\underline{\theta}}}(i,\underline{y}) \right|^2 f_{\mathcal{N}}(\underline{y})d\underline{y} \\
&= 0.
\end{aligned}$$

The first equality follows because the function inside the integral tends to zero at every point and is upper bounded by the following integrable function,

$$\frac{c'}{\epsilon_0}\left[|\underline{y}|+1\right] 1_{\left\{\left|\frac{\partial f}{\partial \theta}\big|_{\theta,\pi}(i,\underline{y})\right|\leq 1\right\}} + \left(\frac{c''}{\epsilon_0}\left[|\underline{y}|+1\right]\right)^2 1_{\left\{\left|\frac{\partial f}{\partial \theta}\big|_{\theta,\pi}(i,\underline{y})\right|> 1\right\}}.$$

We will now upper bound this partial derivative for all $(\underline{\theta},\pi_{\underline{\theta}})$. First observe that, because $\pi_{\underline{\theta}}(i,\underline{y}) \leq 1$ for all $(i,\underline{y})$, from (6.10),

$$\begin{aligned}
&\left| \frac{\partial f}{\partial \underline{\theta}}\bigg|_{(\underline{\theta},\pi_{\underline{\theta}})}(i,\underline{y}) \right| \\
&= \left| \sum_{l,j} \int_{\underline{y}',\underline{v}} \frac{\partial\left(\Pi_{q=1}^n P_{\underline{\theta}}(\hat{s}_{k+n_0-q}|\underline{x}(q))\right)}{\partial \underline{\theta}} P(\underline{S}_{k+1}^{k+n_0})\pi_{\underline{\theta}}(j,\underline{y}')f_{\mathcal{N}}(\underline{v})f_{\mathcal{N}}(\underline{y}')d\underline{v}d\underline{y}' \right| \\
&\leq \left(c_1 \sum_{r=1}^n \sum_{l,j} \int_{\underline{v},\underline{y}'} 1_{\left\{|e_{\underline{\theta}}(\underline{x}(r))|\leq\epsilon_0\right\}} f_{\mathcal{N}}(\underline{v})f_{\mathcal{N}}(\underline{y}')d\underline{v}d\underline{y}' \right) + c_2|\underline{y}| + c_3 E(|N_k|),
\end{aligned}$$

for some appropriate constants c_1, c_2, c_3. Then using $\left(\sum_{k=1}^n a_k\right)^2 \leq n\sum_{k=1}^n a_k^2$, $|x|^2 \leq |x|$

(when $|x| \leq 1$), we get,

$$\begin{aligned}
\left| \frac{\partial f}{\partial \underline{\theta}} \bigg|_{(\underline{\theta}, \pi_{\underline{\theta}})} \right|^2 &= \left(\sum_i \left(\int_{\underline{y}} \left| \frac{\partial f}{\partial \underline{\theta}} \bigg|_{(\underline{\theta}, \pi_{\underline{\theta}})} (i, \underline{y}) \right|^2 \frac{f_{\mathcal{N}}}{|\mathcal{S}|}(\underline{y}) d\underline{y} \right)^{1/2} \right)^2 \\
&\leq c_1' \sum_{r=1}^{n} \sum_{l,j,i} \int_{\underline{v}, \underline{y}', \underline{y}} 1_{\left\{ \left| e_{\underline{\theta}}(\underline{x}(r)) \right| \leq \epsilon_0 \right\}} f_{\mathcal{N}}(\underline{v}) f_{\mathcal{N}}(\underline{y}') f_{\mathcal{N}}(\underline{y}) d\underline{v} d\underline{y}' d\underline{y} \\
&\qquad + c_2' \sum_i \int_{\underline{y}} \left| \underline{y} \right|^2 f_{\mathcal{N}}(\underline{y}) d\underline{y} + c_3' \left[E(|N_k|) \right]^2 \\
&= c_1'' \sum_l P\left(\left\{ \left| \underline{b_l} + \underline{\theta}_f^t N_k \right| \leq \epsilon_0 \right\} \right) + c_2'' \sigma^2, \qquad (6.19)
\end{aligned}$$

where the constants b_l take values $\underline{S}_k^t \underline{Z\theta} + \underline{\theta}_b^t \underline{\hat{S}}_{k-1}$. ■

Appendix C

Proof of Theorem 6.3 : Let

$$\begin{aligned}
f_1(\underline{\theta}, \epsilon_0) &:= E_{G_k(\underline{\theta})}^{\epsilon_0} \left[err_{\underline{\theta}}(G_k)^2 \right], \\
f_2(\underline{\theta}, \epsilon_0) &:= \left| E_{G_k(\underline{\theta})}^{\epsilon_0} \nabla_{\underline{\theta}} \left[err_{\underline{\theta}}(G_k)^2 \right] \right|.
\end{aligned}$$

Note that for any fixed ϵ_0, LMS attractors will be the zeros, i.e., minima of $f_2(., \epsilon_0)$ while the DFE Wiener filters are the minima of the MSE cost function $f_1(., \epsilon_0)$. Also note that $\epsilon_0 = 0$ corresponds to the original decoder.

Let $\{\epsilon_{0n}\}$ be any sequence converging to 0. Let $\Omega = \{\epsilon_{0n}\}$. Take a compact set C large enough such that the Wiener filter is inside it (as $\underline{\theta}$ is increased to infinity, eventually MSE will start increasing and will tend to infinity).

One can follow steps similar to Theorem 6.1 and show that the stationary density $\pi_{\underline{\theta}_n}^{\epsilon_{0n}}$ converges to $\pi_{\underline{\theta}}^0$ as $(\epsilon_{0n}, \underline{\theta}_n) \to (0, \underline{\theta})$. Similarly, one can also show that both functions f_1, f_2 are jointly continuous in $(\underline{\theta}, \epsilon_0) \in C \times \Omega$.

The domain of the parameter $\underline{\theta}$ for every ϵ_0, say $D(\epsilon_0)$, is the same compact set C and hence the correspondence $\epsilon_0 \mapsto D(\epsilon_0)$ is compact and continuous ([51]). Then by the maximum theorem (p. 235, [51]),

$$D_{1n}^* := \arg\min_{\underline{\theta}\in C} f_1(\underline{\theta}, \epsilon_{0n}),$$
$$D_{2n}^* := \arg\min_{\underline{\theta}\in C} f_2(\underline{\theta}, \epsilon_{0n}),$$

are compact valued upper semi-continuous correspondences on Ω. Thus by Proposition 9.8, p. 231, [51] there exists a subsequence of LMS attractors $\underline{\theta}_{n_k}^{LMS}$ converging to an LMS attractor of the original decoder, $\underline{\theta}_0^{LMS}$. Once again by the same proposition there exists a further subsequence such that the DFE Wiener filters $\underline{\theta}_{n_{k_l}}^*$ converge to a DFE Wiener filter of the original decoder, $\underline{\theta}_0^*$. Thus there exists a sequence (after renaming) $\epsilon_{0n} \to 0$ such that

$$\begin{aligned} \underline{\theta}_n^{LMS} &\to \underline{\theta}_0^{LMS} \text{ and} \\ \underline{\theta}_n^* &\to \underline{\theta}_0^*. \end{aligned}$$

■

Appendix D

Proof of Theorem 6.4 : By the last assumption of Section 6.1, we have $\underline{S}_k^t \underline{Z\theta} + \underline{\theta}_b^t \underline{\hat{S}}_{k-1} \neq 0$ for all values of $\underline{S}_k, \underline{\hat{S}}_{k-1}$ at an LMS attractor. This implies the same (in fact the sign of the term, $\underline{S}_k^t \underline{Z\theta} + \underline{\theta}_b^t \underline{\hat{S}}_{k-1}$, for each $\underline{S}_k, \underline{\hat{S}}_{k-1}$ remains same) in a small neighborhood of the LMS attractor by continuity. Thus, when $\sigma^2 = 0$ (the noiseless case), for the original decoder at an LMS attractor (call it $\underline{\theta}_0^*$), we have,

$$\frac{\partial P_{\underline{\theta}}}{\partial \underline{\theta}}(i, \underline{y}|j, \underline{y}') = 0 \text{ for all values of } (i, \underline{y}) \;\; (j, \underline{y}').$$

Thus following the proof of Theorem 6.2 we can show that the gradient of the stationary density, $\nabla_{\underline{\theta}} \pi_{\underline{\theta}}$ exists and equals zero at the LMS-DFE attractor of the original decoder for the noiseless case. Hence at $\underline{\theta}_0^*$,

$$\nabla_{\underline{\theta}} E_{G_k(\underline{\theta})}\left[err_{\underline{\theta}}(G_k)^2\right] = E_{G_k(\underline{\theta})}\left[\nabla_{\underline{\theta}}\left(err_{\underline{\theta}}(G_k)^2\right)\right].$$

Thus in this case, the DFE Wiener filter coincides with the LMS-DFE attractor, $\underline{\theta}_0^*$. Choose $\epsilon_{02} > 0$ such that,

$$\epsilon_{02} < \left| (\underline{S}_k^t Z \underline{\theta}_0^* + \hat{\underline{S}}_{k-1}^t \underline{\theta}_{0b}^* \right|$$

for all values of $\underline{S}_k$ and $\hat{\underline{S}}_{k-1}$.

The DFE Wiener filter ($\underline{\theta}^{*,\epsilon_0}$) and the LMS-DFE attractor ($\underline{\theta}^{LMS,\epsilon_0}$) coincide and equal $\underline{\theta}_0^*$ for a noiseless system having a perturbed decoder, with $\epsilon_0 \leq \epsilon_{02}$. This happens because the perturbed decoder coincides with the original decoder for $\epsilon_0 \leq \epsilon_{02}$, when there is no noise.

Fix $\epsilon_0 \leq \epsilon_{02}$. Then, $w(\underline{\theta}^{*,\epsilon_0}, 0, 0) = 0$ and the partial derivative,

$$\begin{aligned} \left.\frac{\partial w}{\partial \underline{\theta}}\right|_{(\underline{\theta}^*_{\epsilon_0},0,0)} &= \frac{ds}{d\underline{\theta}} = E_{f_{\mathcal{N}}}\left[\nabla_{\underline{\theta}}\left(\nabla_{\underline{\theta}}\left(err_{\underline{\theta}}(G_k)^2\right)\pi_{\underline{\theta}}(G_k)\right)\right], \\ &= E_{\underline{\theta}}\left[\nabla_{\underline{\theta}}\nabla_{\underline{\theta}}\left(err_{\underline{\theta}}(G_k)^2\right)\right], \\ &= 2R_{xx}(\underline{\theta}^{*,\epsilon_0}), \end{aligned}$$

where $R_{xx}(\underline{\theta}^{*,\epsilon_0})$ is the autocorrelation matrix of the vector $X_k(\underline{\theta})$, under stationarity, at $\underline{\theta}^{*,\epsilon_0}$. As $\underline{\theta}^{*,\epsilon_0}$ is a Wiener filter, the above partial derivative will be invertible (all the eigenvalues of the derivative should be negative for the equilibrium point to be an attractor).

Continuity of the above partial derivative with respect to $\sigma^2, \eta, \underline{\theta}$ can be seen as before. Applying Implicit function theorem at $(\underline{\theta}^{*,\epsilon_0}, 0, 0)$, one gets a $\delta > 0$, and a continuous function $q(\sigma^2, \eta)$ such that $q(0,0) = \underline{\theta}^{*,\epsilon_0}$ and $w(q(\sigma^2,\eta), \sigma^2, \eta) = 0$, for all (σ^2, η) such that $|(\sigma^2, \eta)| \leq \delta$. ■

Remark : The above theorem also provides the following useful conclusion. For all $\sigma^2 \leq \delta$, the zeros of $w(., \sigma^2, 0)$ exist and equal $q(\sigma^2, 0)$. These zeros are continuous in σ^2. One can see that these zeros will indeed be LMS attractors as invertability of the derivative of the function $f()$ at $\sigma^2 = 0$ guaranties its invertibility in a small neighborhood of $\sigma^2 = 0$.

Appendix E

Lemma 6.1 *Let* $\pi_{\underline{\theta}_n} \xrightarrow{L_1} \pi_{\underline{\theta}}$. *Let* $|f(\underline{\theta}_n, \underline{x})| \leq g_1(\underline{x})$, $|f(\underline{\theta}_n, \underline{x})|^2 \leq g_2(\underline{x})$ *for all* n, *where* g_1, g_2 *are integrable functions (with respect to measure* μ*). Also let* f *continuous and* $\left|\pi_{\underline{\theta}_n}(\underline{x})\right| \leq C < \infty$ *for all* $\underline{x}$. *Then as* $n \to \infty$,

$$\int f(\underline{\theta}_n, \underline{x}) \pi_{\underline{\theta}_n}(\underline{x}) \mu(d\underline{x}) \to \int f(\underline{\theta}, \underline{x}) \pi_{\underline{\theta}}(\underline{x}) \mu(d\underline{x}).$$

Proof : We have

$$\begin{aligned}
&\left| \int \left(f(\underline{\theta}_n, \underline{x}) \pi_{\underline{\theta}_n}(\underline{x}) - f(\underline{\theta}, \underline{x}) \pi_{\underline{\theta}}(\underline{x}) \right) \mu(d\underline{x}) \right| \\
&\qquad \leq \int \left| \left(f(\underline{\theta}_n, \underline{x}) \pi_{\underline{\theta}_n}(\underline{x}) - f(\underline{\theta}, \underline{x}) \pi_{\underline{\theta}}(\underline{x}) \right) \right| \mu(d\underline{x}) \\
&\qquad \leq \int |f(\underline{\theta}_n, \underline{x})| \left| \pi_{\underline{\theta}_n}(\underline{x}) - \pi_{\underline{\theta}}(\underline{x}) \right| \mu(d\underline{x}) + \int |f(\underline{\theta}_n, \underline{x}) - f(\underline{\theta}, \underline{x})| \, \pi_{\underline{\theta}}(\underline{x}) \mu(d\underline{x}) \\
&\qquad \leq \left(\int |f(\underline{\theta}_n, \underline{x})|^2 \mu(d\underline{x}) \right)^{1/2} \left(\int \left| \pi_{\underline{\theta}_n}(\underline{x}) - \pi_{\underline{\theta}}(\underline{x}) \right|^2 \mu(d\underline{x}) \right)^{1/2} \\
&\qquad\qquad + \int |f(\underline{\theta}_n, \underline{x}) - f(\underline{\theta}, \underline{x})| \, \pi_{\underline{\theta}}(\underline{x}) \mu(d\underline{x}).
\end{aligned}$$

The first term on the right converges to zero because,

$$\left(\int \left| \pi_{\underline{\theta}_n}(\underline{x}) - \pi_{\underline{\theta}}(\underline{x}) \right|^2 \mu(d\underline{x}) \right) \leq 4C^2 \left(\int \left| \pi_{\underline{\theta}_n}(\underline{x}) - \pi_{\underline{\theta}}(\underline{x}) \right| \mu(d\underline{x}) \right).$$

The second term converges to zero by continuity of the function $f(.,.)$ in $\underline{\theta}$ and by the bounded convergence theorem, ■

Lemma 6.2 *Let* $\pi_{\underline{\theta}}$ *represent the Radon-Nikodym derivative of measure* $\Pi_{\underline{\theta}}$ *with respect to the common measure* μ *for all* $\underline{\theta} \in \mathcal{R}^m$, *for some* m. *Assume* $\pi_{\underline{\theta}} \leq 1$ *everywhere for all* $\underline{\theta}$. *If* $\nabla_{\underline{\theta}} \pi_{\underline{\theta}}$ *exists in* L_2 *norm, then*

$$\nabla_{\underline{\theta}} E_{\underline{\theta}}(g(\underline{\theta})) = E_{\mu} \nabla_{\underline{\theta}}(g(\underline{\theta}) \pi(\underline{\theta})),$$

where $g(\underline{\theta}, .)$ is square integrable, continuously differentiable in $\underline{\theta}$ and bounded by a square integrable function uniformly in a neighborhood of $\underline{\theta}$.

