Autumn Semester 2020 **December 31, 2021**

IE 605: Engineering Statistics

Solutions: Tutorial 1

Solution 1

Ω= $\{\{i, j, k\} \ni$ i,j,k $\in \{1, 2, 3, 4, 5, 6\}\}$
F = 2^{Ω}
P(Same no. on exactly 2 dice) $= \frac{\binom{3}{2}\times 6\times 5}{6^3} = 5/12.$

Solution 2

1. $A \cup B \cup C$
2. $(A \cup B \cup C)^c \cup (A \cup B)^c \cup (A \cup C)^c \cup (C \cup B)^c$
3. $(A \cup B \cup C)^c$
4. $A \cap B \cap C$
5. $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
6. $A \cap B \cap C^c$
7. $A \cup (A \cup B)^c$

Solution 3

Ω = $\{HH, THH, HTHH, TTHH, ..\}$
There are two configurations for exactly 4 tosses i.e. $\{TTHH, HTHH\}$
$P(exactly\ \ 4\ \ tosses) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4$ = $\frac{1}{8}$

Solution 4

$n(geniuses)$ = 60
$n(chocolate\ lovers)$ = 70
$n(geniuses \cap chocolate\ lovers)$ = 40
So that, $n(geniuses \cup chocolate\ lovers)$ = $70 + 60 - 40$ = 90
$P(geniuses \cup chocolate\ lovers)^c$ = $\frac{100-90}{100}$ = 0.10

Solution 5

Let $P(odd\ face)$ = 2p;
$P(even\ face)$ = p;
As probability of sample space will be 1;
so,

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$
$$2p + p + 2p + p + 2p + p = 1$$
$$p = \frac{1}{9}$$

Now,

$$P(1) + P(2) + P(3) = 4p = \frac{4}{9}$$

Solution 6

1. Given $A \subset B$;
   Let $D = B/A$ or $B \cap A^c$. then A and D will be mutually exclusive sets;
   hence, P(B) = P(D) + P(A);
   so, P(B) $\geq$ P(A)

2. Let $D_1 = A$
   $D_2 = B \cap A^c$
   $A \cup B = D_1 \cup D_2$, where D_1 and D_2 are mutually exclusives.
   $P(A \cup B) = P(D_1) + P(D_2)$
   $\implies P(A \cup B) = P(A) + P(B \cap B^c)$ ....(i)
   Again, $B = (B \cap A^c) \cup (A \cap B)$, where $B \cap A^c$ and $A \cap B$ are mutually exclusives.
   $\implies P(B) = P(B \cap A^c) + P(A \cap B)$ ....(ii)
   Inserting (ii) in (i) we get, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

3.

$$P(A \cup B \cup C) = P((A \cup B) \cup C)$$
$$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P((A \cap C) \cup (B \cap C))$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)]$$

Therefore,
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

4. $\Omega = A \cup A^c$
   As A and A^c are mutually exclusive
   $P(A \cup A^c) = P(A) + P(A^c)$
   $P(\Omega) = P(A) + P(A^c)$;
   $P(A) + P(A^c) = 1$

5. As Ω and ϕ are mutually exclusive;
   $P(\Omega \cup \phi) = P(\Omega) + P(\phi)$;
   $P(\phi) = 0$

Solution 7

Proof for Axiom 1:

$$P'(A) = P(A|B) = P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

As $P(A \cap B) \geq 0$
hence $P'(A) \geq 0$

Proof for Axiom 2:
$P'(\Omega) = P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

Proof for Axiom 3:
Let A and B are two mutually exclusive set;
$P'(A \cup C) = P((A \cup C)|B)$;
$P'(A \cup C) = \frac{P((A \cup C) \cap B)}{P(B)}$
$P'(A \cup C) = \frac{P((A \cap B) \cup (C \cap B))}{P(A)}$
$P'(A \cup C) = \frac{P(A \cup B)}{P(B)} + \frac{P(C \cap B}{P(B)}$
$P'(A \cup C) = P(A|B) + P(C|B)$
$P'(A \cup C) = P'(A) + P'(C)$

Solution 8

Let, we toss two coin simultaneously,
we are assuming the following events;
A = both coin agree (HH, TT);
B = head appears on coin 1 (HH, HT);
C = head appears on coin 2 (HH , TH);
For independence of two sets A and B ;
$P(A \cap B) = P(A).P(B)$
we can see,

$P(A \cap B) = P(A).P(B) = \frac{1}{4}$
$P(A \cap c) = P(A).P(C) = \frac{1}{4}$
$P(C \cap B) = P(C).P(B) = \frac{1}{4}$
Now , For $P(A \cap B \cap C) = \frac{1}{4}$
but $P(A).P(B).P(C) = \frac{1}{8}$
hence pairwise Independence does not make mutual independence.

