IE 605: Engineering Statistics

Solutions of tutorial 9

Solution 1

1. Since $X_1, X_2, \ldots, X_n$ is a random sample from Bernoulli population with parameter θ,

$$T = \sum_{i=1}^{n} X_i \sim B(n, \theta)$$
$$\Rightarrow \mathbb{E}[T] = n\theta \text{ and } Var(T) = n\theta(1-\theta)$$
$$\mathbb{E}\left[\frac{\sum\limits_{i=1}^{n} X_i \left(\sum\limits_{i=1}^{n} X_i - 1\right)}{n(n-1)}\right] = \mathbb{E}\left[\frac{T(T-1)}{n(n-1)}\right]$$
$$= \frac{1}{n(n-1)}\left[\mathbb{E}\left[T^2\right] - \mathbb{E}[T]\right]$$
$$= \frac{1}{n(n-1)}\left[Var(T) + (\mathbb{E}[T])^2 - \mathbb{E}[T]\right]$$
$$= \frac{1}{n(n-1)}\left[n\theta(1-\theta) + n^2\theta^2 - n\theta\right]$$
$$= \frac{n\theta^2(n-1)}{n(n-1)}$$
$$= \theta^2$$
$$\Rightarrow \left[\frac{\sum\limits_{i=1}^{n} X_i \left(\sum\limits_{i=1}^{n} X_i - 1\right)}{n(n-1)}\right] \text{ is an unbiased estimator of } \theta^2.$$

2. Let us define,

$$T(X) = (-k)^X \text{where } x > 0\text{, so that}$$
$$= \begin{cases} T(x) > 0 & \text{if } X \text{ is even} \\ T(x) < 0 & \text{if } X \text{ is odd.} \end{cases}$$

$$\mathbb{E}[T(X)] = \mathbb{E}\left[(-k)^X\right], k > O$$
$$= \sum_{x=o}^{\infty} (-k)^x \frac{e^{-\theta}\theta^x}{x!} = e^{-\theta} \sum_{x=o}^{\infty} \frac{(-k\theta)^x}{x!} = e^{-\theta}e^{-k\theta} = e^{-(1+k)\theta}$$
$$\Rightarrow T(X) = (-k)^X \text{ is an unbiased estimator for } \exp\{-(1+k)\theta\}, k > O.$$

Solution 2

1.

$$
\begin{aligned}
\log f &= -\log \pi - \log\{1+(x-\theta)^2\} \\
\frac{\partial \log f}{\partial \theta} &= \frac{2(x-\theta)}{[1+(x-\theta)^2]} \\
\mathbb{E}\left[\left(\frac{\partial \log f}{\partial \theta}\right)^2\right] &= \int_{-\infty}^{\infty} \frac{4(x-\theta)^2}{[1+(x-\theta)^2]^2} f(x,\theta)dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{4(x-\theta)^2}{[1+(x-\theta)^2]^2} f(x,\theta)dx
\end{aligned}
$$

Put $x-\theta-tan\phi \Rightarrow dx = sec^2\phi d\phi$

$$
\begin{aligned}
\text{Therefore } \mathbb{E}\left[\left(\frac{\partial \log f}{\partial \theta}\right)^2\right] &= \frac{2}{\pi}\int_0^{\pi/2} \frac{4tan^2\phi}{sec^6\phi} sec^2\phi d\phi = \frac{2}{\pi}\int_0^{\pi/2} \frac{4sin^2\phi}{cos^2\phi} cos^4\phi d\phi \\
&= \frac{2}{\pi}\int_0^{\pi/2} 4sin^2\phi cos^2\phi d\phi = \frac{8}{\pi}\int_0^{\pi/2} (cos^2\phi - cos^4\phi)d\phi \\
&= \frac{8}{\pi}\left[\frac{1}{2}.\frac{\pi}{2} - \frac{3.4}{4.2}\frac{\pi}{2}\right]
\end{aligned}
$$

Using the reduction formula for $\int_0^{\pi/2} cos^n x dx$

$$
= \frac{8}{\pi}\left[\frac{\pi}{4} - \frac{3.\pi}{16}\right]
$$

Hence Cramer-Rao Lower Bound based on n samples is given by,

$$
= \frac{1}{n\mathbb{E}\left[\left(\frac{\partial \log f}{\partial \theta}\right)^2\right]} = \frac{1}{n\left[\frac{1}{2}\right]} = \frac{2}{n}.
$$

2. We have proved Cramer-Rao's inequality,

$$
Var(\hat{\theta}) \geq \frac{[\phi'(\theta)]^2}{I(\theta)}, \text{ where } \mathbb{E}\left[\hat{\theta}\right] = \phi(\theta) \qquad (*)
$$