Proof : Since,

$$\begin{aligned}
\frac{1}{|\underline{\theta}_h|} \int & \left| g(\underline{\theta} + \underline{\theta}_h)\pi_{\underline{\theta}+\underline{\theta}_h} - g(\underline{\theta})\pi_{\underline{\theta}} - \underline{\theta}_h^t \left(g(\underline{\theta}) \nabla_{\underline{\theta}} \pi_{\underline{\theta}} - \nabla_{\underline{\theta}} g(\underline{\theta}) \pi_{\underline{\theta}} \right) \right| \mu(dw) \\
\leq & \frac{1}{|\underline{\theta}_h|} \int \left| g(\underline{\theta}) \left(\pi_{\underline{\theta}+\underline{\theta}_h} - \pi_{\underline{\theta}} - \underline{\theta}_h^t \nabla_{\underline{\theta}} \pi_{\underline{\theta}} \right) \right| \mu(dw) \\
& + \frac{1}{|\underline{\theta}_h|} \int \left| \pi_{\underline{\theta}+\underline{\theta}_h} \left(g(\underline{\theta} + \underline{\theta}_h) - g(\underline{\theta}) - \underline{\theta}_h^t \nabla_{\underline{\theta}} g(\underline{\theta}) \right) \right| \mu(dw) \\
& + \frac{1}{|\underline{\theta}_h|} \int \left| (\pi_{\underline{\theta}+\underline{\theta}_h} - \pi_{\underline{\theta}}) \underline{\theta}_h^t \nabla_{\underline{\theta}} g(\underline{\theta}) \right| \mu(dw),
\end{aligned} \tag{6.20}$$

we will have the result if we show that each of the terms on the right tend to zero as $|\underline{\theta}_h| \to 0$. By Cauchy Schwartz inequality,

$$\begin{aligned}
\lim_{|\underline{\theta}_h| \to 0} \frac{1}{|\underline{\theta}_h|} \int & \left| g(\underline{\theta}) \left(\pi_{\underline{\theta}+\underline{\theta}_h} - \pi_{\underline{\theta}} - \underline{\theta}_h^t \nabla_{\underline{\theta}} \pi_{\underline{\theta}} \right) \right| \mu(dw) \\
\leq & \ |g|_2 \lim_{|\underline{\theta}_h| \to 0} \frac{1}{|\underline{\theta}_h|} \left(\int \left| \left(\pi_{\underline{\theta}+\underline{\theta}_h} - \pi_{\underline{\theta}} - \underline{\theta}_h^t \nabla_{\underline{\theta}} \pi_{\underline{\theta}} \right) \right|^2 \mu(dw) \right)^{1/2}
\end{aligned}$$

The right side tends to zero because the gradient $\nabla_{\underline{\theta}} \pi_{\underline{\theta}}$ exists in L_2 norm.

The second term on the right hand side of (6.20) tends to zero by bounded convergence theorem and mean value theorem (as in Appendix B), because $\pi_{\underline{\theta}} \leq 1$ everywhere for all $\underline{\theta}$ and as ∇g is uniformly bounded in a neighborhood of $\underline{\theta}$ by an integrable function.

The third term of (6.20) tends to zero by Cauchy Schwartz inequality and by the continuity of the stationary density π in L_2 norm, and by uniform boundedness of the function $(\nabla_{\underline{\theta}} g)$ in a neighborhood of $\underline{\theta}$ by a square integrable function. ■

Chapter 7

LMS-DFE versus DFE-WF for a Wireless Channel

In this chapter we study the tracking behavior of an LMS-DFE while tracking a time varying wireless channel. We model the wireless channel by an AR(2) process of Chapter 2. We obtain an ODE approximation for the LMS-DFE. The instantaneous attractors of this ODE are the same as the LMS attractors of the previous chapter, with the channel fixed at the instantaneous value. It is shown in the previous chapter that the LMS attractors are close to that of the DFE-WFs at high SNRs. Hence one may expect that the LMS-DFE while tracking an AR(2) process moves close to the instantaneous WF at high SNRs. Using the ODE approximation of the LMS-DFE and the channel, we show via some examples that the LMS equalizer moves close to the instantaneous Wiener filter after initial transience. We also compare the LMS equalizer with the instantaneous 'optimal' DFE (the commonly used Wiener filter) designed assuming perfect previous decisions and computed using perfect channel estimate (we will call it as IDFE). We show that the LMS-DFE significantly outperforms the IDFE at all practical SNRs almost all the time after initial transience. An interesting observation is that, the improvement is significant even at high SNRs where an IDFE does not suffer from error propagation.

This chapter is organized as follows. Our system model, notations and assumptions are discussed in Section 7.1. In Section 7.2, we obtain an ODE approximation for the tracking trajectory of an LMS-DFE. Section 7.3 provides some examples verifying our claims, while Section 7.4 concludes the chapter. Proofs are provided in the Appendices.

7.1 System model and Notations

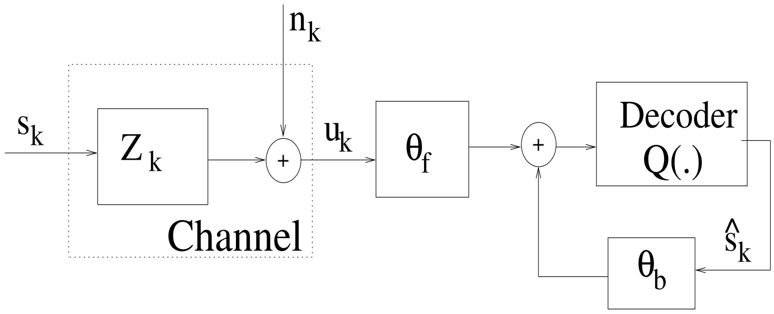

Figure 7.1: Block diagram of a Wireless channel followed by a DFE.

We consider a system with a time varying wireless channel and a DFE (see Figure 7.1). Inputs $\{s_k\}$ enter a time varying finite impulse response channel $\{z_{k,l}\}_{l=0}^{L-1}$, and are corrupted by an additive white Gaussian noise $\{n_k\}$ with variance σ^2. The channel output, u_k, at any time k, is given by,

$$u_k = \sum_{l=0}^{L-1} s_{k-l} z_{k,l} + n_k.$$

The time variations of the channel are modeled by an AR(2) process,

$$\underline{Z}_k = d_1 \underline{Z}_{k-1} + d_2 \underline{Z}_{k-2} + \mu \underline{W}_k \tag{7.1}$$

where $\underline{W}_k$ is an IID vector sequence of Gaussian random variables (Gaussian assumption is not really needed) and $\underline{Z}_k = \begin{bmatrix} z_{k,0}, & z_{k,1} & \cdots & z_{k,L-1} \end{bmatrix}$. The channel outputs u_k pass through a DFE with a hard decoder. The details about the equalizer are given below. We use the following notations and assumptions.

- We assume BPSK modulation, i.e., $s_k \in \{+1, -1\}$.

- Sequences $\{s_k\}$ and $\{n_k\}$ are IID (independent, identically distributed) and independent of each other.
- The equalizer forward filter is given by $\{\theta_{f_l}\}_{l=0}^{N_f-1}$, while the feedback filter is given by $\{\theta_{b_l}\}_{l=1}^{N_b}$.
- $N_L \stackrel{\triangle}{=} N_f + L - 1$.
- The decisions are obtained after hard decoding. Hence decision $\hat{s}_k$ is given by,

$$\begin{aligned} \hat{s}_k &= Q\left(\sum_{l=0}^{N_f-1} \theta_{f_l} u_{k-l} + \sum_{l=1}^{N_b} \theta_{bl} \hat{s}_{k-l}\right) \quad \text{where} \\ Q(x) &:= \begin{cases} +1 & if \quad x \geq 0, \\ -1 & if \quad x < 0. \end{cases} \end{aligned}$$

- For any vector, $\underline{x}$, we use x_l to represent its l^{th} component. $\underline{x}_l^k$, $l \leq k$, represents the vector $\begin{bmatrix} x_k & x_{k-1} & \cdots & x_l \end{bmatrix}^T$.
- The following vector notations are used throughout.

$$\begin{aligned} \underline{S}_k &\stackrel{\triangle}{=} s_{k-N_L+1}^k, & \underline{N}_k &\stackrel{\triangle}{=} n_{k-N_f+1}^k, \\ \underline{U}_k &\stackrel{\triangle}{=} u_{k-N_f+1}^k, & \underline{\hat{S}}_k &\stackrel{\triangle}{=} \hat{s}_{k-N_b+1}^k, \\ \underline{X}_k &\stackrel{\triangle}{=} [\ \underline{U}_k^T \quad \underline{\hat{S}}_{k-1}^T\ ]^T, & \underline{G}_k &\stackrel{\triangle}{=} [\ \underline{S}_k^T \quad \underline{X}_k^T\ ]^T, \\ \underline{\theta}_f &\stackrel{\triangle}{=} \theta_{f0}^{N_f-1}, & \underline{\theta}_b &\stackrel{\triangle}{=} \theta_{b1}^{N_b}, \\ \underline{J}_k &\stackrel{\triangle}{=} [\ \underline{S}_k^T \quad \underline{\hat{S}}_{k-1}^T \quad \underline{N}_k^T\ ]^T, & \underline{\theta} &\stackrel{\triangle}{=} [\ \underline{\theta}_f{}^T \quad \underline{\theta}_b{}^T\ ]^T. \end{aligned}$$

- $\underline{\theta}_k$ represent the time varying equalizer at time k.
- Let $\mathcal{S} := \{+1, -1\}$. For a fixed $(\underline{\theta}, \underline{Z})$, $\{G_k\}$ is a Markov chain made of the channel input $\underline{S}_k$, channel output $\underline{U}_k$ and the decoder output $\underline{\hat{S}}_{k-1}$, when the channel is the fixed vector $\underline{Z}$ and the equalizer is fixed at $\underline{\theta}$. $\underline{G}_k$ takes values in $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b} \times \mathcal{R}^{N_f}$, where $\mathcal{R}$ is the set of real numbers. We represent throughout this chapter the current and previous state values of this Markov chain by the ordered pairs $(i, \underline{y})$,

$(j, \underline{y}')$ respectively. Here i, j take values from the discrete part of the state space, $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b}$, while $\underline{y}, \underline{y}'$ take values in $\mathcal{R}^{N_f}$.

- $Z\underline{\theta} = \{z\theta_l\}_{l=0}^{N_L-1}$ represents the convolution of the channel $\{z_l\}$and forward filter $\underline{\theta}_f$.

- $B(\underline{\theta}, \delta)$, $\bar{B}(\underline{\theta}, \delta)$ are the open and closed balls respectively, with center $\underline{\theta}$ and radius δ.

- $\bar{K}(\epsilon, M) := \left\{(\underline{\theta}_1, \underline{\theta}_2) \in \mathcal{R}^{N_f} \times \mathcal{R}^{N_b} : \epsilon < |\underline{\theta}_1| \leq M, |\underline{\theta}_2| \leq M\right\}$.

- $E_{j,\underline{y}'}^{\underline{\theta},\underline{Z}}$ represents the expectation of the Markov chain $\{G_k\}$ for fixed channel, equalizer pair $(\underline{\theta}, \underline{Z})$, when the initial condition is $j, \underline{y}'$.

- $E_{j,\underline{y}';\underline{\theta},\underline{Z}}$ represents the expectation of the Markov chain $\{G_k, \underline{\theta}_k, Z_k, Z_{k-1}\}$ with initial condition $(j, \underline{y}', \underline{\theta}, \underline{Z})$.

- $E_{(j,\underline{y}');(i,\underline{y})}^{\underline{\theta},\underline{Z};\underline{\theta}',\underline{Z}'}$ represents the expectation of the Markov chain pair $\{(G_k, G_k')\}$ under the initial condition $(j, \underline{y}', i, \underline{y})$. Here $\{G_k\}$ is the Markov chain for fixed channel-equalizer pair $(\underline{\theta}, \underline{Z})$ with initial condition $(j, \underline{y}')$, while $\{G_k'\}$ is the one for channel-equalizer pair $(\underline{\theta}', \underline{Z}')$ with initial condition $(i, \underline{y})$. When both the initial conditions are same, it is simply represented by $E_{i,\underline{y}}^{\underline{\theta},\underline{Z};\underline{\theta}',\underline{Z}'}$. When the channel-equalizer pair is same but with different initial conditions, it is represented by $E_{(i,\underline{y});(j,\underline{y}')}^{\underline{\theta},\underline{Z}}$.

- ${P^{\underline{\theta},\underline{Z}}}^k(.|.)$, $\Pi^{\underline{\theta},\underline{Z}}$, $E^{\underline{\theta},\underline{Z}}$ respectively represent the $k-$step transition function, the stationary distribution, and the expectation wrt to the stationary measure (existence will be shown) of the Markov chain $\{G_k\}$ for a fixed channel, equalizer pair $(\underline{\theta}, \underline{Z})$.

- We use a DFE, $\underline{\theta}$, to track the wireless channel modeled by an AR(2) process, $\{\underline{Z}_k\}$. The LMS algorithm is used to continuously update the equalizer $\underline{\theta}$ to cater to the time varying channel.

$$\begin{aligned} \underline{\theta}_{k+1} &= \underline{\theta}_k - \mu H_{1\underline{\theta}_k}(\underline{G}_k) \text{ where} \\ H_{1\underline{\theta}}(\underline{G}) &\stackrel{\triangle}{=} \underline{X}\left(\underline{X}^t\underline{\theta} - s\right). \end{aligned} \tag{7.2}$$

One can easily extend this theory to any finite alphabet input source. Currently, the theory to follow considers an optimal equalizer for delay 0. However, the entire theory will go through for any arbitrary delay. Indeed in Section 7.3, an example with an optimal equalizer for delay 1, is presented.

7.2 ODE Approximation

We can rewrite the channel AR(2) process (7.1) as,

$$Z_{k+1} = (1-d_2)Z_k + d_2 Z_{k-1} + \mu(W_k + \eta Z_k). \tag{7.3}$$

We will show below that the trajectory $(\underline{\theta}_k, \underline{Z}_k)$ given by equations (7.2), (7.3) can be approximated by the solution of the following system of ODEs,

$$\begin{aligned}
(1+d_2)\,\dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \in (-1,1], \\
\frac{d^2 Z(t)}{dt^2} &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 = -1, \\
\frac{d^2 Z(t)}{dt^2} + \eta_1\,\dot{Z}\,(t) &= [E(W) + \eta Z(t)]\,, \quad \text{if } d_2 \text{ is close to } -1,
\end{aligned} \tag{7.4}$$

$$\dot{\theta}\,(t) = h_1(\underline{\theta}(t), \underline{Z}(t)), \tag{7.5}$$

where

$$\begin{aligned}
h(Z) &\stackrel{\triangle}{=} E(W_k + \eta Z) = E(W_1) + \eta Z, \\
\eta &= \frac{d_1 + d_2 - 1}{\mu}, \ \eta_1 = \frac{1+d_2}{\sqrt{\mu}} \\
h_1(\underline{\theta}, \underline{Z}) &\stackrel{\triangle}{=} E^{\underline{\theta},\underline{Z}}\left[\underline{X}_k\left(\underline{X}_k^t \underline{\theta} - s_k\right)\right] = -R_{xx}(\underline{\theta}, \underline{Z})\theta + R_{xs}(\underline{\theta}, \underline{Z}), \\
R_{xx}(\underline{\theta}, \underline{Z}) &= E^{\underline{\theta},\underline{Z}}\left(\underline{X}\underline{X}^t\right), \\
R_{xs}(\underline{\theta}, \underline{Z}) &= E^{\underline{\theta},\underline{Z}}(\underline{X}s).
\end{aligned}$$

By the following Lemma, the above system of ODEs have unique global solutions that are bounded for any finite time.

Lemma 7.1 *The ODE (7.5) has a unique solution, which satisfies,*

$$|\theta(t)| \leq c_0 + c_1 e^{c_2 t},$$

for appropriate positive constants c_0, c_1 and c_2.

Proof : For convenience, we reproduce the ODE (7.5),

$$\dot{\theta}\ (t) \;=\; -R_{xx}(\underline{\theta}(t), \underline{Z}(t))\theta(t) + R_{xs}(\underline{\theta}(t), \underline{Z}(t)),$$

It is easy to see that $|R_{xx}(\underline{\theta}(t), \underline{Z}(t))| \leq C_1 |Z(t)|^2 + C_2$, $\quad |R_{xs}(\underline{\theta}(t), \underline{Z}(t))| \leq C |Z(t)|$ for all t, for some positive constants C_1, C_2 and C. The above inequalities follow by boundedness of the decisions $\hat{s}$. By boundedness of the channel ODE solution curves (given in Chapter 2) given below, $|R_{xs}(\underline{\theta}(t), \underline{Z}(t))| \leq C_3(T)$, $|R_{xx}(\underline{\theta}(t), \underline{Z}(t))| \leq C_4(T)$, for all $t \leq T$ for any finite time T for some positive constants $C_3(T)$, $C_4(T)$ depending only on T. Thus, for any vector θ, the inner product,

$$\begin{aligned}\left\langle \dot{\underline{\theta}}\ (t), \underline{\theta} \right\rangle &\leq C_3(T)|\underline{\theta}|^2 + C_4(T)|\underline{\theta}| \\ &= [C_3(T)|\underline{\theta}| + C_4(T)]\, |\underline{\theta}|.\end{aligned}$$

Therefore by Global existence theorem (pp 169 - 170 of [42]), the ODE (7.5), has a unique solution for any finite time and the solution is bounded by the solution of the following scalar ODE (after choosing the initial conditions properly),

$$\dot{k}\ (t) = C_3(T)k(t) + C_4(T),$$

whose solution is given by,

$$k(t) = c_1 e^{C_3(T)t} + C_4(T),$$

for some appropriate constant c_1. ■

Let $Z(t, t_0, Z), \theta(t, t_0, \theta)$ represent the solutions of the ODEs (7.4), (7.5) with initial conditions $Z(t_0) = Z$, $\theta(t_0) = \theta$ and $\dot{Z}\ (t_0) = 0$, whenever the channel is approximated by a second order ODE. We prove Theorem 7.1, using the Theorem A.2 of Appendix I, provided at the end of the thesis.

