IE 605:Engineering Statistics

Tutorial 3 Solution

Solution 1

- Say $Z = min(X, Y)$
  $P(Z > k) = P(min(X, Y) > k) = P(X > k)P(Y > k) = e^{-\lambda_1 * k} e^{-\lambda_2 * k}$
  $= e^{-(\lambda_1+\lambda_2)*k}$
  $P(Z \leq k) = 1 - e^{-(\lambda_1+\lambda_2)*k}$
  I.e. Z $\sim$ exp($\lambda_1 + \lambda_2$).

- Say $Z = max(X, Y)$
  $P(Z \leq k) = P(max(X, Y) \leq k) = P(X \leq k)P(Y \leq k)$
  $= (1 - e^{-\lambda_1 k})(1 - e^{-\lambda_2 k})$.

Solution 2

Given: X, Y, Z are discrete random variable.

The joint pmf of X, Y, Z is, $p_{X,Y,Z}(x, y, z)$ where $x \in A_1, y \in A_2, z \in A_3$ and A_1, A_2, A_3 are the support of x, y, z respectively.

- To show: $\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|Y]]$.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Z|Y]] &= \sum_y \mathbb{E}[Z|Y].p_Y(y) \qquad \text{where } p_Y(y) \text{ is marginal pmf of } Y \in A_2 \\
&= \sum_y p_Y(y) \sum_x \sum_z z.p_{(X,Z)|Y}(x, z|y) \\
&= \sum_x \sum_y \sum_z z.p_Y(y) p_{(X,Z)|Y}(x, z|y) \\
&= \sum_x \sum_y \sum_z z.p_Y(y) \frac{p_{X,Z,Y}(x, y, z)}{p_Y(y)} \\
&= \sum_x \sum_y \sum_z z.p_{X,Z,Y}(x, y, z) \\
&= \mathbb{E}[Z].
\end{aligned}$$

- To show: $\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|X,Y]]$.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Z|X,Y]] &= \sum_x \sum_y \mathbb{E}[Z|X,Y] .p_{X,Y}(x,y) \\
&= \sum_x \sum_y p_{X,Y}(x,y) \sum_z z.p_{Z|X,Y}(z|x,y). \\
&= \sum_x \sum_y \sum_z z.p_{X,Y}(x,y) \frac{p_{X,Z,Y}(x,y,z)}{p_{X,Y}(x,y)} \\
&= \sum_x \sum_y \sum_z z.p_{X,Z,Y}(x,y,z) \\
&= \mathbb{E}[Z].
\end{aligned}$$

Solution 3

$$E[X|Y=3] = \sum_{x=0}^{3} xP(X=x|Y=3) = \sum_{x=1}^{3} xP(X=x|Y=3)$$

$$P(X=1|Y=3) = \frac{P(X=1 \cap Y=3)}{P(Y=3)} = \frac{\binom{6}{3} * (\frac{5}{14})^3 * \binom{3}{1} * (\frac{3}{14}) * (\frac{6}{14})^2}{\binom{6}{3} * (\frac{5}{14})^3 * (\frac{9}{14})^3} = \frac{4}{9}$$

$$P(X=2|Y=3) = \frac{P(X=2 \cap Y=3)}{P(Y=3)} = \frac{\binom{6}{3} * (\frac{5}{14})^3 * \binom{3}{2} * (\frac{3}{14})^2 * (\frac{6}{14})}{\binom{6}{3} * (\frac{5}{14})^3 * (\frac{9}{14})^3} = \frac{2}{9}$$

$$P(X=3|Y=3) = \frac{P(X=3 \cap Y=3)}{P(Y=3)} = \frac{\binom{6}{3} * (\frac{5}{14})^3 * \binom{3}{3} * (\frac{3}{14})^3}{\binom{6}{3} * (\frac{5}{14})^3 * (\frac{9}{14})^3} = \frac{1}{27}$$