Solution 9

Given $A_i \in \mathcal{F}, i \geq 1$.
To show: $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$
Consider $B_i = A_i - \cup_{j=1}^{i-1} A_j$, such that $\{B_i\}_{i \geq 1}$ are disjoint.
By construction, $B_i \subset A_i$ and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$
Using Ques 6(1), $P(B_i) \leq P(A_i)$
So, $P(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

Solution 10

Let A be the event that the product is manufactured at company A, B be the event that the product is manufactured at company B, D be the event that the product is defective.

1. P(sample is defective) = $P(A)P(D|A) + P(B)P(D|B) = 0.26$
2. P(defective sample is manufactured at company A) = $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{12}{13}$

Solution 11

Let F denote the event that the attempt fails, W be the event that the server is working, NW be the event that the server is not working.

1. P(first attempt fails) = $P(F|NW)P(NW) + P(F|W)P(W) = 0.28$.
2. P(server was working given that first access attempt fails) = $\frac{P(F \cap W)}{P(F)} = \frac{P(F|W)P(W)}{P(F)} = 2/7$.
3. Let F_1, F_2 denote the event that the first, second access attempt fails respectively.
$P(F_2|F_1) = \frac{P(F_1, F_2)}{P(F_1)} = \frac{P(W)P(F_1|W)P(F_2|W) + P(NW)P(F_1|NW)P(F_2|NW)}{P(F_1)} = 208/280$.
4. P(server is working given that first and second attempt fails) = $\frac{P(W \cap F_1 \cap F_2)}{P(F_1 \cap F_2)} = \frac{P(F_1 \cap F_2|W)P(W)}{P(F_1 \cap F_2)} = 8/208$.

Solution 12

We will use first principle of mathematical induction for this question.
For n = 2, refer to ques 6. Let the claim be true for n = r. We will thus prove that it also holds for n = r + 1.

$$
\begin{aligned}
P(\cup_{i=1}^{r+1} E_i) &= P((\cup_{i=1}^{r} E_i) \cup E_{r+1}) \\
&= P(\cup_{i=1}^{r} E_i) + P(E_{r+1}) - P((\cup_{i=1}^{r} E_i) \cap E_{r+1}) \\
&= P(\cup_{i=1}^{r} E_i) + P(E_{r+1}) - P(\cup_{i=1}^{r} (E_i \cap E_{r+1})) \\
&= \sum_{i=1}^{r} P(E_i) - \sum_{1 \le i < j \le r} P(E_i \cap E_j) + \cdots + (-1)^{r+1} P(\cap_{i=1}^{r} E_i) + P(E_{r+1}) \\
&- P(\cup_{i=1}^{r} (E_i \cap E_{r+1})) \\
&= \sum_{i=1}^{r+1} P(E_i) - \sum_{1 \le i < j \le r} P(E_i \cap E_j) + \cdots + (-1)^{r+1} P(\cap_{i=1}^{r} E_i) - \{\sum_{i=1}^{r} P(E_i \cup E_{r+1}) \\
&- \sum_{1 \le i < j \le r} P((E_i \cap E_{r+1}) \cap (E_j \cap E_{r+1})) + \cdots + (-1)^{r+1} P(\cap_{i=1}^{r+1} E_i)\} \\
&= \sum_{i=1}^{r+1} P(E_i) - \{\sum_{1 \le i < j \le r} P(E_i \cap E_j) + \sum_{i=1}^{r} P(E_i \cap E_{r+1})\} + \cdots + (-1)^{r+2} P(\cap_{i=1}^{r+1} E_i) \\
&= \sum_{i=1}^{r+1} P(E_i) - \sum_{1 \le i < j \le r+1} P(E_i \cap E_j) + \cdots + (-1)^{r+2} P(\cap_{i=1}^{r+1} E_i)
\end{aligned}
$$

Thus, the claim holds true for any finite value of n.