Theorem 7.1 *For any finite $T > 0$, for all $\delta > 0$ and for any initial condition $(G, \underline{\theta}, \underline{Z})$, with $d_2 Z_{-1} + d_1 Z_0 = Z$, $\dot{Z}\ (t_0) = 0$, whenever the channel is approximated by a second order ODE and $\theta_0 = \theta$,*

$$P_{G,Z,\theta} \left\{ \sup_{\left\{1 \le k \le \frac{T}{\mu^\alpha}\right\}} |(Z_k, \underline{\theta}_k) - (Z(\mu^\alpha k, 0, Z), \theta(\mu k, 0, \underline{\theta}))| \ge \delta \right\} \to 0,$$

as $\mu \to 0$, uniformly for all $Z, \theta \in Q$, if Q is contained in the bounded set containing the solution of the ODEs (7.4), (7.5) till time T. In the above $\alpha = 1$ if $Z(.,.,.)$ is solution of a first order ODE and equals $1/2$ otherwise.

Proof : Refer to Appendix A.

Thus we obtain the ODE approximation for the LMS-DFE tracking an AR(2) process. The approximating ODE (7.5) suggests that, its instantaneous attractors will be same as the LMS-DFE attractors when the channel is fixed at the instantaneous value of the channel ODE (7.4) (as in the previous chapter, Chapter 6). In the previous chapter, these LMS-DFE attractors are shown to be close to the DFE-WF at high SNRs. Hence the ODE suggests that the LMS-DFE may move close to the instantaneous DFE-WFs. We will in fact see that this is true for the examples we study in the next section.

One of the uses of the above ODE approximation is that, one can approximately obtain the performance (e.g., BER, MSE) of LMS-DFE at any time by using the trajectory of this ODE. Of course, obtaining BER theoretically is still a problem because the BER of a system with a fixed known channel and a fixed DFE is still not available. But our ODE approximation is still useful because one can obtain the performance (transient as well as stationary) of the LMS-DFE with only one simulation, which would not be possible otherwise.

7.3 Examples

We use the ODE approximation of the previous section to obtain some interesting conclusions. The ODE approximation gives accurate deterministic approximation of the LMS-DFE and the channel trajectory for practical values of step sizes. Hence as commented above, using these ODEs one can get good estimates of instantaneous performance measures like, Bit Error Rate (BER) and Mean Square Error (MSE) for almost all realizations of the LMS-DFE and the channel trajectory. Using the channel ODE, one can also obtain the same performance measures for instantaneous IDFE. Then one can compare the IDFE and the LMS-DFE along the entire time axis.

In all the examples below, we estimate the DFE-WF as in Chapter 6 (directly using steepest descent algorithm by approximating the gradient of the MSE with difference of estimated MSEs at two close points divided by the distance between the same two points). In the figures, the solid line, the dash dot line and the dash dash line represent the true coefficient trajectory, the ODE approximation and the IDFE trajectory respectively, while the stars represent the DFE-WFs.

To begin with we consider a stable channel (for which all the poles are inside the unit circle) in Figure 7.2,

$$Z_k \quad = \quad .4995 Z_{k-1} + 0.5 Z_{k-2} + 0.0001 W_k.$$

Here W_k is a Gaussian IID random vector with independent components of unit variance and its mean is a constant multiple of,

$$\begin{bmatrix} 0.26 & 0.34 & 0.25 & 0.064 & -0.13 & -0.19 & -0.16 & 0 & 0.064 & 0.064 \end{bmatrix}.$$

We consider a five tap feed-forward filter and a five tap feed-back filter. The LMS step-size equals $\mu = 0.001$ (In theory, it is shown that step-size of LMS μ is also equal to the channel step-size. However one can absorb the difference into one of the $H_1()$, $H()$ functions.) The noise variance $\sigma^2 = 0.05$. We plot the channel in the first subfigure. Th equalizer coefficient trajectories along with their ODE approximations are plotted in the next four subfigures of Figure 7.2. We start the LMS and the ODEs at time $t = 0$ with the instantaneous IDFE. We also plot the instantaneous DFE-WF and the IDFEs in the same sub figures. We can see that the ODE approximation is quite accurate for all the co-efficients. The approximation for the feed-forward coefficients is better than for the feed-back coefficients. We also see that the LMS-DFE is very close to the instantaneous DFE-WF after some initial transience. Furthermore, the IDFE trajectory is away from the DFE-WF in most of the cases. We also plot the instantaneous BER and MSE of the IDFE and the LMS-DFE (both calculated from corresponding ODE approximations) in the last two sub figures of Figure 7.2. One can see a huge improvement (upto 35%) of LMS-DFE (also of DFE-WF) over the IDFE both in terms of BER, MSE after the initial transience (Figure 7.2). On the other hand, performance of the LMS-DFE is quite close to that of the DFE-WF.

Next, we consider a marginally stable channel in Figure 7.3. Here $d_1 = 1.9999998$, $d_2 = -2$, $\mu_{ch} = 1e^{-7}$ (One can see from channel solution curves, (2.2) of Chapter 2, that $\sqrt{\frac{d_1+d_2-1}{\mu}}$ gives the period of oscillations). W_k is generated as before. Again there are five taps in the feed-forward filter and five in the feed-back filter. The step size of the LMS equals, $\mu_{LMS} = 0.001$. Here the channel trajectory is approximated by a cosine waveform. From Figure 7.3, we can make the same observations as in the stable case. In particular we see that the LMS-DFE has BER and MSE upto 50% less than for the IDFE. We also see that the LMS-DFE is always (after initial transience) very close to the DFE-WF, while the IDFE stays quite away. This again explains the poor performance of the IDFE (in terms of BER, MSE) over the LMS-DFE after initial transience.

We consider two more examples one for stable channel in Figure 7.4 and the second for a marginally stable channel in Figure 7.5. Here we consider a 3 tap channel followed by a (2,2) tap DFE. One can make similar observations as above. However the improvement (in BER, MSE) of LMS-DFE/DFE-WF over IDFE is slightly lower ($\approx 18\%$).

Finally, in Figure 7.6, we consider a stable channel with d_2 close to -1 (from the figure, it actually looks like a marginally stable channel but its magnitude is reducing at a very small rate, $\frac{1+d_2}{\sqrt{\mu}}$, as d_2 is very close to -1). In this case, as is shown theoretically, a better ODE approximation is obtained by a second order ODE. Here, the channel trajectory is approximated by an exponentially reducing cosine waveform. We considered the AR(2) process, which approximates the fading channel with band limited and U-shaped spectrum and received with $f_d T = 0.001$ ($f_d T$ represents the product of maximum Doppler frequency and the actual data sampling time). One can see that, the LMS-DFE once again tracks the instantaneous DFE-WF after initial transience (in this case more than half of the first cycle, as this is a fast varying channel) and that the IDFE is poor in comparison with the LMS-DFE and the DFE-WF.

7.4 Conclusions

We study an LMS-DFE tracking a wireless channel, approximated by an AR(2) process. We considered a long-standing problem of tracking the true MSE optimal DFE. We approximated the LMS-DFE trajectory with the solution of a system of ODEs. Using this ODE approximation, we show that the LMS-DFE comes close to the instantaneous DFE-WF after the initial transience. We also see that the performance measures BER and MSE of the LMS-DFE are quite close to that of the DFE-WF after the transient period. We thus conclude that the LMS-DFE can be used to track the DFE-WF.

Furthermore, we also compared the LMS-DFE with IDFE, the popular WF designed assuming perfect past decisions (also designed from perfect channel estimate). IDFE is shown to be far away from the DFE-WF (also from the LMS-DFE) trajectory throughout the entire time axis. Its performance (BER, MSE) is substantially worse than that of the DFE-WF and the LMS-DFE.

Appendix A

Proof of Theorem 7.1 : We considered a general system (1)-(2) in Appendix I (provided at the end of the thesis) and proved the ODE approximation for this in Theorem

A.2. The channel, equalizer pair, $(\underline{\theta}_k, \underline{Z}_k)$ given by equations (7.2), (7.3), is a specific example of the general system (2), (1). Thus Theorem 7.1 is proved if we show that $(\underline{\theta}_k, \underline{Z}_k)$ given by (7.2), (7.3) satisfies the assumptions **A**.1-**A**.3 of Chapter 2 and **B**.1-**B**.4 (after replacing the assumption **B**.3.c.ii with **B**.3.c.ii$'$) of Appendix I (the solution of the system of ODEs is bounded for any finite time by Lemma 7.1).

The AR(2) process $\{\underline{Z}_k\}$ in (7.3) clearly satisfies the assumptions **A.**1 - **A.**3 as is shown in Chapter 2. If $(\underline{\theta}_k, \underline{Z}_k)$ stay constant and equal $(\underline{\theta}, \underline{Z})$, then $\{G_k\}$ is a Markov chain whose transition probabilities $P^{\underline{\theta},\underline{Z}}(G, \mathbf{A})$ are a function of $(\underline{\theta}, \underline{Z})$ alone. Thus condition **B**.1 is satisfied. **B**.2 is also satisfied as for any compact set Q and for any $\theta \in Q$,

$$|H_{1\theta}(\underline{G})| \leq 2 \left[max \left\{ 1, \sup_{\theta \in Q} |\theta| \right\} \right] (1 + |G|^2).$$

The condition **B**.4 is trivially met as for any $n > N_f + N_b + 1$, the expectation does not depend upon the initial condition G but is bounded based on the compact set Q and because of the Gaussian random variable N and discrete random variable $S, \hat{S}$.

In the previous chapter, we showed that for every fixed channel, equalizer pair $(\underline{\theta}, \underline{Z})$, a unique stationary measure of the Markov chain $\{\underline{J}_k\}$ (we referred it as $\{G_k\}$ in that chapter) exists and is continuous in $\underline{\theta}$. One can show the joint continuity wrt $(\underline{\theta}, \underline{Z})$ pair, on similar lines by redefining M_1 of upper bound (6.14) of the Appendix A in the previous chapter by,

$$M_1 \stackrel{\triangle}{=} \min_{\underline{S}_k, \hat{\underline{S}}_{k-1}, \underline{\theta} \in \bar{B}(\underline{\theta}_0, \epsilon), \underline{Z} \in \bar{B}(\underline{Z}_0, \epsilon_0)} \underline{Z\theta}^t \underline{S}_k + \underline{\theta}_b^t \hat{\underline{S}}_{k-1}.$$

We can show the same for the Markov chain, $\{G_k\}$ also, as $G_k = \Gamma(\underline{J}_k)$ for some fixed one-one, onto C^∞ function Γ, whenever the channel and equalizer values are fixed. Further in Chapter 6, it is shown that $h_1(\underline{\theta}, \underline{Z}) \stackrel{\triangle}{=} E^{\underline{\theta},\underline{Z}} H_{1\underline{\theta}}$ exists for every channel equalizer pair, $(\underline{\theta}, \underline{Z})$. We will show that,

$$\nu_{\underline{\theta},\underline{Z}}(\underline{G}) \stackrel{\triangle}{=} \sum_{k \geq 0} P^{\underline{\theta},\underline{Z}^k} (H_1(\theta, \underline{G}) - h_1(\underline{\theta}, \underline{Z})),$$

exists and will satisfy the conditions **B**.3. This will complete the proof.

Verification of Assumptions B.3.b and B.3.c.i :

We will use the Proposition 2, p.253, [4] and Proposition 4, p. 257, [4] to verify the required assumptions. For convenience, the statement of both the Propositions (and the assumptions under which they are valid) are reproduced in Appendix C as Proposition 7.1, Proposition 7.2 respectively.

We will first verify the assumptions $X.1$ to $X.4$ of Appendix C. Towards this, one can clearly see that for all initial conditions $j, \underline{y}'$, $i, \underline{y}$ and all equalizers $\underline{\theta}$,

$$E^{\underline{\theta},\underline{Z}}_{j,\underline{y}'}\left(|U_n|^p\right) \leq \begin{cases} C & if \quad p > N_f \\ C' \left|\underline{y}'\right|^p & if \quad p \leq N_f, \end{cases} \tag{7.6}$$

$$E^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}\left(|U_n - U_n'|^p\right) \leq \begin{cases} C & if \quad p > N_f \\ C' \left|\underline{y} - \underline{y}'\right|^p & if \quad p \leq N_f, \end{cases} \tag{7.7}$$

whenever the channel $\underline{Z} \in B(0, \epsilon')$ for any ϵ'.

Thus the Markov chain $\{G_k\}$ satisfies the assumptions $(X.1)$, $(X.2)$ for any fixed channel, equalizer pair $(\underline{\theta}, \underline{Z})$. In Lemma 7.2 and 7.3 of Appendix B, we show that $\{G_k\}$ satisfies the assumptions $(X.3)$, $(X.4)$. Here all the constants in the upper bounds are same as long as $(\underline{\theta}, \underline{Z}) \in \bar{K}(\epsilon, M) \times \bar{B}(0, \epsilon')$. Hence Proposition 7.2 follows and then we can apply Proposition 7.1. This Proposition also needs the inequality (7.6) . By this Proposition,

$$h_1(\underline{\theta}, \underline{Z}) = \lim_{k \to \infty} P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(j, \underline{y}'), \tag{7.8}$$

exists for every $(\underline{\theta}, \underline{Z})$, and for any initial condition $j, \underline{y}'$. Note that $P^{\underline{\theta},\underline{Z}^k}$ as mentioned in Section 7.1 represents the k-step transition function of the Markov chain, $\{G_k\}$, with channel, equalizer fixed at $\underline{\theta}, \underline{Z}$. The function h_1 is actually equal to the expected value of function $H_{1\underline{\theta}}$ under stationary measure as is shown above. Also, by the same Proposition, for some constants $C < \infty$, $q > 0$ and $\rho < 1$,

$$\left| h_1(\underline{\theta}, \underline{Z}) - P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(i, \underline{y}) \right| \leq C\rho^k(1 + |y|^q). \tag{7.9}$$

We also get the existence of,

$$\nu_{\underline{\theta},\underline{Z}}(\underline{G}) \;\; \stackrel{\triangle}{=} \;\; \sum_{k \geq 0} P^{\underline{\theta},\underline{Z}^k}(H_1(\theta, \underline{G}) - h_1(\underline{\theta}, \underline{Z})),$$

for all channel, equalizer pairs $(\underline{\theta}, \underline{Z})$, which satisfy the assumption **B**.3.b. Finally, by uniformity of all the inequalities for any $\underline{\theta} \in K(\epsilon, M)$, $\underline{Z} \in B(0, \epsilon')$, assumption **B**.3.c.i is satisfied, i.e.,

$$\left| \nu_{\underline{\theta},\underline{Z}}(i, \underline{y}) \right| \leq C_6(1 + |y|^q), \text{ with } C_6 < \infty,$$

uniformly in $\underline{\theta} \in K(\epsilon, M)$, $\underline{Z} \in B(0, \epsilon')$.

Verification of Assumption B.3.a :

One can easily see that for $(\underline{\theta}, \underline{Z})$, $(\underline{\theta}', \underline{Z}')$ from a compact set, Q, there exists a constant C depending upon Q such that, for all $k > N_f$,

$$\left| H_{1\underline{\theta}}(G_k(\underline{\theta}, \underline{Z})) - H_{1\underline{\theta}'}(G_k(\underline{\theta}', \underline{Z}')) \right| \;\; \leq \;\; C \left| (\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}') \right| (1 + |\underline{N}_k|). \tag{7.10}$$

Hence for all $k > N_f$,

$$E^{\underline{\theta},\underline{Z};\underline{\theta}',\underline{Z}'}_{(i,\underline{y});(j,\underline{y}')} \left| H_{1\underline{\theta}}(G_k(\underline{\theta}, \underline{Z})) - H_{1\underline{\theta}'}(G_k(\underline{\theta}', \underline{Z}')) \right| \;\; \leq \;\; C' \left| (\underline{\theta}, \underline{Z}) - (\underline{\theta}', \underline{Z}') \right|.$$

Using limit (7.8) and the upper bound (7.10) we get,

$$|h_1(\underline{\theta}, \underline{Z}) - h_1(\underline{\theta}', \underline{Z}')| \;\; = \;\; \left| \lim_{k \to \infty} \left(P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(j, \underline{y}') - P^{\underline{\theta}',\underline{Z}'^k} H_{1\underline{\theta}'}(j, \underline{y}') \right) \right|$$

$$
\begin{aligned}
&= \left| \lim_{k\to\infty} E^{\underline{\theta},\underline{Z};\underline{\theta}',\underline{Z}'}_{(i,\underline{y});(j,\underline{y}')} \left\{ H_{1\underline{\theta}}(G_k(\underline{\theta},\underline{Z})) - H_{1\underline{\theta}'}(G_k(\underline{\theta}',\underline{Z}')) \right\} \right| \\
&\leq C' \left| (\underline{\theta},\underline{Z}) - (\underline{\theta}',\underline{Z}') \right|,
\end{aligned}
$$

whenever $(\underline{\theta},\underline{Z})$ $(\underline{\theta}',\underline{Z}')$ are in a compact set Q.

