IE 605: Engineering Statistics

Solutions of tutorial 4

Solution 1

Part 1: MGF of Exponential(λ) distribution is $\frac{\lambda}{\lambda-u}$. Therefore, this is Exponential distribution with $\lambda = 2$.
Part 1: MGF of Binomial(n,p) distribution is $(pe^t+q)^n$. Therefore, this is Binomial distribution with $p = \frac{1}{3}$ and $n = 1050$.
Part 3: MGF of Poisson (λ) distribution is $e^{\lambda(e^u-1)}$. Therefore, this is Poisson distribution with $\lambda = 3.5$.

Solution 2

2

The conditional density of X, given that Y = 1, is given by

$$\begin{aligned} f_{X|Y}(x|1) &= \frac{f(x,1)}{f_Y(1)} \\ &= \frac{\frac{1}{2}e^{-x}}{\int\limits_0^\infty \frac{1}{2}e^{-x}} \\ &= e^{-x} \end{aligned}$$

Hence, by the definition of $\mathbb{E}\left[g(X)\right]$ we have,

$$\begin{aligned} \mathbb{E}\left[e^{X/2} \mid Y=1\right] &= \int\limits_0^\infty e^{x/2} f_{X|Y}(x|1)dx \\ &= \int\limits_0^\infty e^{x/2}e^{-x}dx \\ &= 2 \end{aligned}$$

Solution 3

Part 1: If X and Y are independent then

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

So,

$$E[X|Y=y] = \int x f_{X|Y}(x|y)dx = \int x f_X(x)dx = E[X]$$

Similarly, it can be proved for discrete case.

Part 3: We must compute $E[Xg(Y)|Y=y]$. Given that $Y=y$, the possible values of $Xg(Y)$ are $xg(y)$ where x varies over the range of X. The probability of the value $xg(y)$ given that $Y=y$ is just $P(X=x|Y=y)$. So

$$\begin{aligned}E[Xg(Y)|Y=y] &= \sum_x xg(y)P(X=x|Y=y)\\ &= g(y)\sum_x xP(X=x|Y=y)\\ &= g(y)E[X|Y=y]\end{aligned}$$

Solution 4

0.0175

Let X_i be the indicator random variable for the ith bit in the packet. That is, $X_i = 1$ if the i^{th} bit is received in error, and $X_i = 0$ otherwise. Then the X_i's are i.i.d. and $Xi \sim Bernoulli(p = 0.1)$.

If Y is the total number of bit errors in the packet, we have $Y = X_1 + X_2 + \cdots + X_n$. Since $Xi \sim Bernoulli(p = 0.1)$, we have

$E[X_i] = \mu = p = 0.1, Var(X_i) = \sigma^2 = p(1-p) = 0.09$

Using the CLT, we have

$$\begin{aligned}P(Y > 120) &= P\left(\frac{Y-n\mu}{\sqrt{n}\sigma} > \frac{120-n\mu}{\sqrt{n}\sigma}\right)\\ &= P\left(\frac{Y-n\mu}{\sqrt{n}\sigma} > \frac{120-n\mu}{\sqrt{90}}\right)\\ &\approx 1-\Phi\left(\frac{20}{\sqrt{90}}\right)\\ &= 0.0175\end{aligned}$$

Solution 5

0.1587

If we let X_i denote the lifetime of the ith battery to be put in use, then we desire $p = P\{X_1 + \cdots + X_{25} > 1100\}$, which is approximated as follows:

$$\begin{aligned}p &= P\left\{\frac{X_1 + \cdots + X_{25} - 1000}{20\sqrt{25}} > \frac{1100-1000}{20\sqrt{25}}\right\}\\ &= P\{N(0,1) > 1\}\\ &= 1-\Phi(1)\end{aligned}$$

$$= 0.1587$$

Solution 6

0.2405

We have

$$\begin{aligned} E[X_i] &= (0.6)(1) + (0.4)(-1) \\ &= \frac{1}{5} \\ E[X_i^2] &= (0.6)(1)^2 + (0.4)(-1)^2 \\ &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} Var(X_i) &= 1 - \frac{1}{25} = \frac{24}{25} \\ \text{and } \sigma_{X_i} &= \frac{2\sqrt{6}}{5} \end{aligned}$$

Now,

$$\begin{aligned} E[Y] &= 25 \times \frac{1}{5} = 5 \\ Var(Y) &= 25 \times \frac{24}{25} = 24 \\ \text{and } \sigma_Y &= 2\sqrt{6} \end{aligned}$$

$$\begin{aligned} P(3.5 \leq Y \leq 6.5) &= P\left(\frac{3.5-5}{2\sqrt{6}} \leq \frac{Y-5}{2\sqrt{6}} \leq \frac{6.5-5}{2\sqrt{6}}\right) \\ &= P\left(-0.3062 \leq \frac{Y-5}{2\sqrt{6}} \leq +0.3062\right) \\ &\approx \Phi(0.3062) - \Phi(-0.3062) \\ &= 2\Phi(0.3062) - 1 \\ &\approx 0.2405 \end{aligned}$$

Solution 7

74

Let X_i be the number of sandwiches that the ith person needs, and let

$$Y = X_1 + X_2 + \cdots X_{64}$$

The goal is to find y such that $P(Y \leq y) \geq 0.95$.
Now, $E[X_i] = (0.25)(0) + (0.5)(1) + (0.25)(2) = 1$,