Now,

$$
\begin{aligned}
\mathbb{E}\left[\hat{\theta}-\theta\right]^2 &= \mathbb{E}\left[\hat{\theta}-\phi(\theta)+\phi(\theta)-\theta\right]^2 \\
&= \mathbb{E}\left[\hat{\theta}-\phi(\theta)\right]^2 + [\theta-\phi(\theta)]^2 + 2[\phi(\theta)-\theta].\mathbb{E}\left[\hat{\theta}-\phi(\theta)\right] \\
&= V(\hat{\theta}) + [\theta-\phi(\theta)]^2
\end{aligned}
$$

Therefore $\mathbb{E}\left[\hat{\theta}-\theta\right]^2 \geq \dfrac{[\phi'(\theta)]^2}{I(\theta)} + [\theta-\phi(\theta)]^2$ [Using (*)] $\qquad (**)$

Let $\hat{\theta}$ be a 'biased' estimator of θ with bias given by $b(\theta)$,

$$\text{i.e., } \mathbb{E}\left[\hat{\theta}\right] = \theta + b(\theta) = \phi(\theta) \text{ (say)}$$
$$\Rightarrow \phi(\theta) - \theta = b(\theta)$$

From (**), we get

$$\mathbb{E}\left[\hat{\theta}-\theta\right]^2 \geq \frac{\left[1+\frac{\partial}{\partial\theta}b(\theta)\right]^2}{I(\theta)} + [b(\theta)]^2 > 0,$$

where $I(\theta) = n\int\limits_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log f\right)^2 f(x,\theta)dx > 0.$

This proves the result.

Solution 3

1. Let $X_1, X_2, \ldots, X_n$ be n random samples drawn from the distribution. The joint distribution is given by,

$$f(x|\theta) = \prod_{i=1}^{n} \frac{\alpha}{\beta^\alpha} x_i^{\alpha-1} I_{[0,\beta]}(x_i)$$
$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} I_{(-\infty,\beta]}(x_{(n)}) I_{[0,\infty)}(x_{(1)}) = L(\alpha,\beta|x) \text{ (the likelihood function).}$$

By the Factorization Theorem, $\left(\prod\limits_{i=1}^{n} X_i, X_{(n)}\right)$ are sufficient.

2. For any fixed $\alpha, L(\alpha,\beta|x) = 0$ if $\beta < x_{(n)}$, and $L(\alpha,\beta|x)$ a decreasing function of β if $\beta \geq x_{(n)}$. Thus, $X_{(n)}$ is the MLE of β. For the MLE of α calculate,

$$\frac{\partial}{\partial\alpha}\log L = \frac{\partial}{\partial\alpha}\left[n\log\alpha - n\alpha\log\beta + (\alpha-1)\log\prod_i x_i\right] = \frac{n}{\alpha} - n\log\beta + \log\prod_i x_i$$

Set the derivative equal to zero and use $\hat{\beta} = X_{(n)}$ to obtain

$$\hat{\alpha} = \frac{n}{n \log X_{(n)} - \log \prod_i X_i} = \left[\frac{1}{n} \sum_i \left(\log X_{(n)} - \log X_i \right) \right]^{-1}$$

The second derivative is $n/\alpha^2 < 0$, so this is the MLE.

3. According to the data set given,

$$X_{(n)} = 25.0, \log \prod_i X_i = \sum_i \log X_i = 43.95 \Rightarrow \hat{\beta} = 25.0, \hat{\alpha} = 12.59.$$

Solution 4

1. Let $X_1, X_2, \ldots, X_n$ be n random samples drawn from the distribution. The joint distribution is given by,

$$\begin{aligned} f(x|\theta) &= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\ &= \theta^n \left(\prod_{i=1}^{n} x_i \right)^{\theta-1} = L(\theta|x) \text{ (the likelihood function)} \\ \frac{d}{d\theta} \log L &= \frac{d}{d\theta} \left[n \log \theta + (\theta - 1) \log \prod_{i=1}^{n} x_i \right] = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i \end{aligned}$$

Set the derivative equal to zero and solve for θ to obtain $\hat{\theta} = \left(-\frac{1}{n} \sum_{i=1}^{n} \log x_i \right)^{-1}$. The second derivative is $n/\theta^2 < 0$, so this is the MLE. To calculate the variance of $\hat{\theta}$, note that $Y_i = -\log X_i \sim exponential(1/\theta)$, so $-\sum_{i=1}^{n} \log X_i \sim gamma(n, 1/\theta)$. Thus $\hat{\theta} = \frac{n}{T}$, where $T \sim gamma(n, 1/\theta)$. We can either calculate the first and second moments directly, or use the fact that $\hat{\theta}$ is inverted gamma (page 51). We have

$$\mathbb{E}\left[\frac{1}{T}\right] = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} t^{n-1} e^{-\theta t} dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1},$$

$$\mathbb{E}\left[\frac{1}{T^2}\right] = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t} dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)},$$

and thus,

$$\mathbb{E}\left[\hat{\theta}\right] = \frac{n}{n-1}\theta \text{ and } Var(\hat{\theta}) = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \text{ as } n \to \infty.$$