Verification of Assumption B.3.c.ii$'$:

Note that,

$$
P^{\underline{\theta},\underline{Z}}\nu_{\underline{\theta},\underline{Z}}(j,\underline{y}') = \sum_{k\geq 1} \left\{ P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(j,\underline{y}') - h(\underline{\theta},\underline{Z}) \right\}.
$$

Hence, for any $(\underline{\theta},\underline{Z})$, $(\underline{\theta}',\underline{Z}')$ pair and any $j, \underline{y}'$,

$$
\begin{aligned}
\left| P^{\underline{\theta},\underline{Z}}\nu_{\underline{\theta},\underline{Z}}(j,\underline{y}') - P^{\underline{\theta}',\underline{Z}'}\nu_{\underline{\theta}',\underline{Z}'}(j,\underline{y}') \right| \leq\; & \left| \sum_{k=1}^{n} \left(P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(j,\underline{y}') - P^{\underline{\theta}',\underline{Z}'^k} H_{1\underline{\theta}}(i,\underline{y}) \right) \right| \\
& + (n-1) \left| h_1(\underline{\theta},\underline{Z}) - h_1(\underline{\theta}',\underline{Z}') \right| \\
& + \left| \sum_{k=1}^{n} \left(P^{\underline{\theta}',\underline{Z}'^k} H_{1\underline{\theta}}(j,\underline{y}') - P^{\underline{\theta}',\underline{Z}'^k} H_{1\underline{\theta}'}(j,\underline{y}') \right) \right| \\
& + \left| \sum_{k\geq n} \left\{ P^{\underline{\theta},\underline{Z}^k} H_{1\underline{\theta}}(j,\underline{y}') - h(\underline{\theta},\underline{Z}) \right\} \right| \\
& + \left| \sum_{k\geq n} \left\{ P^{\underline{\theta}',\underline{Z}'^k} H_{1\underline{\theta}'}(j,\underline{y}') - h(\underline{\theta}',\underline{Z}') \right\} \right|.
\end{aligned}
$$

The second term is bounded by a constant multiple of the term, $n\left|(\underline{\theta},\underline{Z}) - (\underline{\theta}',\underline{Z}')\right|$, as h_1 is locally Lipschitz (proved in the previous para). Using the upper bound (7.9), one can see that the fourth and fifth terms are bounded by a constant multiple of the term $\rho^n(1 + \left|\underline{y}'\right|^q)$. Without loss of generality, we can further choose $q \geq 2$. The third term can be bounded because $\left|H_{1\underline{\theta}}(i,\underline{y}) - H_{1\underline{\theta}'}(i,\underline{y})\right| \leq C \left|\underline{y}\right|^2 \left|\underline{\theta} - \underline{\theta}'\right|$ whenever $\underline{\theta}, \underline{\theta}'$ come from a compact set and because $P_{j,\underline{y}'}^{\underline{\theta}',\underline{Z}'^k} \left|\underline{y}\right|^4 \le C'\left(1+\left|\underline{y}'\right|^4\right)$ for any k. Hence we get,

$$\begin{aligned}\left|P^{\underline{\theta},\underline{Z}}\nu_{\underline{\theta},\underline{Z}}(j,\underline{y}') - P^{\underline{\theta}',\underline{Z}'}\nu_{\underline{\theta}',\underline{Z}'}(j,\underline{y}')\right|^2 &\le B_1 n\sum_{k=1}^{n}\left|P^{\underline{\theta},\underline{Z}^k}H_{1\underline{\theta}}(j,\underline{y}') - P^{\underline{\theta}',\underline{Z}'^k}H_{1\underline{\theta}}(j,\underline{y}')\right|^2 \\ &\quad + \left(B_2 n^2\left|(\underline{\theta},\underline{Z}) - (\underline{\theta}',\underline{Z}')\right|^2 + B_3\rho^{2n}\right)(1+\left|\underline{y}'\right|^{2q}).\end{aligned}$$

Fix $\epsilon > 0$, $M > 0$, $\epsilon' > 0$. Define $\tau \stackrel{\triangle}{=} \inf_n \left\{(\underline{\theta}_n,\underline{Z}_n) \notin \bar{K}(\epsilon,M)\times\bar{B}(0,\epsilon')\right\}$. By Lemma 7.4 and 7.5 for any m,

$$\begin{aligned}&E_{i,\underline{y},\underline{\theta}_0,\underline{Z}_0}\Big\{I(m+1\le\tau)\left|P^{\underline{\theta}_{m+1},\underline{Z}_{m+1}}\nu_{\underline{\theta}_{m+1},\underline{Z}_{m+1}}(G_{m+1}) - P^{\underline{\theta}_m,\underline{Z}_m}\nu_{\underline{\theta}_m,\underline{Z}_m}(G_{m+1})\right|^2\Big\} \\ &\le B_1 n\sum_{k=1}^{n} E_{i,\underline{y},\underline{\theta}_0,\underline{Z}_0}\Big\{I(m+1\le\tau)\left|P^{\underline{\theta}_{m+1},\underline{Z}_{m+1}}{}^k H_{1\underline{\theta}_{m+1}}(G_{m+1}) - P^{\underline{\theta}_m,\underline{Z}_m}{}^k H_{1\underline{\theta}_{m+1}}(G_{m+1})\right|^2\Big\} \\ &\quad + E_{i,\underline{y},\underline{\theta}_0,\underline{Z}_0}\Big\{I(m+1\le\tau)\left(1+\left|\underline{U_{m+1}}\right|^{2q}\right)\left(B_2 n^2\left|(\underline{\theta}_{m+1},\underline{Z}_{m+1}) - (\underline{\theta}_m,\underline{Z}_m)\right|^2 + B_3\rho^{2n}\right)\Big\} \\ &\le B_1 n^2 C_5\mu^{0.5}\left(1+\left|\underline{y}\right|^4\right) + B_2 n^2 C_6\mu^{0.5}\left(1+\left|\underline{y}\right|^{2q}\right) + B_3\rho^{2n}\left(1+\left|\underline{y}\right|^{2q}\right) \\ &\le B\left(n^2\mu^{0.5}+\rho^{2n}\right)\left(1+\left|\underline{y}\right|^{2q}\right).\end{aligned}$$

Now, we choose $n = \left\lceil log\mu^{0.5}.\left(log\rho^2\right)^{-1}\right\rceil$, where $\lceil x\rceil$ represents the smallest integer $\ge x$. Then,

$$log\rho^{2n} \;\ge\; log\mu^{0.5}.$$

Hence we have for some constant C depending upon ρ,

$$n^2\mu^{0.5} + \rho^{2n} \le C\left(1+\left|log\mu^{0.5}\right|^2\right)\mu^{0.5} + \mu^{0.5}.$$

Then for any $\lambda < 0.5$, the upper bound

$$\sup_{0\le\mu\le\mu_0}\left(1+\left|log\mu^{0.5}\right|^2\right)\mu^{0.5-\lambda} < \infty.$$

This follows because the limit, $\lim_{x\to 0} x^{\alpha}(log(x))^2 = 0$ whenever $\alpha > 0$ (by applying L'Hospital's rule twice). Hence there exists a constant $B'(\lambda)$ not depending upon μ such that,

$$E_{i,\underline{y},\underline{\theta}_0,\underline{Z}_0}\Big\{I(m+1\leq\tau)\left|P^{\underline{\theta}_{m+1},\underline{Z}_{m+1}}\nu_{\underline{\theta}_{m+1},\underline{Z}_{m+1}}(G_{m+1}) - P^{\underline{\theta}_m,\underline{Z}_m}\nu_{\underline{\theta}_m,\underline{Z}_m}(G_{m+1})\right|^2\Big\}$$
$$\leq \; B'(\lambda)\mu^{\lambda}\left(1+\left|\underline{y}\right|^{2q}\right).$$

This completes the verification of Assumption **B**.3 and hence the theorem follows. ■

Appendix B

Lemma 7.2 *Let* $A(n) = \left\{\underline{\hat{S}}_k \neq \underline{\hat{S}}'_k; k = 1,2,\cdots,n\right\}$. *Given* ϵ, M, ϵ', *there exist positive* $C_2 < \infty$, *and* $\rho < 1$ *such that, for all* $\underline{Z} \in \bar{B}(0,\epsilon')$, $\underline{\theta} \in \bar{K}(\epsilon, M)$ *and all* n,

$$P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}(A(n)) \leq C_2\rho^n.$$

Proof : Let $r \stackrel{\triangle}{=} 1+N_f+N_b$, $B(n)\stackrel{\triangle}{=}A(nr)$. Then,

$$\begin{aligned}
p_n &\stackrel{\triangle}{=} P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}(B(n)) \\
&= E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left[I(B(n))I(B(n-1))\right] \\
&\stackrel{a}{=} E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left[E\left[I(B(n))I(B(n-1))|G_{(n-1)r},..,G_1,G'_{(n-1)r},..,G'_1\right]\right] \\
&= E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\Big[E\Big[I\left(\{\underline{\hat{S}}'_k \neq \underline{\hat{S}}_k; (n-1)r \leq k \leq nr\}\right) \\
&\qquad\qquad \big|G_{(n-1)r},..,G_1,G'_{(n-1)r},..,G'_1\Big]\, I(B(n-1))\Big] \\
&\stackrel{b}{=} E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\Big[E\Big[I\left(\{\underline{\hat{S}}'_k \neq \underline{\hat{S}}_k; (n-1)r \leq k \leq nr\}\right) \\
&\qquad\qquad \big|G_{(n-1)r},G'_{(n-1)r}\Big]\, I(B(n-1))\Big] \\
&= E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left[I(B(n-1))P^{\underline{\theta},\underline{Z}}_{G_{(n-1)r},G'_{(n-1)r}}(B(1))\right].
\end{aligned}$$

Here equality a follows by conditional expectation, while, equality b follows by Markovian property. For any pair of initial conditions $(i,\underline{y}),(j,\underline{y}')$ and for all $\underline{Z} \in \bar{B}(0,\epsilon')$ and all

$\underline{\theta} \in \bar{K}(\epsilon, M)$, with ρ_1 given by Lemma 7.6,

$$\begin{aligned} P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left(\left\{\underline{\hat{S}}_k \neq \underline{\hat{S}}'_k; k=1,2,\cdots,r\right\}\right) &= 1 - P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left(\cup_{k=1}^{r}\left\{\underline{\hat{S}}_k = \underline{\hat{S}}'_k\right\}\right) \\ &\leq 1 - P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left(\left\{\underline{\hat{S}}_r = \underline{\hat{S}}'_r\right\}\right) \\ &\leq 1-\rho_1. \end{aligned}$$

This also shows that, $p_1 \leq (1-\rho_1)$. Hence, for any pair of initial conditions $(i,\underline{y}),(j,\underline{y}')$ and for all $\underline{Z} \in \bar{B}(0,\epsilon')$ and all $\underline{\theta}, \in \bar{K}(\epsilon, M)$

$$\begin{aligned} p_n &\leq E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left[I(B(n-1))(1-\rho_1)\right] \\ &\leq (1-\rho_1)E^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}\left[I(B(n-1))\right] \\ &\leq (1-\rho_1)p_{n-1}. \end{aligned}$$

Therefore, $p_n \leq (1-\rho_1)^n$. Thus for any general n,

$$P^{\underline{\theta},\underline{Z}}_{(i,\underline{y});(j,\underline{y}')}(A(n)) \leq (1-\rho_1)^{n/r} \quad \blacksquare$$

Lemma 7.3 *For any $\underline{\theta}, \underline{Z}$, for any pair of initial conditions $(j,\underline{y}')$, $(i,\underline{y})$ and for any $n > N_f + N_L + N_b$,*

$$P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}\left(\left\{\underline{\hat{S}}_{n-1} = \underline{\hat{S}}'_{n-1}, \underline{\hat{S}}_n \neq \underline{\hat{S}}'_n\right\}\right) = 0.$$

Proof : Note that,

$$P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}\left(\left\{\underline{\hat{S}}_{n-1} = \underline{\hat{S}}'_{n-1}, \underline{\hat{S}}_n \neq \underline{\hat{S}}'_n\right\}\right) = P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}\left(\left\{\underline{\hat{S}}_{n-1} = \underline{\hat{S}}'_{n-1}, \hat{s}_n \neq \hat{s}'_n\right\}\right).$$

For any $n > N_f + N_L + N_b$ on $\left\{\underline{\hat{S}}_{n-1} = \underline{\hat{S}}'_{n-1}\right\}$, the equalizer output/ decoder input will be same once the channel and equalizer are fixed and hence the probability of the next decision being different (based on different initial conditions) is zero. $\blacksquare$

Lemma 7.4 *There exists a constant C_5 such that for all n, for all initial conditions $(i,\underline{y})$,*

$(\underline{\theta}_0, \underline{Z}_0) \in \bar{K}(\epsilon, M) \times B(0, \epsilon')$,

$$E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) \left| \left(P^{\underline{\theta}_{m+1},\underline{Z}_{m+1}}{}^n - P^{\underline{\theta}_m,\underline{Z}_m}{}^n \right) H_{1\underline{\theta}_{m+1}}(G_{m+1}) \right|^2 \right\} \leq C_5 \mu^{0.5} \left(1 + \left| \underline{y} \right|^4 \right).$$

Proof : By direct calculations we get (in all $\{G_n\}$ represents the Markov chain for the first channel equalizer pair, while $\{G'_n\}$ represents the same for the second pair),

$$\begin{aligned}
&E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) \left| \left(P^{\underline{\theta}_{m+1},\underline{Z}_{m+1}}{}^n - P^{\underline{\theta}_m,\underline{Z}_m}{}^n \right) H_{1\underline{\theta}_{m+1}}(G_{m+1}) \right|^2 \right\} \\
&\quad = E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) \left| E_{G_{m+1}}^{\underline{\theta}_{m+1},\underline{Z}_{m+1};\underline{\theta}_m,\underline{Z}_m} \left[H_{1\underline{\theta}_{m+1}}(G_n) - H_{1\underline{\theta}_{m+1}}(G'_n) \right] \right|^2 \right\} \\
&\quad \leq E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) E_{G_{m+1}}^{\underline{\theta}_{m+1},\underline{Z}_{m+1};\underline{\theta}_m,\underline{Z}_m} \left| H_{1\underline{\theta}_{m+1}}(G_n) - H_{1\underline{\theta}_{m+1}}(G'_n) \right|^2 \right\} \\
&\quad \leq C E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \Big\{ I(m+1 \leq \tau) \left| (\underline{\theta}_{m+1}, \underline{Z}_{m+1}) - (\underline{\theta}_m, \underline{Z}_m) \right|^2 \\
&\qquad\qquad E_{G_{m+1}}^{\underline{\theta}_{m+1},\underline{Z}_{m+1};\underline{\theta}_m,\underline{Z}_m} \left(1 + |\underline{N}_n|^4 \right) \Big\} \\
&\quad \leq C E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) \left| (\underline{\theta}_{m+1}, \underline{Z}_{m+1}) - (\underline{\theta}_m, \underline{Z}_m) \right|^2 \left(1 + \left| \underline{N}_{m+1} \right|^4 \right) \right\} \\
&\quad \leq C' \mu^{0.5} (1 + |y|^4).
\end{aligned}$$

The second inequality follows using upper bound (7.10) (on $\{m+1 \leq \tau\}$ the channel equalizer trajectory $\{\underline{\theta}_k, \underline{Z}_k\}_{k \leq m+1}$ is in a compact set), while the last inequality follows as in Lemma 7.5 (by just repeating the steps as in Lemma 7.5). ■

Lemma 7.5 *For any given ϵ, ϵ', M, there exists a constant $C_6 > 0$ such that for all initial conditions $(i, \underline{y})$, $(\underline{\theta}_0, \underline{Z}_0) \in (\bar{K}(\epsilon, M) \times \bar{B}(0, \epsilon'))$ and for any $q > 0$,*

$$E_{i,\underline{y};\underline{\theta}_0,\underline{Z}_0} \left\{ I(m+1 \leq \tau) \left| (\underline{\theta}_{m+1}, \underline{Z}_{m+1}) - (\underline{\theta}_m, \underline{Z}_m) \right|^2 \left(1 + \left| \underline{U}_{m+1} \right|^q \right) \right\} \leq C_6 \mu^{0.5} \left(1 + \left| \underline{y} \right|^q \right).$$