$$E[X|Y=3] = 1 * \frac{4}{9} + 2 * \frac{2}{9} + 3 * \frac{1}{27} = 1$$

Solution 4

$X_1 + X_2 = m$
Since X_1 and X_2 are independent,
$\mathbb{P}\{X_1 + X_2 = m\} = \binom{n_1+n_2}{m} p^m (1-p)^{n_1+n_2-m}$.
$P(X_1 = k|X_1 + X_2 = m) = \frac{P(X_1=k \cap X_1+X_2=m)}{P(X_1+X_2=m)}$
$P(X_1 = k|X_1 + X_2 = m) = \frac{P(X_1=k)P(X_2=m-k)}{P(X_1+X_2=m)}$ as X_1, X_2 are independent
$P(X_1 = k|X_1 + X_2 = m) = \frac{(\binom{n_1}{k} p^k (1-p)^{n_1-k}) \binom{n_2}{m-k} p^{m-k} (1-p)^{n_2-m+k})}{(\binom{n_1+n_2}{m}) p^m (1-p)^{n_1+n_2-m}}$
$P(X_1 = k|X_1 + X_2 = m) = \frac{\binom{n_1}{k}\binom{n_2}{m-k}}{(\binom{n_1+n_2}{m})}$.

Solution 5

Let $X \sim U([-1,1])$ and $Y = X^2$. Then, it is evident that X, Y are not independent and $Cov(X,Y) = E((X - E(X)(Y - E(Y)) = E(X^3) = 0$.

Solution 6

Given two random variables X, Y such that their joint distribution is as follows:

$$f_{X,Y}(x,y) = \begin{cases} c(1+xy) & \text{if } 2 \le x \le 3, 1 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

1. We know that, $\int_{y=1}^{2}\int_{x=2}^{3} f(x,y)dxdy = 1$. Thus, on substitution we get, c = 4/19

2.

$$\begin{aligned} f_X(x) &= \int_{y=1}^{2} f_{X,Y}(x,y)dy \\ &= \begin{cases} \frac{4}{19}(1+3x/2) & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

   Similarly,

$$\begin{aligned} f_Y(y) &= \int_{x=2}^{3} f_{X,Y}(x,y)dx \\ &= \begin{cases} \frac{4}{19}(1+5y/2) & \text{if } 1 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{2(1+xy)}{2+5y} & \text{if } 2 \le x \le 3, 1 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Solution 7

Let X denote the number of accidents that a randomly chosen policyholder has next year. Letting Y be the Poisson mean number of accidents for this policyholder, then conditioning on Y yields

$$\begin{aligned} P\{X=n\} &= \int_0^\infty P\{X=n|Y=\lambda\}g(y)dy \\ &= \int_0^\infty e^{-\lambda}\frac{\lambda^n}{n!}\lambda e^{-\lambda}dy \\ &= \frac{1}{n!}\int_0^\infty e^{-2\lambda}\lambda^{n+1}dy \end{aligned}$$

Since,

$$h(y) = \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!}, \lambda > 0$$

is the density function of a gamma $(n+2, 2)$ random variable, its integral is 1. Therefore,

$$1 = \int_0^\infty \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!} d\lambda = \frac{2^{n+2}}{(n+1)!} \int_0^\infty \lambda^{n+1} e^{-2\lambda}$$

$$\implies P\{X = n\} = \frac{n+1}{2^{n+2}}$$

Solution 8

Let N_1 denote the number of women and N_2 the number of men who visit the academy today. Also, let $N = N_1 + N_2$ be the total number of people who visit. Conditioning on N gives,

$$\mathbb{P}\{N_1 = n, N_2 = m\} = \sum_{i=0}^{\infty} \mathbb{P}\{N_1 = n, N_2 = m | N = i\} \mathbb{P}\{N = i\}$$

Because $\mathbb{P}\{N_1 = n, N_2 = m | N = i\} = 0$ when $i \neq n+m$, the preceding equation yields

$$\mathbb{P}\{N_1 = n, N_2 = m\} = \mathbb{P}\{N_1 = n, N_2 = m | N = n+m\} e^{\lambda} \frac{\lambda^{n+m}}{(n+m)!}$$

