IE605: Engineering Statistics

Lecture 14

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Previous Lecture:

- Hypothesis testing
- Likelihood Ratio Test (LRT)

This Lecture:

- Method of Evaluating Tests
- Power Functions
- Uniformly Most Powerful (UMP) test
- Neyman-Pearson Lemma

Error in Hypothesis Testing

$$H_0 : \theta \in \Theta_0 \quad \text{null hypotesis}$$
$$H_1 : \theta \in \Theta_0^c \quad \text{alternate hypotesis}$$

- Rejection set $\mathcal{R} = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\}$. If $\boldsymbol{x} \in \mathcal{R}$ hypothesis H_0 is rejected, otherwise H_1 is accepted
- In accepting or rejecting hypothesis the experimenter could be making mistake
- How to control the error?

Types of Error

There could be two types of error

- Type I error: If $\theta \in \Theta_0$ (H_0 is true), but the hypothesis test incorrectly **rejects** the null hypotesis H_0
- Type II error: If $\theta \in \Theta_0^c$ (H_1 is true), but the hypothesis test incorrectly **accepts** the null hypotesis H_0

Truth/Decision	Accept	Reject
H_0	Correct	**Type I error**
H_1	**Type II error**	Correct

Power Function

- If $\theta \in \Theta_0$, $P_\theta(\boldsymbol{X} \in \mathcal{R})$ gives probability of Type I error
- If $\theta \in \Theta_0^c$, $1 - P_\theta(\boldsymbol{X} \in \mathcal{R})$ gives probability of Type II error.

> Power Function of a hypothesis test with rejection region $\mathcal{R}$ is a function of θ given by $\beta(\theta) = P_\theta(\boldsymbol{X} \in \mathcal{R})$

For an 'ideal' hypothesis test

- waht should be the value of $P_\theta(\boldsymbol{X} \in \mathcal{R})$ iif $\theta \in \Theta_0$?
- What should be the value of $P_\theta(\boldsymbol{X} \in \mathcal{R})$ if $\theta \in \Theta_0^c$?
- The ideal case cannot be attained in practical problems
- A good hypothesis test should be such that $P_\theta(\boldsymbol{X} \in \mathcal{R}) \to 1$ when $\theta \in \Theta_0$ and $P_\theta(\boldsymbol{X} \in \mathcal{R}) \to 0$ when $\theta \in \Theta_0^c$

Type I vs Type II error

Example: $X \in Bin(5, \theta)$

$$H_0 : \theta \leq 1/2 \quad \text{null hypotesis}$$
$$H_1 : \theta > 1/2 \quad \text{alternate hypotesis}$$

- Test 1: Reject H_0 iff all success are observed

$$\beta_1(\theta) = \Pr(\{1,1,1,1,1\}) = \theta^5$$

  Type 1 error: $\beta_1(\theta) \leq (1/2)^5 = 0.0312$ for all $\theta \leq 1/2$
  Type II error: $1 - \beta_1(\theta) > 0.87$

- Test 2: Reject H_0 if $3, 4, 5$ success are observed

$$\beta_2(\theta) = \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{3}\theta^4(1-\theta) + \theta^5$$

Example:

Let $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n\}$ is iid drwan from population $\mathcal{N}(\theta, \sigma^2)$ with known σ^2. We want to test the hypothesis

$$\begin{aligned} H_0 &: \theta \leq \theta_0 \quad \text{null hypotesis} \\ H_1 &: \theta > \theta_0 \quad \text{alternate hypotesis} \end{aligned}$$

$$\mathcal{R} = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\} = \left\{\boldsymbol{x} : \frac{\bar{\boldsymbol{x}} - \theta_0}{\sigma/\sqrt{n}} \geq c\right\}$$

$$\begin{aligned} \beta(\theta) &= P\left(\frac{\bar{\boldsymbol{x}} - \theta_0}{\sigma/\sqrt{n}} \geq c\right) \\ &= P\left(\frac{\bar{\boldsymbol{x}} - \theta}{\sigma/\sqrt{n}} \geq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\ &= P\left(Z \geq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$.

Example Contd.

Suppose we want Type I error to be at most 0.1 and Type II error to be at most 0.2 when $\theta > \theta_0 + \sigma$. What should be the value of c and n?

$$\beta(\theta) = P\left(Z \geq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

- Increasing in θ
- set $\beta(\theta_0) = 0.1$ and $\beta(\theta_0 + \sigma) = 0.2$

- For a fixed sample, it is impossible to make both types of errors arbitrarily small.
- To find a good test, it is common to restrict Type I at a fixed level.

Size and Levels

Definition: For $0 \leq \alpha \leq 1$, a test with power function $\beta(\theta)$ is a size α test if $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$.

Definition: For $0 \leq \alpha \leq 1$, a test with power function $\beta(\theta)$ is a level α test if $\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$.

- Set of level α tests contains set of size α tests
- By fixing α we are only controlling Type I Error, not Type II error.

Unbiased Test

A test with power function $\beta(\theta)$ is unbiased if we have $\beta(\theta') \geq \beta(\theta'')$ for any $\theta' \in \Theta_0^c$ and $\theta'' \in \Theta_0$

Example: Let $\boldsymbol{X} = \{X_1, X_2, \ldots, X_n\}$ is iid drwan from population $\mathcal{N}(\theta, \sigma^2)$ with known σ^2. We want to test the hypothesis

$$H_0 : \theta \leq \theta_0 \quad \text{null hypotesis}$$
$$H_1 : \theta > \theta_0 \quad \text{alternate hypotesis}$$

$$\beta(\theta) = P\left(Z \geq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

Since $\beta(\theta)$ is increasing. It is clear that $\beta(\theta)$ is unbiased.

Most Powerful test

- There can be class of level α tests
- Level α tests have type I Error probability at most α in $\theta \in \Theta_0$
- A good test in such a class would also have a small Type II Error probability

Definition: Let $\mathcal{C}$ be a class of test for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$. A test in class $\mathcal{C}$ with power function $\beta(\theta)$ is a uniformly most powerful (UMP) class $\mathcal{C}$ test if for every power function $\beta'(\theta)$ in $\mathcal{C}$ we have $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$.

Neyman-Pearson Lemma

Lemma: Consider $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Using a rejection region $\mathcal{R}$ that satsifies

$$\begin{aligned} \boldsymbol{x} \in \mathcal{R} \quad &\text{if} \quad f(\boldsymbol{x}|\theta_1) > \kappa f(\boldsymbol{x}|\theta_0) \\ \boldsymbol{x} \in \mathcal{R}^c \quad &\text{if} \quad f(\boldsymbol{x}|\theta_1) < \kappa f(\boldsymbol{x}|\theta_0) \end{aligned}$$

for some $\kappa \geq 0$, and Let $\alpha = P_{\theta_0}(\boldsymbol{X} \in \mathcal{R})$. Then

- Any test satisfying above rejection region is a UMP level α test (necessary condiiton)
- Every UMP level α test satisfies the above rejection regions (sufficient condition)