Proof : By Chebyshev's inequality,

$$\begin{aligned}
&E_{i,\underline{y};\underline{\theta},\underline{Z}} \left\{ I(m+1 \leq \tau) \left| (\underline{\theta}_{m+1}, \underline{Z}_{m+1}) - (\underline{\theta}_m, \underline{Z}_m) \right|^2 \left(1 + \left| \underline{U}_{m+1} \right|^q \right) \right\} \\
&\quad \leq \left(E_{i,\underline{y};\underline{\theta},\underline{Z}} \left[I(m+1 \leq \tau) \left| (\underline{\theta}_{m+1}, \underline{Z}_{m+1}) - (\underline{\theta}_m, \underline{Z}_m) \right|^4 \right] \right)^{1/2} \\
&\qquad\qquad \left(E_{i,\underline{y};\underline{\theta},\underline{Z}} \left[I(m+1 \leq \tau) \left(1 + \left| \underline{U}_{m+1} \right|^{2q} \right) \right] \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C'\left(1+\left|\underline{y}\right|^{q}\right)\left(E_{i,\underline{y};\underline{\theta},\underline{Z}}\left\{I(m+1\leq\tau)\left[\left|Z_{m+1}-Z_m\right|^4+\left|\underline{\theta}_{m+1}-\underline{\theta}_m\right|^4\right]\right\}\right)^{1/2} \\
&\leq \; C'\left(1+\left|\underline{y}\right|^{q}\right)\left\{\left(E_{i,\underline{y};\underline{\theta},\underline{Z}}\left[I(m+1\leq\tau)\left|Z_{m+1}-Z_m\right|^4\right]\right)^{1/2}\right. \\
&\qquad\qquad\qquad\qquad \left.+\left(E_{i,\underline{y};\underline{\theta},\underline{Z}}\left[I(m+1\leq\tau)\left|\underline{\theta}_{m+1}-\underline{\theta}_m\right|^4\right]\right)^{1/2}\right\}.
\end{aligned}
$$

By the upper bound (6) of Chapter 4,

$$
E_{i,\underline{y};\underline{\theta},\underline{Z}}\left[I(m+1\leq\tau)\left|Z_{m+1}-Z_m\right|^4\right]\leq N\mu,
$$

where $N<\infty$ is a positive constant. We have the following also because, $\underline{\theta}_{m+1}-\underline{\theta}_m = \mu H(G_m)$,

$$
\begin{aligned}
&E_{i,\underline{y};\underline{\theta},\underline{Z}}\left\{I(m+1\leq\tau)\left|(\underline{\theta}_{m+1},\underline{Z}_{m+1})-(\underline{\theta}_m,\underline{Z}_m)\right|^2\left(1+\left|\underline{U}_{m+1}\right|^q\right)\right\} \\
&\qquad\qquad\qquad\qquad \leq \; C''\left(1+\left|\underline{y}\right|^q\right)(\mu^{0.5}+\mu^2). \quad \blacksquare
\end{aligned}
$$

Lemma 7.6 *Given ϵ, M, ϵ' there exists a $\rho_1>0$ such that for any pair of initial conditions $(j,\underline{y}'),(i,\underline{y})$ for all $n\geq N_f+N_b$ for all $\underline{\theta}\in\bar{K}(\epsilon,M)$ and for all $\underline{Z}\in\bar{B}(0,\epsilon')$ we have,*

$$
P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}(\underline{\hat{S}}_n=\underline{\hat{S}}'_n)\geq\rho_1>0.
$$

Proof : Let

$$
M_n \overset{\triangle}{=} \inf_{\underline{Z}\in\bar{B}(0,\epsilon'),\underline{\theta}\in\bar{K}(\epsilon,M),\underline{\hat{S}}\in\mathcal{S}^{N_b},\underline{S}\in\mathcal{S}^{N_L}} \sum_{l=0}^{N_L-1} z\theta_l s_l+\sum_{l=1}^{N_b}\theta_{bl}\hat{s}_l.
$$

Note that $P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}(\underline{\hat{S}}_n=\underline{\hat{S}}'_n)$ represents the probability of the decisions $\underline{\hat{S}}_n$ with channel equalizer pair $\underline{\theta},\underline{Z}$ equal to the decisions, $\underline{\hat{S}}'_n$, of the same channel equalizer pair when the system is started with the initial condition $(j,\underline{y}')$, $(i,\underline{y})$ respectively. Hence,

$$
P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}(\underline{\hat{S}}_n=\underline{\hat{S}}'_n) \;\;\geq\;\; P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}(\cap_{l=0}^{N_b-1}\{\hat{s}_{n-l}=\hat{s}'_{n-l}=1\})
$$

$$
\begin{aligned}
&\geq P^{\underline{\theta},\underline{Z}}_{(j,\underline{y}');(i,\underline{y})}\left(\cap_{l=0}^{N_b-1}\left\{\underline{\theta}_f^t.N_{n-l} > -M_n\right\}\right) \\
&= P\left(\cap_{l=0}^{N_b-1}\left\{\underline{\theta}_f^t.N_{n-l} > -M_n\right\}\right).
\end{aligned}
$$

The last equality follows for all $n \geq N_f + N_b$. The event in the last probability is a finite intersection of inverse images of non-empty open sets and one can easily see that its probability is strictly positive for every $\underline{\theta} \in \bar{K}(\epsilon, M)$ as,

$$
\left\{\underline{\theta}_f^t.N_k > -M_n\right\} \supset \cap_{l=0;\underline{\theta}_{fl}\neq 0}^{N_f}\left\{n_{k-l} > \frac{-M_n}{\underline{\theta}_{fl} N_f}\right\}.
$$

Furthermore, the map $\underline{\theta} \mapsto P\left(\cap_{l=0}^{N_b-1}\left\{\underline{\theta}_f^t.N_{n-l} > -M_n\right\}\right)$ is continuous (by bounded convergence theorem and by almost sure continuity of the indicator function). Hence one can get a $\rho_1 > 0$ such that the Lemma follows (as continuous maps take compact sets to compact sets and since maximum/minimum are attained in a compact set). ■

Appendix C

In this section we provide the statements of the Propositions of [4] that we used in Appendix A. In all these $\{G_k\}$ is a general Markov chain whose state has a discrete component and a continuous component as in the case of our DFE example. We are using the same notations as defined in Section 7.1. We state the required propositions of [4] in our setup.

To begin with, we state the Proposition 2, p. 253, [4]. The following definitions are used. For any given function, g on $\mathcal{S}^{N_L} \times \mathcal{S}^{N_b} \times \mathcal{R}^{N_f}$, for any $p \geq 0$ we set,

$$
\begin{aligned}
||g||_{\infty,p} &= \sup_{i,\underline{y}} \frac{\left|g(i,\underline{y})\right|}{1+\left|\underline{y}\right|^p}, \\
[g]_p &= \sup_{\underline{y}\neq\underline{y}',i} \frac{\left|g(i,\underline{y}) - g(i,\underline{y}')\right|}{\left|\underline{y}-\underline{y}'\right|\left(1+\left|\underline{y}\right|^p+\left|\underline{y}'\right|^p\right)}, \\
Li(p) &= \left\{g; [g]_p < \infty\right\}, \\
N_p(g) &= \sup\left\{||g||_{\infty,p+1}\,[g]_p\right\}.
\end{aligned}
$$

Proposition 7.1 (Proposition 2, p. 253, [4]) : *Suppose that there exist positive constants* K_1*,* q*,* μ *and* $0 \leq \rho < 1$ *such that for all* $g \in Li(p)$*,* $\underline{y} \in \mathcal{R}^{N_f}$*,* $(i, \underline{y})$*,* $j, \underline{y'}$ *:*

$$\begin{aligned} \left|P^n g(i, \underline{y}) - P^n g(j, \underline{y'})\right| &\leq K_1 \rho^n N_p(g) \left(1 + \left|\underline{y}\right|^p + \left|\underline{y'}\right|^p\right), \\ \sup_j \int P(i, d\underline{y}|j, \underline{y'}) \left|\underline{y}\right|^p &\leq \mu \left(1 + \left|\underline{y'}\right|^p\right). \end{aligned}$$

Then there exists a constant K_2 *depending only on* K_1*,* μ*,* ρ*, and for all* $g \in Li(p)$ *a number* Γg *such that for all* $i, \underline{y}$*,*

(i) $\left|P^n g(i, \underline{y}) - \Gamma g\right| \leq K_2 N_p(g) \rho^n \left(1 + \left|\underline{y}\right|^q\right)$ *and*

(ii) $\nu = \sum_{n \geq 0} \left(P^n g - g\right)$ *satisfies* $(I - P)\,\nu = g - \Gamma g$*.*

The assumptions for Proposition 4, p. 257, [4] given in p.256 are,

X.1 For all $p \in \mathcal{N}$, there exists a constant $C > 0$ such that :

$$\sup_i E_{i,\underline{y}} \left|\underline{U}_n\right|^p \leq C \left(1 + \left|\underline{y}\right|^p\right).$$

X.2 There exist positive constants K_1, $\rho_1 < 1$ such that,

$$\sup_{i,j,\underline{y'}} E_{i,\underline{y};j,\underline{y'}} \left(\left|G_n - G'_n\right|^2\right) \leq K_1 \rho_1^n \left(1 + \left|\underline{y}\right|^2 + \left|\underline{y'}\right|^2\right).$$

X.3 There exist positive constants K_2, s_1, $\rho_2 < 1$ such that,

$$P_{i,\underline{y};j,\underline{y'}} \left((\underline{S}_k, \underline{\hat{S}_{k-1}}) \neq (\underline{S}'_k, \underline{\hat{S}_{k-1}}'), k = 0, 1, \cdots, n\right) \leq K_2 \rho_2^n \left(1 + \left|\underline{y}\right|^{s_1} + \left|\underline{y'}\right|^{s_1}\right).$$

X.4 There exist $r \in \mathcal{N}$, $K_3 \in \mathcal{R}_+$, $0 \leq \rho_3 < 1$ such that for all $n \geq r$, $(i, \underline{y})$, $(j, \underline{y'})$:

$$\begin{aligned} P_{i,\underline{y};j,\underline{y'}} &\left((\underline{S}_{n-1}, \underline{\hat{S}_{n-2}}) = (\underline{S}'_{n-1}, \underline{\hat{S}_{n-2}}'), (\underline{S}_n, \underline{\hat{S}_{n-1}}) \neq (\underline{S}'_n, \underline{\hat{S}_{n-1}}')\right) \\ &\leq K_3 \rho_3^n \left(1 + \left|\underline{y}\right|^{s_2} + \left|\underline{y'}\right|^{s_2}\right). \end{aligned}$$

Proposition 7.2 (Proposition 4, p. 257, [4]) : *If the process* $(\zeta_n)_{n\geq 0} \stackrel{\triangle}{=} (G_n, G'_n)_{n\geq 0}$ *satisfies* $X.1$ *to* $X.4$*; then for all* $p > 0$*, there exist positive constants* K*,* q*,* $\rho < 1$*, depending only on* K_i*,* s_i*,* ρ_i*, such that for all* $g \in Li(p)$*,* n*,* $(i, \underline{y})$*,* $(j, \underline{y'})$ *:*

$$\left| E_{i,\underline{y};j,\underline{y'}} \left(g(G_n) - g(G'_n) \right) \right| \leq K N_p(g) \rho^n \left(1 + \left| \underline{y} \right|^s + \left| \underline{y'} \right|^s \right).$$

In particular, if each of the processes $\{G_n\}$ *and* $\{G'_n\}$ *is a Markov chain with transition function* P*, then for all* $g \in Li(p)$ *we have,*

$$\left| P^n g(i, \underline{y}) - P^n g(j, \underline{y'}) \right| \leq K N_p(g) \rho^n \left(1 + \left| \underline{y} \right|^s + \left| \underline{y'} \right|^s \right).$$

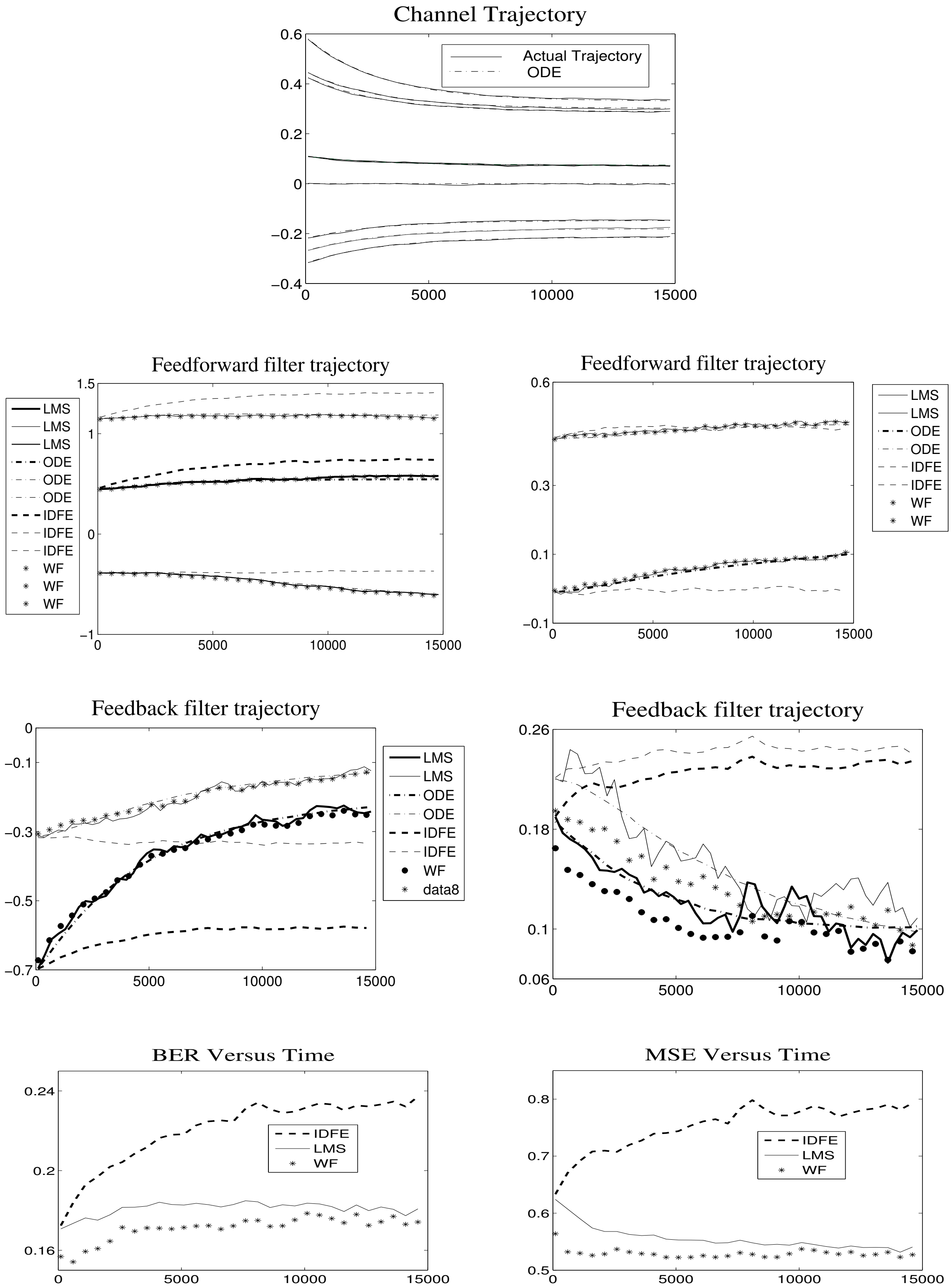

Figure 7.2: A Stable channel with $d_1 = 0.4995$, $d_2 = 0.5$, $\mu = 1e^{-4}$ and mean $= c[0.26,\ 0.34,\ 0.25,\ 0.064,\ -0.13,\ -0.19,\ -0.16,\ 0,\ 0.064,\ 0.064]$.