Given that $n+m$ people visit it follows, because each of these $n+m$ is independently a woman with probability p, that the conditional probability that n of them are women (and m are men) is just the binomial probability of n successes in $n+m$ trials. Therefore,

$$\begin{aligned}
\mathbb{P}\{N_1 = n, N_2 = m\} &= \binom{n+m}{n} p^n (1p)^m e^{\lambda} \frac{\lambda^{n+m}}{(n+m)!} \\
&= \frac{(n+m)!}{n!m!} p^n (1p)^m e^{\lambda p} e^{\lambda (1p)} \frac{\lambda^n \lambda^m}{(n+m)!} \\
&= e^{\lambda p} \frac{(\lambda p)^n}{n!} e^{\lambda (1p)} \frac{(\lambda (1p))^m}{m!}
\end{aligned}$$

Because the preceding joint probability mass function factors into two products, one of which depends only on n and the other only on m, it follows that N_1 and N_2 are independent. Moreover, because

$$\begin{aligned}
\mathbb{P}\{N_1 = n\} &= \sum_{i=0}^{\infty} \mathbb{P}\{N_1 = n, N_2 = m\} \\
&= e^{\lambda p} \frac{(\lambda p)^n}{n!} \sum_{i=o}^{\infty} e^{\lambda (1-p)} \frac{(\lambda (1-p)^m}{m!}
\end{aligned}$$

and similarly,

$$\mathbb{P}\{N_2 = m\} = e^{\lambda (1-p)} \frac{(\lambda (1-p)^m}{m!}$$

we can conclude that N_1 and N_2 are independent Poisson random variables with respective means λp and $\lambda(1-p)$. Therefore, this example establishes the important result that when each of a Poisson number of events is independently classified either as being type 1 with probability p or type 2 with probability $1-p$, then the numbers of type 1 and type 2 events are independent Poisson random variables.

Solution 9

Given: A chicken lays n eggs. Each egg independently does or doesn't hatch, with probability p of hatching.

Let N: number of eggs which hatch out of n eggs.

$\implies N \sim Bin(n,p)$.

For each egg that hatches, the chick does or doesn't survive (independently of the other eggs), with probability s of survival.

Let X: the number of chicks which survive out of N chicks.

$\implies X|N \sim Bin(N,s)$.

Let Y: the number of chicks which hatch but don't survive (so X + Y = N).

To find: PMF of X

$$
\begin{aligned}
p_X(x) = \mathbb{P}\{X = x\} &= \sum_{n'=x}^{n} p_{X|N}(x|N=n')p_N(n') \\
&= \sum_{n'=x}^{n} \binom{n'}{x} s^x(1-s)^{n'-x}\binom{n}{n'}p^{n'}(1-p)^{n-n'} \\
&= \sum_{n'=x}^{n} \frac{n'!}{x!(n'-x)!}s^x(1-s)^{n'-x}.\frac{n!}{n'!(n-n')!}p^{n'}(1-p)^{n-n'}
\end{aligned}
$$

Let $n'-x=r$. Then, $n'=x \implies r=0$ and $n'=n \implies r=n-x$

$$
\begin{aligned}
&= \sum_{r=0}^{n-x} \frac{n!}{x!r!(n-r-x)!}s^x(1-s)^r p^{r+x}(1-p)^{n-r-x} \\
&= \frac{n!}{x!(n-x)!}(ps)^x(1-p)^{n-x}\sum_{r=0}^{n-x}\frac{(n-x)!}{r!(n-r-x)!}(1-s)^r p^r(1-p)^{-r} \\
&= \binom{n}{x}(ps)^x(1-p)^{n-x}\sum_{r=0}^{n-x}\binom{n-x}{r}\left(\frac{(1-s)p}{1-p}\right)^r \\
&= \binom{n}{x}(ps)^x(1-p)^{n-x}\left(1+\frac{(1-s)p}{1-p}\right)^{n-x} \\
&= \binom{n}{x}(ps)^x(1-p)^{n-x}\left(\frac{1-p+(1-s)p}{1-p}\right)^{n-x} \\
&= \binom{n}{x}(ps)^x(1-p)^{n-x}\left(\frac{1-ps}{1-p}\right)^{n-x} \\
&= \binom{n}{x}(ps)^x(1-ps)^{n-x} \implies X \sim Bin(n,ps).
\end{aligned}
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