IE 605: Engineering Statistics

Solutions: Tutorial 2

Solution 1

(a) If E is an event independent of itself, then $P(E) = P(E \cap E) = P(E)P(E)$. This can happen if $P(E) = 0$. If $P(E) \neq 0$ then canceling a factor of P(E) on each side yields P(E) = 1. In summary, either $P(E) = 0$ or $P(E) = 1$.

(b) In general, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If the events A and B are independent, then $P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.4 - (0.3)(0.4) = 0.58$. On the other hand, if the events A and B are mutually exclusive, then $P(A \cap B) = 0$ and therefore $P(A \cup B) = 0.3 + 0.4 = 0.7$.

(c) If $P(A) = 0.6$ and $P(B) = 0.8$, and A and B are independent events, then $P(A \cup B) = P(A) + P(B) - P(A).P(B) \leq 1$. Therefore, the two events could be independent.

However, if A and B were mutually exclusive, then $P(A)+P(B) = P(A \cup B) \leq 1$, so it would not possible for A and B to be mutually exclusive if $P(A) = 0.6$ and $P(B) = 0.8$.

Solution 2

- (a) X: lifetime of a certain car battery (rounded up to an integer number of years)

  The pmf of X is given as $p_X(k) = 0.2$, if $3 \leq k \leq 7$ and $p_X(k) = 0$, otherwise.

  (i) $\mathbb{P}\{X > 3\} = 1 - \mathbb{P}\{X \leq 3\} = 1 - p_X(3) = 1 - 0.2 = 0.8$.

  (ii) $\mathbb{P}\{X > 8 \mid X > 5\} = \frac{\mathbb{P}\{\{X>8\} \cap \{X>5\}\}}{\mathbb{P}\{X>5\}} = \frac{\mathbb{P}\{X>8\}}{\mathbb{P}\{X>5\}} = 0$.

- (b) Let A: the shot is success.

  $\mathbb{P}\{A\} = p$ (given), and $\mathbb{P}\{A^c\} = 1 - p$.

  Let Y: no. of shots required for the first success.

  $\mathbb{P}\{Y = y\} = p(1-p)^{y-1}, y = 1, 2, ...,$ and $\mathbb{P}\{Y = y\} = 0$, otherwise.

  (i)

$$
\begin{aligned}
\mathbb{P}\{Y > 3\} &= \mathbb{P}\{Y = 4\} + \mathbb{P}\{Y = 5\} + \dots \\
&= p(1-p)^3 + p(1-p)^4 + \dots \\
&= p(1-p)^3\{1 + (1-p) + (1-p)^2 + \dots\}
\end{aligned}
$$

$$= (1-p)^3.$$

(ii)

$$\begin{aligned}\mathbb{P}\{Y > 8 \mid Y > 5\} &= \frac{\mathbb{P}\{\{Y > 8\} \cap \{Y > 5\}\}}{\mathbb{P}\{Y > 5\}} \\ &= \frac{\mathbb{P}\{Y > 8\}}{\mathbb{P}\{Y > 5\}} \\ &= \frac{p(1-p)^8 + p(1-p)^9 + \dots}{p(1-p)^5 + p(1-p)^6 + \dots} \\ &= (1-p)^3.\end{aligned}$$

(iii) Geometric Distribution.

Solution 3

(a)There are 6 choices of choosing a face and for each face there are 4 choices to choose an additional corner to color 5 corners so that atleast one face has 4 blue corners. Therefore, there are $6 \times 4 = 24$ ways.

There are $\binom{8}{5} = 56$ ways to select 5 out of 8 corners.

Let B : atleast one face of the cube has all four corners coloured blue.

C : exactly 5 corners of the cube are coloured.