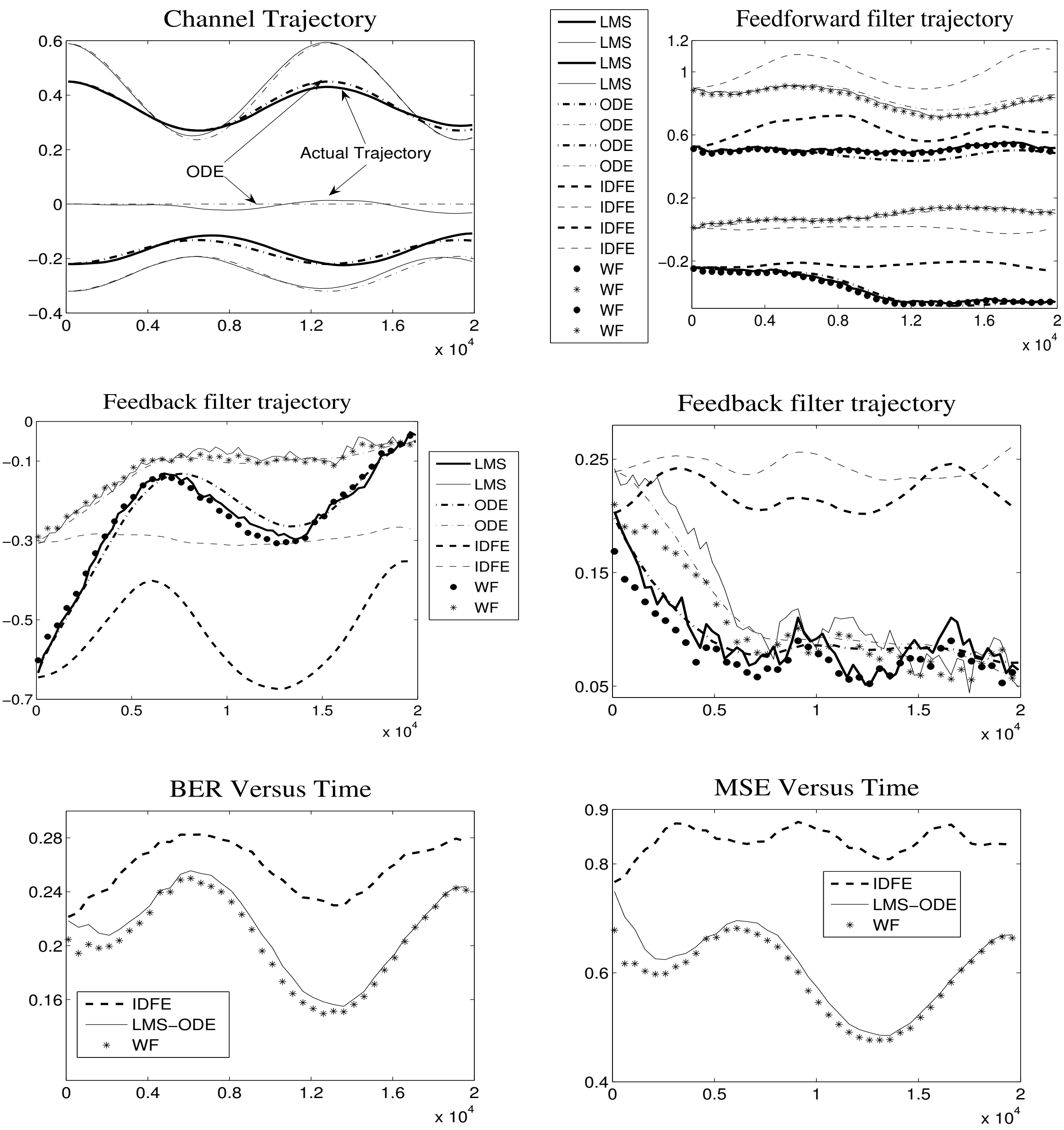

Figure 7.3: A Marginally stable channel with $d_1 = 1.9999998$, $d_2 = -1$, $\mu = 1e^{-7}$ and mean $= c[0.26,\ 0.34,\ 0.25,\ 0.064,\ -0.13,\ -0.19,\ -0.16,\ 0,\ 0.064,\ 0.064]$.

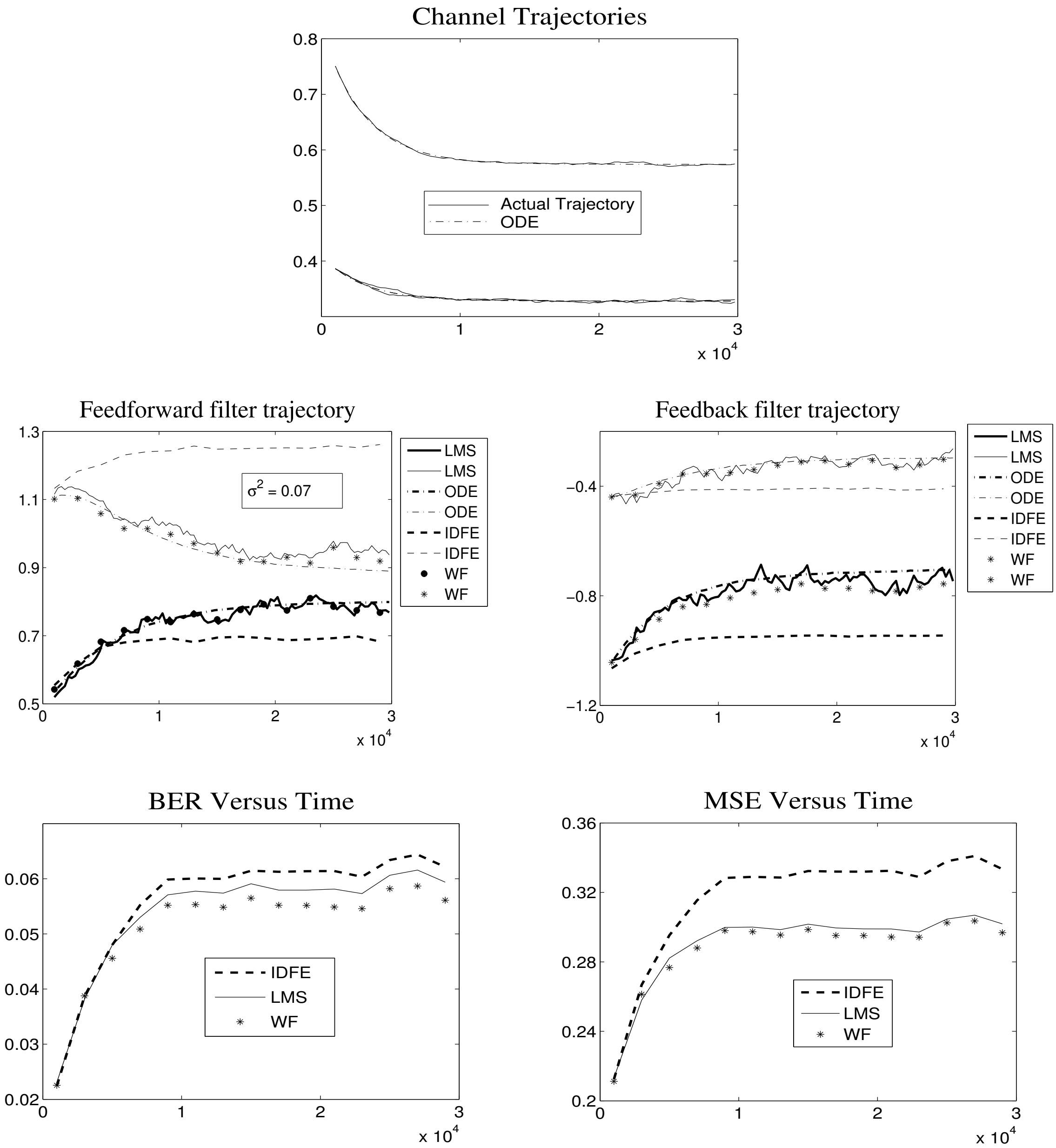

Figure 7.4: A Stable channel with mean a constant multiple of $[0.41, .82, .41]$, $d_1 = 0.4995$, $d_2 = 0.5$ and $\mu = 0.0005$.

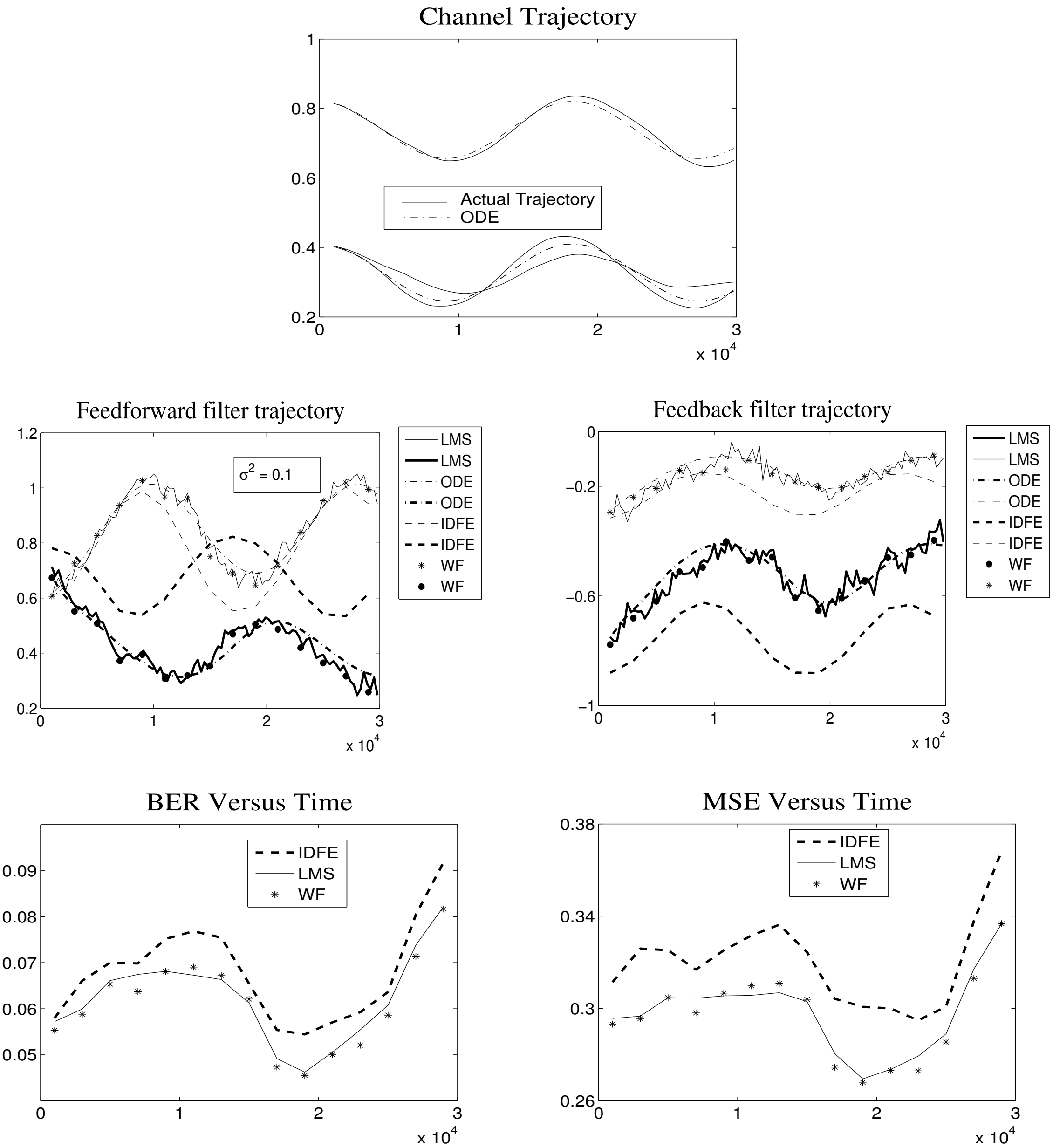

Figure 7.5: A Marginally stable channel with mean a constant multiple of $[0.41, .82, .41]$, $d_1 = 1.9999998$, $d_2 = -1$ and $\mu = 1e^{-7}$.

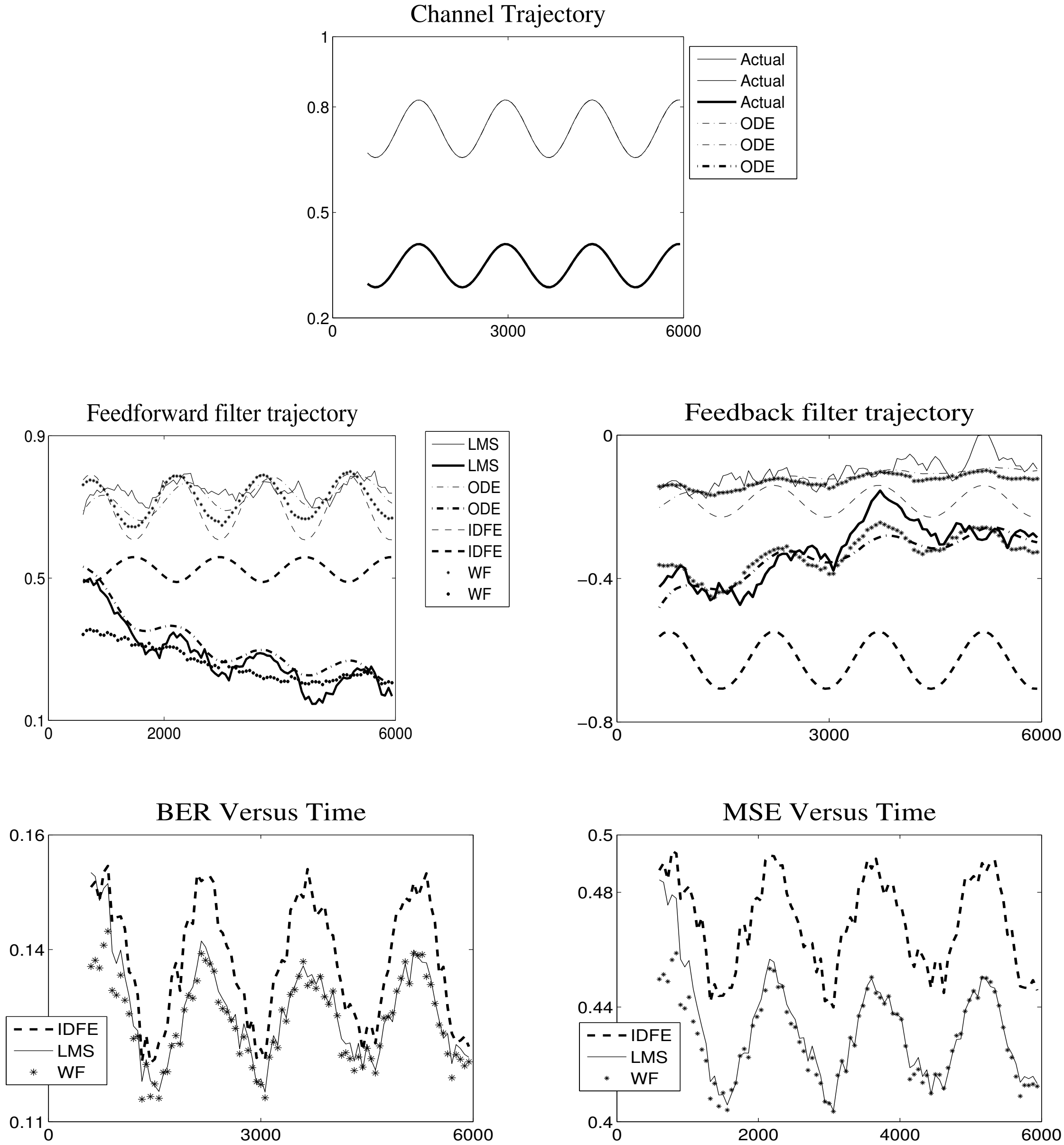

Figure 7.6: A Stable channel with mean a constant multiple of [0.41, .82, .41]. It is obtained for $f_d T = 0.001$, i.e., $d_1 = 1.999982$, $d_2 = -.9999947$ and $\mu = 1.3997e - 010$.

Chapter 8

Conclusions

In this thesis we attempted to address various issues related to an equalizer working in a wireless environment. Our goal was to obtain an optimal wireless equalizer for a given wireless scenario. We considered a slow fading channel. We considered two notions of optimality in this regard.

Traditionally the equalizers were designed to optimize the *MSE* (the BW lost due to training sequence is not considered). This approach was well suited for time invariant (e.g., wireline) channels. One could send required amount of training sequence in the beginning, obtain a 'good' equalizer and then switch over to the data transmission mode. However with time varying nature of the wireless channels, training sequence needs to be sent frequently and the optimality of the above approach is questionable. This was the first issue considered in our thesis.

With the advent of wireless communications, people started using the *information theoretic measures* (which consider the BW lost due to training) to optimize the wireless components. We used this notion of optimality in the first part of our thesis to come up with the best (blind, semi-blind or training based) equalizer for a given wireless scenario.

Any practical system uses training algorithms and they are still optimal as long as the criterion for optimality is MSE. Some of these equalizers were well understood for a fixed channel (some long-standing issues for a fixed channel are also addressed in this thesis). However, the time varying wireless channels bring in the requirement for understanding the theoretical tracking behavior (e.g., the study of how well a training equalizer tracks the time varying optimal equalizer). In the second part of the thesis we considered the tracking behavior of the MSE optimal training equalizers.

Blind/Semi-blind versus Training equalizers

We compared blind/semi-blind equalizers with training based algorithms in Chapter 3. The difficulty is in comparing the loss in accuracy of the blind algorithms with that of loss in data rate in training based methods. As mentioned above, information capacity is the most appropriate measure for this comparison. Defining a 'composite' channel for each equalizer, we compared the three algorithms based on the capacity of this channel. We obtained easily computable tight lower bounds on this capacity.

Via some examples, using the above measure, we observed that the semi-blind/blind methods perform superior to training methods in LOS conditions (≈50 to 70% improvement in transmit power) even when they have not converged to the equilibrium point. But for Rayleigh fading, the semi-blind methods are worse than the training based and the blind methods become completely useless. We also obtained the optimum number of training symbols (for training and semi-blind equalizers).