$E[X_i^2] = (0.25)(0^2) + (0.5)(1^2) + (0.5)(2^2) = 1.5.$
Thus, $Var(X_i) = E[X_i^2] - (E[X_i])^2 = 1.5 - 1 = \frac{1}{2} \implies \sigma_{X_i} = \frac{1}{\sqrt{2}}.$
Then, $E[Y] = 64 \times 1 = 64,$
$Var(Y) = 64 \times 12 = 32 \implies \sigma_Y = 4\sqrt{2}.$
Now, we can use the CLT to find y

$$\begin{aligned} P(Y \leq y) &= P\left(\frac{Y - 64}{4\sqrt{2}} \leq \frac{y - 64}{4\sqrt{2}}\right) \\ &= \Phi\left(\frac{y - 64}{4\sqrt{2}}\right) \end{aligned}$$

We have

$$\Phi\left(\frac{y - 64}{4\sqrt{2}}\right) = 0.95$$

Therefore,

$$\begin{aligned} \left(\frac{y - 64}{4\sqrt{2}}\right) &= \Phi^{-1}(0.95) \\ &\approx 1.6449 \end{aligned}$$

Thus, $y = 73.3.$
Therefore, if you make 74 sandwiches, you are 95% sure that there is no shortage.
Note that the numerical value of $\Phi^1(0.95)$ by can be computed on MATLAB by running the norminv(0.95) command.

Solution 8

0.00043
Let W be the total weight, then $W = X_1 + X_2 + \cdots + X_n$, where $n = 100$.
We have

$$\begin{aligned} E[W] &= n\mu = (100)(170) = 17000 \\ Var(W) &= 100Var(X_i) = 100(30)^2 = 90000. \end{aligned}$$

Thus, $\sigma_W = 300.$
We have

$$\begin{aligned} P(W > 18000) &= P\left(\frac{W - 17000}{300} > \frac{18000 - 17000}{300}\right) \\ &= P\left(\frac{W - 17000}{300} > \frac{10}{3}\right) \\ &= 1 - \Phi\left(\frac{10}{3}\right) \quad \text{(by CLT)} \\ &\approx 0.00043 \end{aligned}$$

Solution 9

Let X_i be Poisson with mean 1, and $\mathbb{E}[X_i] = 1$ and $Var(X_i) = 1$. We define $S_n = X_0 + X_1 + \cdots + X_n$, and S_n follows Poisson distribution with mean n (Verify by MGF).

Therefore, $\mathbb{P}\{S_n \le n\} = e^{-n} \sum\limits_{k=0}^{n} \frac{n^k}{k!}$.

Use the central limit theorem to show that $\lim\limits_{n\to\infty} e^{-n} \sum\limits_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$.

$$\begin{aligned}
\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} &= \lim_{n\to\infty} \mathbb{P}\{S_n \le n\} \\
&= \lim_{n\to\infty} \mathbb{P}\left\{\frac{S_n - n}{\sqrt{n}} \le \frac{n-n}{\sqrt{n}}\right\} \\
&= \lim_{n\to\infty} \mathbb{P}\{Z_n \le 0\} \qquad \text{where } Z_n \sim N(0,1) \\
&= \frac{1}{2}.
\end{aligned}$$

Solution 10

Here,

$$\begin{aligned}
P(X \ge \theta n) &= P(X - np \ge \theta n - np) \\
&\le P\big(|X - np| \ge n\theta - np\big) \\
&\le \frac{Var(X)}{(n\theta - np)^2} \quad \text{(using Chebyshev's inequality)} \\
&= \frac{p(1-p)}{n(\theta - p)^2}
\end{aligned}$$

For, $p = 1/2$ and $\theta = 3/4$, we get

$$P(X \ge \frac{3n}{4}) \le \frac{4}{n}. \tag{1}$$

Solution 11

This is the binomial distribution with $n = 20$ and $p = \frac{1}{5}$, and $E[X] = np = 20 \times \frac{1}{5}$. By Markov's inequality, we have

$$P\{X \ge 16\} \le \frac{E[X]}{16} = \frac{1}{4}$$

Now, the actul probabilty is given by

$$P\{X \ge 16\} = \sum_{k=16}^{20} {}^{20}C_k \left(\frac{1}{5}\right)^k \cdot \left(\frac{4}{5}\right)^{20-k} = 1.38 \times 10^{-8}$$

So, it can be seen that Markov's inequality is not a very good estimate as it doesn't get close enough to the true value.

Solution 12

Let X be any RV, and suppose that the MGF of $X, M(t) = \mathbb{E}\left[e^{tX}\right]$, exists for every $t > 0$.

To show: For any $t > 0, \mathbb{P}\left\{tX > s^2 + logM(t)\right\} < e^{-s^2}$.

$$\begin{aligned}
\mathbb{P}\left\{tX > s^2 + logM(t)\right\} &= \mathbb{P}\left\{e^{tX} > e^{s^2 + logM(t)}\right\} \\
&= \mathbb{P}\left\{e^{tX} > e^{s^2}.M(t)\right\} && \text{Taking exponent on both sides} \\
&\leq \frac{\mathbb{E}\left[e^{tX}\right]}{e^{s^2}.M(t)} && \text{By Markov's Inqeuality} \\
&= \frac{M(t)}{e^{s^2}.M(t)} \\
\implies \mathbb{P}\left\{tX > s^2 + logM(t)\right\} &\leq e^{-s^2}
\end{aligned}$$