$\mathbb{P}\{B \mid C\} = \frac{\mathbb{P}\{B \cap C\}}{\mathbb{P}\{C\}} = 24/56 = 3/7.$

(b)By counting the number of ways that B can happen for different numbers of blue corners we find

$$P(B) = \binom{6}{1}p^4(1-p)^4 + \left(\binom{6}{1} \times \binom{4}{1}\right) p^5(1-p)^3 + \left(\binom{8}{6} - 4\right) p^6(1-p)^2 + \binom{8}{7}p^7(1-p) + p^8$$

Solution 4

- 1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.
  Therefore,

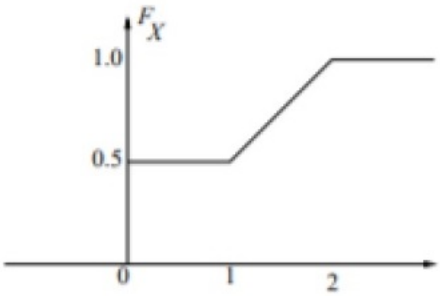
$$\begin{aligned}\mathbb{P}\left\{X^2 > 5\right\} &= \mathbb{P}\left\{|X| > \sqrt{5}\right\} \\ &= \mathbb{P}\left\{X < -\sqrt{5}\right\} + \mathbb{P}\left\{X \geq \sqrt{5}\right\} \\ &= F(-\sqrt{5}) + 1 - F(\sqrt{5}) \\ &= e^{-5}/2\end{aligned}$$

- 2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$. But, $F(0) = 0.5 + 1 > 1$, which is not possible. Hence, $F(x)$ is not cdf.

- 3. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$. And, $F(0) = F(0^+)$ and $F(10) = F(10^+)$. They are right continuous also. Hence, $F(x)$ is cdf. $\mathbb{P}\left\{X^2 > 5\right\} = 0 + 1 - \mathbb{P}\left\{X \leq \sqrt{5}\right\} = 1 - (0.5 + \sqrt{5}/20) = 0.5 - \sqrt{5}/20$.

Solution 5

Given X have the CDF shown as:



We can write CDF as:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leq x < 1 \\ \text{x/2} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

1. $P(X \leq 0.8) = F(0.8) = 0.5$
2. $E(X) = (0 \times 1/2) + \int_1^2 0.5 * x \, dx = 3/4$
3. $Var(X) = E(X^2) - E(X)^2 = \int_1^2 x^2/2 \, dx - 9/16 = 29/48$

Solution 6

$$\begin{aligned} P(X \geq 0.4|X \leq 0.8) &= P(0.4 \leq X \leq 1|X \leq 0.8) \\ &= \frac{P(0.4 \leq X \leq 0.8)}{P(x \leq 0.8)} \\ &= \frac{(0.8^2 - 0.4^2)}{0.8^2} \\ &= \frac{3}{4} \end{aligned}$$

Solution 7

For the following outcome in the original sample space I_E equals 1
$E = \{H, H, H, H, H\}$

$P(I_E = 1) = P(E\ occurs) = \frac{1}{2^5}$.

Solution 8

Let X: no. of heads, and n: no. of tosses. Then $X \sim Bin(n, p)$ where $p = P(\text{Head occurs}) = 0.7$.

Following is the pmf of X

$P(X = 0) = 0.3^3 = 0.027$
$P(X = 1) = \binom{3}{1} * 0.3^2 * 0.7 = 0.189$
$P(X = 2) = \binom{3}{2} * 0.7^2 * 0.3 = 0.441$
$P(X = 3) = 0.7^3 = 0.343$

Solution 9

$$P(X = b) = \begin{cases} 1/2 & \text{if } b = 0 \\ 1/2 & \text{if } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution 10

In order for X to equal n, the first $n - 1$ flips must have $r - 1$ tails, and then the n-th flip must land heads. By independence the desired probability is thus

$$\binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \times p = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

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Solution 11

Formula signifies negative binomial random number. It is discrete probability distribution of the number of successes in a sequence of independent and identically distributed Bernoulli trials before a specified (non-random) number of failures occurs.

Solution 12

Firstly we will calculate the value of "c" as follows:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$
$$\int_{-\infty}^{0} 0dx + \int_{0}^{\infty} ce^{-2x}dx = 1$$
$$c[\frac{e^{-2x}}{-2}]_0^{\infty} = 1$$
$$\frac{c}{-2}[0 - 1] = 1$$
$$c = 2$$

Next, we calculate $P(X > 2)$:

$$\begin{aligned} P(x > 2) &= \int_2^\infty ce^{-2x} \\ &= 2[\frac{e^{-2x}}{-2}]_2^\infty \\ &= e^{-4} \end{aligned}$$