We modified the semi-blind algorithm, where the step size is adapted with respect to the training based channel estimate. The modified-semi-blind algorithm shows significant SNR improvement of $30\%, 20\%$ and 8.6% at low K-factors with $K = 0.9$, 0.1 and 0 (Rayleigh channel) respectively over the training method. One may achieve better results by further investigation.

We finally conclude that a semi-blind algorithm can easily outperform the training based equalizers for any wireless scenario (slow fading channel).

Training Equalizers

Training based equalizers are most commonly used even for a wireless system. They are still optimal as long one just considers the performance of the equalizer as a standalone component. Hence it is important to understand their (tracking) performance in a wireless environment. Also, some of the training equalizers (e.g. MMSE DFE) are not well understood theoretically even when working with a time invariant channel. The second part of our thesis concentrates on these issues.

Fixed Channel

For a fixed channel using implicit function theorem, we obtained the existence of DD-attractors (decision directed attractors) close to the WF at high SNRs (Chapter 5). We used similar techniques and compared the LMS-DFE attractors with that of the DFE Wiener filters at high SNRs (Chapter 6). Our conclusions for a fixed channel can be summarized as:

- A decision directed LMS linear equalizer converges close to the WF whenever the SNR is high and when it is properly initialized (using training sequence). Hence, we conclude that a DD-LMS-LE can be used to obtain the WF under high SNR conditions (after using a small amount of training sequence to 'open the eye'). However, at low SNRs, the DD attractors are away from the WF.

- A training based LMS decision feedback equalizer converges close to the WF at high SNRs. Thus, we conclude that at least under high SNR, LMS can be used to obtain the optimal DFE, i.e., the DFE Wiener filter. We also show that the 'Optimal' DFE obtained by ignoring the decision errors can perform much worse than the LMS DFE (and DFE-WF) even under high SNR (even after designing the former with perfect channel estimate).

Wireless Channel

The tracking behavior of a wireless component (e.g., channel estimator) is obtained till now, by theoretically modeling the wireless channel either as a first order AR process (e.g., Random Walk model) or as a deterministic periodic process. Block fading model is also used to study a slow fading channel. However, an AR(2) process models a fading channel better and is sufficient most of the times while considering the receiver design. We used an AR(2) process to model the wireless channel while studying the tracking performance of various equalizers.

We first obtained an ODE approximation for an AR(2) process (Chapter 2). Using this, we showed that an AR(2) process can be approximated with an exponential, cosine, polynomial, hyperbolic or an exponentially raising/decaying cosine or hyperbolic waveforms.

We then obtained an ODE approximation for a general system, whose components may depend on two previous values (like in an AR(2) process). Using this, we obtained the ODE approximation for an LMS based LE, DD-LE or DFE while tracking an AR(2) process. Using this ODE approximation (also using the fixed channel analysis) we concluded the following :

- A training based LMS-LE tracks the instantaneous WF (Chapter 4).
  - The error between the instantaneous WF and the LMS trajectory is shown to reduce polynomially/exponentially to zero with time for stable/unstable channels.
  - The error remains bounded for a marginally stable channel. Via, some examples, we show that the LMS-LE tracks even a marginally stable channel quite reasonably (the error does not tend to zero but remains bounded).
  - For stable channels, the MSE of an LMS-LE converges to the instantaneous MMSE exponentially.
- A decision directed LMS-LE can be used to track the instantaneous WF only under high SNR conditions, after proper initialization using training sequence (Chapter 5).
  - A decision directed LMS linear equalizer stays close to the instantaneous WF whenever the SNR is high and when it is properly initialized.
  - At low SNRs, the DD attractors are constantly away from the instantaneous WF (as in case of the fixed channel).

- A training based LMS decision feedback equalizer can be used to track the DFE-WF (Chapter 7).
  - It stays close to the instantaneous WF at all time, after an initial transience, at high SNR (actually at all practical SNRs).
  - The performance of an LMS-DFE (the BER and the MSE) is close to that of the DFE-WF at all time once again after the initial transience.
  - Further the BER and MSE performance of an 'Optimal' DFE obtained by ignoring the decision errors stays constantly inferior to that of the LMS DFE (and DFE-WF) at all time, after an initial transient period (even after designing the former with perfect channel estimate).

New Results

In achieving the above results, we obtained new general results, which can be useful in other applications.

In Appendix I, we obtained an ODE approximation for an adaptive system whose components may depend on two previous values (Theorems A.1 and A.2 of Appendix I). This is the first time that such an ODE approximation is obtained. Unlike the available ODE approximations, a second order ODE also approximates the adaptive system depending on the system parameters.

In Chapter 6, we obtain differentiability of the stationary density of a Markov Chain with respect to a parameter (which parameterizes the Markov chain) in L_2 norm (Theorem 6.2). We used Implicit function theorem to obtain this. In recent years a great deal of attention has been devoted to the computation of derivatives of performance indicators in stochastic systems under stationarity. The differentiability of the stationary density (in L_2 norm) allows the computation of derivatives of performance indicators that satisfy the hypothesis of Lemma 6.2 of Chapter 6. Moreover, the computation of derivatives allows one to take an additional step and develop optimization procedures for the performance indicator of interest (e.g., the MSE optimal DFE).

Future Directions

We compared the blind/semi-blind, CMA, algorithm with training based algorithm using Information theoretic arguments in Chapter 3. One can try comparing the DD-LMS with blind/semi-blind CMA (also with training equalizer) using capacity as measure. Using this one can try understanding when a DD-LMS is better than a CMA. We are currently working towards this problem.

We considered a block-fading model for the wireless channel, while comparing the three equalizers. This model only covers a slow fading channel. One can instead compare the three algorithms after modeling the wireless channel as an AR(2) process.

We obtained an ODE approximation for AR(2) process. Here we assumed the AR parameters to be fixed. A more general model is obtained by assuming some dynamics for the AR parameters also. We will attempt getting an ODE approximation for this model and use it for obtaining a better theoretical tracking performance. Further, we suggested an ODE approximation for AR(p) process. A higher order AR process better approximates the wireless channel and hence one can attempt obtaining an ODE approximation for higher order AR processes.

We showed that the error between the LMS-LE and the instantaneous WF converges to zero for stable/unstable channels. We also showed that the LMS-LE tracks the WF for a marginally stable channel (only via some examples, theoretically we could just say that the error remains bounded). One can try to bring in the concept of degree of non-stationarity ([36]) in this regard and obtain a better understanding of the tracking performance. We are currently working towards this.

For an LMS-DFE and DD-LMS-LE tracking an AR(2) process, we obtained an ODE approximation and used it to draw some interesting conclusions via examples. One can try concluding the same directly using the ODEs like in the case of an LMS-LE.

One could try extending the DD-LMS-LE result to a DFE (i.e., one can try understanding the relation between DD-LMS-DFE attractors and the DFE-WFs).

The ODE approximation we obtained were for finite time intervals. It would be good to obtain these for the infinite time intervals.

For comparing blind/semi-blind and training equalizers we considered MIMO fading channels. Our results for LMS equalizers should also be extended to MIMO systems. We are currently working on this problem.

Appendix I : ODE approximation of a General System

We consider the following general system,

$$Z_{k+1} = (1-d_2)Z_k + d_2 Z_{k-1} + \mu H(Z_k, W_k), \quad (1)$$

$$\theta_{k+1} = \theta_k + \mu H_1(Z_k, \theta_k, G_{k+1}), \quad (2)$$

where equation (1) satisfies all the conditions in **A**.1–**A**.3 of Chapter 2 and the equation (2) satisfies the assumptions **B**.1–**B**.4 given in the next para. We will show that the above equations can be approximated by the solution of the ODE's,

$$\begin{aligned}
(1+d_2)\,\dot{Z}\,(t) &= h(Z(t)), \quad \text{if } d_2 \in (-1,1], \\
\frac{d^2 Z(t)}{dt^2} &= h(Z(t)), \quad \text{if } d_2 = -1, \\
\frac{d^2 Z(t)}{dt^2} + \eta_1\,\dot{Z}\,(t) &= h(Z(t)), \quad \text{if } d_2 \text{ is close to } -1, \quad (3)
\end{aligned}$$

$$\dot{\theta}\,(t) = h_1(Z(t), \theta(t)), \quad (4)$$

where the function h_1 is defined in the assumptions given below and $h(Z) = E[H(Z,W)]$, with $\eta_1 = \frac{1+d_2}{\sqrt{\mu}}$. We use this result to obtain the ODE approximation for an LMS (LE, DDLE or DFE) tracking an AR(2) process.

We make the following assumptions, which are similar to that in [4]. Let D be an open subset of $\mathcal{R}^d$.

B.1 There exists a family $\{P_{Z,\theta}\}$ of transition probabilities $P_{Z,\theta}(G, \mathbf{A})$ such that, for any Borel subset $\mathbf{A}$ we have

$$P[G_{n+1} \in \mathbf{A}|\mathcal{F}_n] = P_{Z_n,\theta_n}(G_n, \mathbf{A})$$

where $\mathcal{F}_k \stackrel{\triangle}{=} \sigma(\theta_0, Z_0, Z_1, W_1, W_2, \cdots, W_k, G_0, G_1, \cdots, G_k)$. This in turn implies that the tuple $(G_k, \theta_k, Z_k, Z_{k-1})$ forms a Markov chain.

B.2 For any compact subset Q of D, there exist constants C_1, q_1 such that for all $(Z, \theta) \in D$ we have

$$|H_1(Z, \theta, G)| \;\; \leq \;\; C_1(1 + |G|^{q_1}).$$

B.3 There exists a function h_1 on D, and for each $\theta, Z \in D$ a function $\nu_{Z,\theta}(.)$ such that

(a) h_1 is locally Lipschitz on D.

(b) $(I - P_{\theta,Z})\nu_{\theta,Z}(G) = H_1(\theta, Z, G) - h_1(\theta, Z)$.

(c) For all compact subsets Q of D, there exist constants C_3, C_4, q_3, q_4 and $\lambda \in [0.5, 1]$, such that for all $\theta, Z, \theta', Z' \in Q$

i. $|\nu_{\theta,Z}(G)| \leq C_3(1 + |G|^{q_3})$,

ii. $|P_{\theta,Z}\nu_{\theta,Z}(G) - P_{\theta',Z'}\nu_{\theta',Z'}(G)| \leq C_4 \left(1 + |G|^{q_4}\right) \left|(\theta, Z) - (\theta', Z')\right|^{\lambda}$.

B.4 For any compact set Q in D and for any $q > 0$, there exists a $\mu_q(Q) < \infty$, such that for all $n, G,\ A = (Z, \theta) \in \mathcal{R}^d$

$$E_{G,A}\left\{I(\theta_k, Z_k \in Q, k \leq n)\left(1 + |G_{n+1}|^q\right)\right\} \;\; \leq \;\; \mu_q(Q)\left(1 + |G|^q\right),$$

where $E_{G,A}$ represents the expectation taken with $G_0, \theta_0, Z_0, = G, Z, \theta$.

Let $Z(t, t_0, Z), \theta(t, t_0, \theta)$ represent the solutions of the ODE's (3), (4) with initial conditions $Z(t_0) = Z$, $\theta(t_0) = \theta$. For second order ODEs, the additional initial condition is given by $\dot{Z}\,(t_0) = 0$. Let Q_1 and Q_2 be any two compact subsets of D, such that $Q_1 \subset Q_2$ and we can choose a $T > 0$ such that, there exist an $\delta_0 > 0$ satisfying

$$d\left((Z(t,0,Z), \theta(t,0,\theta)), Q_2^c\right) \geq \delta_0, \tag{5}$$

for all $(Z, \theta) \in Q_1$ and all t, $0 \leq t \leq T$.

We prove Theorem A.1, following the approach used in [4] and using the ODE approximations of Chapter 2.

Theorem A.1 *Assume,* $E|H(Z,W)|^4 \leq C_1(Q)$ *for all* Z *in any given compact set* Q *of* D*. Also assume* **A**.*1*–**A**.*3 of Chapter 2 and* **B**.*1*–**B**.*4. Furthermore, pick compact sets* Q_1*,* Q_2*, and positive constants* T*,* δ_0 *satisfying (5). Then for all* $\delta \leq \delta_0$ *and for any initial condition* G*, with* $Z_{-1} = Z_0 = Z$*,* $\dot{Z}\,(t_0) = 0$ *(whenever* $Z(.,.,.)$ *is solution of a second order ODE), and* $\theta_0 = \theta$*,*

$$P_{G,Z,\theta}\left\{ \sup_{1 \leq k \leq \left\lfloor \frac{T}{\mu^\alpha} \right\rfloor} \left|(Z_k, \underline{\theta}_k) - (Z(k\mu^\alpha, 0, Z), \underline{\theta}(k\mu, 0, \theta))\right| \geq \delta \right\} \to 0$$

as $\mu \to 0$ *uniformly for all* $Z, \theta \in Q_1$*. If* $Z(.,.,.)$ *is solution of a first order ODE then* $\alpha = 1$ *and otherwise* $1/2$*.*

Proof : For $d_2 \in (-1, 1]$, the proof is presented in Appendix II.

With $d_2 = -1$, following the steps as in the above case and using Theorem 2.2 of Chapter 2, we can show that the general system ((1), (2)) is approximated by the ODEs ((3), (4)) (we are not repeating those steps here).

With d_2 close to -1, the theorem follows by following the steps as in the case of $d_2 \in (-1, 1]$ and using Theorem 2.3 of Chapter 2. ■

Now we replace the assumption **B.3**.c.ii with the following
B.3.c.ii$'$: $E_{G,A}\{\left|P_{\theta_k, Z_k,}\nu_{\theta_k, Z_k,}(G_k) - P_{\theta_k, Z_k,}\nu_{\theta_{k-1}, Z_{k-1},}(G_k)\right|^2 I(k < \tau(Q))\} \leq C_4\,(1 + |G|^{q_4})\,\mu^\lambda$.
This result is used for approximating an LMS-DFE in Chapter 7.

Theorem A.2 *Assume,* $E|H(Z,W)|^4 \leq C_1(Q)$ *for all* Z *in any given compact set* Q *of* D*. Also assume* **A**.*1*–**A**.*3 of Chapter 2 and* **B**.*1*–**B**.*4 after replacing the assumption* **B.3**.*c.ii with* **B.3**.*c.ii*$'$*. Furthermore, pick compact sets* Q_1*,* Q_2*, and positive constants* T*,* δ_0 *satisfying (5). Then for all* $\delta \leq \delta_0$ *and for any initial condition* G*, with* $Z_{-1} = Z_0 = Z$*,* $\dot{Z}\,(t_0) = 0$ *(whenever* $Z(.,.,.)$ *is solution of a second order ODE), and* $\theta_0 = \theta$*,*

$$P_{G,Z,\theta}\left\{\sup_{1\leq k\leq \lfloor \frac{T}{\mu^\alpha}\rfloor} |(Z_k, \underline{\theta}_k) - (Z(k\mu^\alpha, 0, Z), \underline{\theta}(k\mu, 0, \theta))| \geq \delta\right\} \rightarrow 0$$

as $\mu \rightarrow 0$ *uniformly for all* $Z, \theta \in Q_1$*. If* $Z(.,.,.)$ *is solution of a first order ODE then* $\alpha = 1$ *and otherwise* $1/2$*.*

Proof : This theorem is similar to Theorem A.1. The only change being that assumption **B**.3(c)ii is replaced with **B**.3(c)ii$'$.

Only the proof of Lemma A.1, of Appendix II changes because of the change in the above assumption. Here equation (9) is directly given by the new Assumption **B.**3.c.ii$'$ and hence the statement of Lemma A.1 gets modified as,

$$E\mathcal{U}_1^2 \leq K\mu^\lambda \left(1 + |G|^s\right).$$

This does not alter the proof of the theorem. ■

Appendix II

Proof of Theorem A.1:

If the adaptation (1) does not contain the term Z_{k-1}, then we can combine the two adaptations as a single vector and consider the single adaptation and provide convergence to a combined ODE. Here things are not straight forward as the equation (1) contains Z_{k-1}. But by modifying the proof given in Benveniste et al. [1] at a few places, we will show that the equations (1), (2) can be approximated by the solution of the required system of the ODE's. The proof also uses the intermediate results given in Theorem 2.1 of Chapter 2.

The expectations are with respect to G, Z, θ and we use a simpler notation E in place of $E_{G,Z,\theta}$.

All the terms like $\epsilon_k, \alpha_k, \tau$ etc, used in Theorem 2.1 will also be used here once again for the AR process with the same meaning and the equivalent terms for the equalizer are defined as $\epsilon_{1k}, \alpha_{1k}, \tau_1$ etc.