Solution 13

Let B = event both are girls; E = event oldest is girl and L = event at least one is a girl.
Part (a):

$$P(B|E) = \frac{P(BE)}{P(E)} = \frac{P(B)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Part (b):

$$P(L) = 1 - P(\text{no girls}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Then,

$$P(B|L) = \frac{P(BL)}{P(L)} = \frac{P(B)}{P(L)} = \frac{1/4}{3/4} = \frac{1}{3}$$

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Solution 14

Let $E =$ event at least one is a six.

$$P(E) = \frac{\text{number of ways to get E}}{\text{number of sample points}} = \frac{11}{36}$$

Let $D =$ event two faces are different

$$P(D) = 1 - \text{Prob(two faces the same)} = 1 - \frac{6}{36} = \frac{5}{6}$$

Then,

$$P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}$$

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Solution 15

Let $E =$ event same number on exactly two of the dice; $S =$ event all three numbers are the same; $D =$ event all three numbers are different. These three events are mutually exclusive and define the whole sample space. Thus,

$$1 = P(D) + P(S) + P(E)$$

Now,

$$P(S) = \frac{6}{216} = \frac{1}{36}$$

For even D, we have six possible values for first die, five for second, and four for third. Therefore, Number of ways to get $D = 6 \cdot 5 \cdot 4 = 120$.

$$\implies P(D) = \frac{120}{216} = \frac{20}{36}$$

Then,

$$\begin{aligned} P(E) &= 1 - P(S) - P(D) \\ &= 1 - \frac{1}{36} - \frac{20}{36} \\ &= \frac{5}{12}. \end{aligned}$$

Solution 16

Let $C =$ event person is color blind. Then,

$$\begin{aligned} P(\text{Male}|C) &= \frac{P(C|\text{Male})P(\text{Male})}{P(C|\text{Male})P(\text{Male}) + P(C|\text{Female})P(\text{Female})} \\ &= \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.0025 \times 0.5} \end{aligned}$$

$$= \frac{2500}{2625} = \frac{20}{21}.$$

Solution 17

Let A_k: points of A in game k.

B_k: points of B in game k.

Let $x_k = A_k - B_k$: difference of points between A and B in game k.

Given: $\mathbb{P}\{A_k\} = p \forall k = 1, 2, ..$

$\mathbb{P}\{\text{Total 2n points}\} = \mathbb{P}\{X_{2n} \in \{-2, 2\}, X_{2n-2} = 0, X_{2n-4} = 0, .., X_0 = 0\}$

$$\begin{aligned}
\mathbb{P}\{X_0 = 0\} &= 1 \\
\mathbb{P}\{X_2 = 0, X_0 = 0\} &= p(1-p) + (1-p)p = 2p(1-p) \\
\mathbb{P}\{X_4 = 0, X_2 = 0, X_0 = 0\} &= p(1-p)p(1-p) + p(1-p)(1-p)p \\
&\quad + (1-p)pp(1-p)(1-p)p(1-p)p \\
&= 2^2 p^2 (1-p)^2
\end{aligned}$$

$X_0 = 0; X_1 = -1$ (if B wins); $X_2 = 0$ (if A wins); $X_3 = 1$ (if A wins); $X_4 = 0$ (if B wins) $\implies (1-p)pp(1-p)$ holds.

Similarly, $X_0 = 0; X_1 = 1$ (if A wins); $X_2 = 0$ (if B wins); $X_3 = -1$ (if B wins); $X_4 = 0$ (if A wins) $\implies p(1-p)(1-p)p$ holds.

Similarly,

$$\mathbb{P}\{X_{2n-2} = 0, ..., X_0 = 0\} = 2^{n-1} p^{n-1} (1-p)^{n-1}$$

Therefore,

$$\mathbb{P}\{X_{2n} \in \{-2, 2\}, X_{2n-2} = 0, X_{2n-4} = 0, .., X_0 = 0\} = 2^{n-1} p^{n-1} (1-p)^{n-1} (p^2 + (1-p)^2)$$

$$\mathbb{P}\{\text{A wins}\} = \sum_{n=1}^{\infty} \{2^{n-1} p^{n-1} (1-p)^{n-1} p^2\} = \frac{p^2}{1 - 2p(1-p)}.$$