Define $\tau_1 = \inf\{n \leq \left\lfloor \frac{T}{\mu} \right\rfloor : (\theta_n, Z_n) \in Q_2^c\}$, $t_n = n\mu$. Let $\theta(t)$ denote solution $\theta(t, 0, \theta)$ with $Z(0) = Z$. Then from equation (4), using Taylor series expansion and assumption **B**.3 a,

$$\theta(t_{k+1}) = \theta(t_k) + \mu h_1(Z(t_k), \theta(t_k)) - \alpha_{1k},$$

where $|\alpha_{1k}| \leq B_1\mu^2$, for some finite constant $B_1(Q_2)$. Define the error process,

$$\begin{aligned} \epsilon_{1n} &\triangleq \theta_{n+1} - \theta_n - \mu h_1(Z_n, \theta_n) \\ &= \mu \left[H_1(Z_n, \theta_n, G_{n+1}) - h_1(Z_n, \theta_n) \right]. \end{aligned}$$

Then for any $k < \tau_1$,

$$\begin{aligned} \theta_k - \theta(t_k) &= \theta_{k-1} - \theta(t_{k-1}) + \mu \left[h_1(Z_{k-1}, \theta_{k-1}) - h_1(Z(t_{k-1}), \theta(t_{k-1})) \right] \\ &\quad + \epsilon_{1k-1} + \alpha_{1k-1}. \end{aligned}$$

Replacing $\theta_{k-1} - \theta(t_{k-1})$ by the previous equation again and continuing we get,

$$\begin{aligned} \theta_k - \theta(t_k) &= \theta_0 - \theta(0) + \mu \sum_{i=0}^{k-1} \left[h_1(Z_i, \theta_i) - h_1(Z(t_i), \theta(t_i)) \right] \\ &\quad + \sum_{i=0}^{k-1} \epsilon_{1i} + \alpha_{1i}. \end{aligned}$$

Using assumption **B**.3 and the bound on α_{1k}, we get on the set $1 \leq k < \tau_1$,

$$|\theta_k - \theta(t_k)| \leq \mu B_2 \sum_{i=0}^{k-1} |(\theta_i, Z_i) - (\theta(t_i), Z(t_i))| + \left| \sum_{i=0}^{k-1} \epsilon_{1i} \right| + kB_1\mu^2,$$

where once again the constant B_2 depends upon Q_2. All the constants introduced henceforth will depend upon the compact set Q_2, but we will not mention that repeatedly. Now combining with step (2.16) of Theorem 2.1, for any $2 \leq k \leq \frac{T}{\mu}$ on the set $1 \leq k < \tau_1$,

$$\begin{aligned} |(\theta_k, Z_k) - (\theta(t_k), Z(t_k))| &= |\theta_k - \theta(t_k)| + |Z_k - Z(t_k)| \\ &\leq K_1|(\theta_1, Z_1) - (\theta(t_1), Z(t_1))| + K_2|(\theta_2, Z_2) - (\theta(t_2), Z(t_2))| \\ &+\mu B_3 \sum_{i=2}^{k-1} |(\theta_i, Z_i) - (\theta(t_i), Z(t_i))| + \mathcal{U} + \mathcal{U}_1 + K_3\mu T, \end{aligned}$$

where $K_1 = max\{\frac{2a_1}{1+d_2}, \mu B_2\}$, $K_2 = max\{\frac{2a_2}{1+d_2}, \mu B_2\}$, $K_3 = max\{\frac{2L}{1+d_2}, B_1\}$, $B_3 = max\{\frac{2L_1(Q_2)}{1+d_2}, B_2\}$ $(a_1, a_2, L_1(Q_2), L$ and $\mathcal{U}$ are defined in step (2.16) of Theorem 2.1) and $\mathcal{U}_1$ is defined by,

$$\mathcal{U}_1 = \sup_{m \leq \frac{T}{\mu}} \mathbf{1}\{m < \tau_1\} \left| \sum_{i=0}^{m-1} \epsilon_{1i} \right|.$$

Using the upper bound (2.13) of Theorem 2.1,

$$\begin{aligned} |(\theta_1, Z_1) - (\theta(t_1), Z(t_1))| &= |\theta_1 - \theta(t_1)| + |Z_1 - Z(t_1)| \\ &\leq \mu M_1. \end{aligned}$$

Using Lemma 2.1 on the set $\{1 \leq k < \tau_1\}$ for any k, with $1 \leq k \leq \frac{T}{\mu}$,

$$|(\theta_k, Z_k) - (\theta(t_k), Z(t_k))| \leq [\mathcal{U} + \mathcal{U}_1 + K_3\mu T + \mu M_1]e^{\{K_1 + K_2 + TB_3\}}.$$

Hence,

$$\sup_{k<\tau_1} |(\theta_k, Z_k) - (\theta(t_k), Z(t_k))|^2 \leq 4[\mathcal{U}^2 + \mathcal{U}_1^2 + K_3^2\mu^2T^2 + \mu^2M_1^2] \\ e^{2\{K_1+K_2+TB_3\}}.$$

We will show below in Lemma A.1 that $E\mathcal{U}_1^2 \leq K\sqrt{\mu}(1+|G|^s)$. Thus, using the bound obtained on $\mathcal{U}$ in Theorem 2.1,

$$E\left[\sup_{1\leq k<\tau_1} |(\theta_k, Z_k) - (\theta(t_k), Z(t_k))|^2\right] \leq B\sqrt{\mu}(1+|G|^s),$$

where constant B depends upon Q_2, d_2, T. Then the theorem follows by Chebyshev's inequality for all $\delta < \delta_0$ (Once again the logic applied in the last step of Theorem 2.1 has to be used here). ■

Lemma A.1 *There exist constants K, s such that,* $E\mathcal{U}_1^2 \leq K\sqrt{\mu}(1+|G|^s)$.

Proof : This almost follows from Proposition 7, p.228 of [4], when θ is replaced with (θ, Z) and their assumptions are replaced with Assumptions **B**.1–**B**.4 and when $\gamma_k = \mu$ for all k. The only change that needs to be made is in proving Lemma 3 on page 225 in [4], where we need to get a bound on $E|Z_k - Z_{k-1}|^4$.

Towards this end, define the adaptive equation,

$$Z_k^{'} = Z_{k-1}^{'} + \mu\frac{H(Z_{k-1}^{'}, W_k)}{1+d_2},$$

using same W_k as in (1), such that its trajectory converges to the ODE (3). Then for any $k < \tau_1$,

$$\begin{aligned} |Z_k - Z_{k-1}| &\leq |Z_k - Z_k^{'}| + |Z_k^{'} - Z_{k-1}^{'}| + |Z_{k-1} - Z_{k-1}^{'}| \\ &\leq |Z_k - Z(t_k)| + |Z_k^{'} - Z(t_k)| + |Z_k^{'} - Z_{k-1}^{'}| \\ &+ |Z_{k-1} - Z(t_{k-1})| + |Z_{k-1}^{'} - Z(t_{k-1})|. \end{aligned}$$

Therefore, using the steps as in Theorem 2.1 and using, the extra assumption $E|H(Z,W)|^4 \leq$

$C_1(Q_2)$ for all $Z \in Q_2$, the Lemma A.2 and the fact that $Z_k^{'}$ can also be approximated with the same ODE solution $Z(t_k)$ we get,

$$E[\ \mathbf{1}\{k < \tau_1\}|Z_k - Z_{k-1}|^4] \ \ \leq N\mu. \tag{6}$$

In the steps to follow, we use the notations and constants in [1], except that, we use $\epsilon_{1k}^{(2)}$ in place of $\epsilon_k^{(2)}$ (defined on page 222 of [4]) and (θ, Z) in place of θ. Let $\phi(.)$ be a C^2 function of θ into $\mathcal{R}$ with bounded second derivatives (We will be using $\phi(.)$ as one of the co-ordinate functions for θ). With $\psi_{\theta,Z}(G) = \phi^{'}(\theta)P_{\theta,Z}\nu_{\theta,Z}(G)$,

$$\begin{aligned}
\epsilon_{1k}^{(2)} &\triangleq \mu\left[\psi_{\theta_k,Z_k,}(G_k) - \psi_{\theta_{k-1},Z_{k-1},}(G_k)\right] \\
&= \mu\phi^{'}(\theta_k)\left[P_{\theta_k,Z_k,}\nu_{\theta_k,Z_k,}(G_k) - P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k)\right] \\
&\quad +\mu\left(\phi^{'}(\theta_k) - \phi^{'}(\theta_{k-1})\right)P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k).
\end{aligned}$$

Lemma 3 of [4] is proved if we show that, for some positive constants $K^{'}$ and $s^{'}$, and for all m, $E_{G,Z,\theta}\left\{\sum_{k=1}^{m\wedge\tau_1-1}|\epsilon_k^{(2)}|\right\}^2 \leq K^{'}\sqrt{\mu}(1+|G|^{s^{'}}$.

Towards this end, we will first get an upper bound for the second term of $\epsilon_{1k}^{(2)}$ on the set $\{k < \tau_1\}$. Choose $M_2 < \infty$ such that $\sup_\theta|\phi^{''}(\theta)| \leq M_2$. Then,

$$\left|\phi^{'}(\theta_k) - \phi^{'}(\theta_{k-1})\right| \leq M_2|\theta_k - \theta_{k-1}|.$$

From equation (2), using **B**.2

$$|\theta_k - \theta_{k-1}|^2 \leq N_3\mu^2(1+|G_k|^{2q_1}), \tag{7}$$

where N_3 depends upon C_1. Using **B**.3 and **B**.4 on the set $\{k < \tau_1\}$,

$$\begin{aligned}
|P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k)| &= |E_{G_k,Z_{k-1},\theta_{k-1}}\left(\nu_{Z_{k-1},\theta_{k-1}}(G_1)\right)| \\
&\leq C_3E_{G_k,\theta_{k-1},Z_{k-1},}(1+|G_1|^{q_3}) \\
&\leq C_3\mu_{q_3}(Q_2)(1+|G_k|^{q_3}).
\end{aligned}$$

Therefore using **B**.4 again,

$$\begin{aligned}
E_{G,Z,\theta} \Big| \ \mathbf{1}\{k<\tau_1\} \left(\phi^{'}(\theta_k)-\phi^{'}(\theta_{k-1})\right) P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k)\Big|^2 \\
&\leq \ \mu^2 M_2^2 N_3 C_3^2 \mu_{q_3}(Q_2)^2 E_{G,Z,\theta}\left[\ \mathbf{1}\{k<\tau_1\}\left(1+|G_k|^{2q_3+2q_1}\right)\right] \\
&\leq \ N_4\mu^2(1+|G|^{s_1}), \qquad (8)
\end{aligned}$$

for appropriate constants N_4, s_1. Once again, using assumption B.3 and equation (7), we have

$$\begin{aligned}
\Big|P_{\theta_k,Z_k,}\nu_{\theta_k,Z_k,}(G_k) - P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k)\Big|^2 \\
&\leq \ C_4^2|(\theta_k,Z_k,)-(\theta_{k-1},Z_{k-1},)|^{2\lambda}(1+|G_k|^{2q_4}) \\
&\leq \ C_4^2 N_3\mu^{2\lambda}(1+|G_k|^{2q_1+\lambda 2q_4}) + C_4^2|Z_k-Z_{k-1}|^{2\lambda}(1+|G_k|^{2q_4}).
\end{aligned}$$

Using Cauchy Schwartz inequality and since $4\lambda \leq 4$,

$$\begin{aligned}
\left(E_{G,Z,\theta}\left[\ \mathbf{1}\{k<\tau_1\}|Z_k-Z_{k-1}|^{2\lambda}(1+|G_k|^{2q_4})\right]\right)^2 \\
&\leq \ E[\ \mathbf{1}\{k<\tau_1\}|Z_k-Z_{k-1}|^{4\lambda}]E_{G,Z,\theta}[\ \mathbf{1}\{k<\tau_1\}(1+|G_k|^{4q_4})] \\
&\leq \ N_5 E[\ \mathbf{1}\{k<\tau_1\}|Z_k-Z_{k-1}|^{4}](1+|G|^{4q_4}) \leq N_5\mu(1+|G|^{4q_4}).
\end{aligned}$$

Therefore,

$$E_{G,Z,\theta}\left[\ \mathbf{1}\{k<\tau_1\}|Z_k-Z_{k-1}|^{2\lambda}(1+|G_k|^{2q_4})\right] \leq N_6\sqrt{\mu}(1+|G|^{2q_4}).$$

Thus as $2\lambda \geq 1$,

$$\begin{aligned}
E_{G,Z,\theta}\left[\ \mathbf{1}\{k<\tau_1\}\left|P_{\theta_k,Z_k,}\nu_{\theta_k,Z_k,}(G_k) - P_{\theta_{k-1},Z_{k-1},}\nu_{\theta_{k-1},Z_{k-1},}(G_k)\right|^2\right] \\
&\leq \ N_7\sqrt{\mu}(1+|G|^{2q_4+2q_1}). \qquad (9)
\end{aligned}$$

Using (8), (9) we finally get for any $m \leq \frac{T}{\mu}$,

$$
\begin{aligned}
E_{G,Z,\theta}\left\{\sum_{k=1}^{m\wedge\tau_1-1}|\epsilon_k^{(2)}|\right\}^2 &\leq E_{G,Z,\theta}\left\{\sum_{k=1}^{m}|\epsilon_k^{(2)}|\ \mathbf{1}\{k<\tau_1\}\right\}^2 \\
&\leq m\sum_{k=1}^{m}E_{G,Z,\theta}\left\{|\epsilon_k^{(2)}|\ \mathbf{1}\{k<\tau_1\}\right\}^2 \\
&\leq m^2\mu^2\sqrt{\mu}\ N_8(1+|G|^{s'}) \\
&\leq N_8T^2\sqrt{\mu}(1+|G|^{s'}) \leq K'\sqrt{\mu}(1+|G|^{s'}),
\end{aligned}
$$

where the constant $s' = max\{2q_1+2q_4, 2q_1+2q_3\}$. ■

Lemma A.2 *For some constant $\tilde{N} < \infty$,*

$$E|\mathcal{U}|^4 \leq \tilde{N}\mu.$$

Proof: For proving the lemma, we need to deviate from **Step** 2 of Theorem 2.1 only at the point, where Doob's martingale inequality is applied to K_k. We neglect constants $\left(1-[-d_2]^{i+1}\right)$ in the following arguments to simplify the steps. The same logic holds even if the constants are included and only the constant will vary.

We once again apply Doob's martingale inequality and obtain,

$$E\left\{\sup_{k\leq m}|K_k|^4\right\} \leq N_1\sup_{k\leq m}E|K_m|^4,$$

for some constant N_1. Thus we have,

$$
\begin{aligned}
E|\mathcal{U}|^4 &\leq N_1\sup_{k\leq\frac{T}{\mu}}E|K_k|^4 \\
&\stackrel{a}{=} N_1\sup_{k\leq\frac{T}{\mu}}\sum_{i=1}^{k-1}E\left|\ \mathbf{1}\{i<\tau\}\epsilon_i\right|^4 \\
&\quad +N_2\sup_{k\leq\frac{T}{\mu}}\sum_{j,i=1,j\neq i}^{k-1}E\left[\ \mathbf{1}\{j<\tau, i<\tau\}\,|\epsilon_i|^2\,|\epsilon_j|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
& +N_3 \sup_{k \le \frac{T}{\mu}} \sum_{j,i=1, j\neq i}^{k-1} E\left[\ \mathbf{1}\{j<\tau, i<\tau\}\epsilon_j^T \epsilon_i \epsilon_i^T \epsilon_j\right] \\
& +N_4 \sup_{k \le \frac{T}{\mu}} \sum_{l,j,i=1, l\neq j\neq i\neq l}^{k-1} E\left[\ \mathbf{1}\{j<\tau, i<\tau, l<\tau\} \left|\epsilon_i\right|^2 \epsilon_j^T \epsilon_l\right] \\
& +N_5 \sup_{k \le \frac{T}{\mu}} \sum_{l,j,i=1, l\neq j\neq i\neq l}^{k-1} E\left[\ \mathbf{1}\{j<\tau, i<\tau, l<\tau\}\epsilon_l^T \epsilon_i \epsilon_i^T \epsilon_j\right] \\
\stackrel{b}{\le}\ & \tilde{N}_1 \mu^3 T + \tilde{N}_2 \mu^2 T^2 + \tilde{N}_3 \mu^2 T^2 + N_4 \mu T^3 + N_5 \mu T^3
\end{aligned}
$$

where the equality a follows, as for any $j_1 \neq j_2 \neq j_3 \neq j_4$ just like in **Step 2** of Theorem 2.1,

$$
E\left[\ \mathbf{1}\{j_1<\tau, j_2<\tau, j_3<\tau, j_4<\tau\}\epsilon_{j_1}^T \epsilon_{j_2} \epsilon_{j_3}^T \epsilon_{j_4}\right] = 0.
$$

Inequality b holds by Assumption $\mathbf{A}$.2 and because $E|H(Z,W)|^4 \le C_1(Q_2)$ for all $Z \in Q_2$.
■

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