2. Because $X \sim Beta(\theta, 1), \mathbb{E}[X] = \frac{\theta}{\theta+1}$ and the method of moments estimator

is the solution to

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\theta}{\theta+1} \Rightarrow \tilde{\theta} = \frac{\sum\limits_{i=1}^{n} X_i}{n - \sum\limits_{i=1}^{n} X_i}.$$

Solution 5

1. For each value of x, the MLE $\hat{\theta}$ is the value of θ that maximizes $f(x|\theta)$. These values are in the following table.

x	0	1	2	3	4
$\hat{\theta}$	1	1	2 or 3	3	3

   At $x = 2$, $f(x|2) = f(x|3) = 1/4$ are both maxima, so both $\hat{\theta} = 2$ or $\hat{\theta} = 3$ are MLEs.

2. The log function is a strictly monotone increasing function. Therefore, $L(\theta|x) > L(\theta'|x)$ if and only if $\log L(\theta|x) > \log L(\theta'|x)$. So the value $\hat{\theta}$ that maximizes $\log L(\theta|x)$ is the same as the value that maximizes $L(\theta|x)$.

Solution 6

1. We know $T = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Then

$$\begin{aligned}
\mathbb{E}\left[T^{p/2}\right] &= \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}}\int_0^\infty t^{\frac{p+n-1}{2}-1}e^{-\frac{t}{2}}dt \\
&= \frac{2^{\frac{p}{2}}\Gamma(\frac{p+n-1}{2})}{\Gamma(\frac{n-1}{2})} \\
\implies \mathbb{E}\left[\left(\frac{(n-1)S^2}{\sigma^2}\right)^{\frac{p}{2}}\right] &= \frac{2^{\frac{p}{2}}\Gamma(\frac{p+n-1}{2})}{\Gamma(\frac{n-1}{2})} \\
\implies \mathbb{E}\left[\left(\frac{n-1}{2}\right)^{\frac{p}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{p+n-1}{2})}S^p\right] &= \sigma^p
\end{aligned}$$

   Hence, $\left(\frac{n-1}{2}\right)^{\frac{p}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{p+n-1}{2})}S^p$ is an unbiased estimator of σ^p.

2. With $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ as the sample variance, we know that,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

   Keeping in mind that for a chi-squared random variable X with n degrees of freedom, mean and variance of X are n and $2n$ respectively, we have

$$\text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\text{or ,} \quad \mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

We also have,

$$\mathbb{E}\left[\left(\frac{(n-1)S^2}{\sigma^2}\right)\right] = (n-1)$$
$$\text{or ,} \, \mathbb{E}\left[S^2\right] = \sigma^2$$

We require an MSE for the parameter σ^2 of the form $\alpha S^2, (\alpha \neq 1)$.

Now,

$$\begin{aligned}
\mathrm{MSE}_\sigma(\alpha S^2) &= \mathrm{Var}_\sigma(\alpha S^2) + \left\{\mathrm{bias}(\alpha S^2)\right\}^2 \\
&= \alpha^2 \, \mathrm{Var}_\sigma(S^2) + \left(\alpha\sigma^2 - \sigma^2\right)^2 \\
&= \alpha^2 \frac{2\sigma^4}{n-1} + \sigma^4 \left(\alpha - 1\right)^2 \\
&= \sigma^4 \left[\frac{2\alpha^2}{n-1} + (\alpha - 1)^2\right] \\
&= \sigma^4 \, \psi(\alpha), \text{ (say)}
\end{aligned}$$

Minimizing $\psi(\alpha)$ by usual calculus, we find that $\alpha = \frac{n-1}{n+1}$ is the point of minima.

Therefore, the required minimum MSE estimator of the form αS^2 is

$$T = \left(\frac{n-1}{n+1}\right) S^2 = \frac{1}{n+1} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

with the minimum MSE being $\sigma^4 \, \psi\left(\frac{n-1}{n+1}\right) = \frac{2\sigma^4}{n+1}$

Solution 7

$\mathbb{E}(T) = \theta, \quad \mathbb{E}(T') = \theta$ and $V(T) < V(T')$
$\therefore \; \mathbb{E}(T - T') = 0$
$\implies T - T' \in U_0$ where $U_0 = \{u(X) : \mathbb{E}[u(X)] = 0, V(u(X)) < \infty \;\; \forall \;\; \theta \in \Omega\}$
From the theorem, we know that if
$U_\Psi = \{T(X) : \mathbb{E}[T(X)] = \Psi(\theta), V(T(X)) < \infty \;\; \forall \;\; \theta \in \Omega\}$
$U_0 = \{u(X) : \mathbb{E}[u(X)] = 0, V(u(X)) < \infty \;\; \forall \;\; \theta \in \Omega\}$
Then $T \in U_\Psi$ is UMVUE of $\Psi(\theta)$ iff $cov(u, T) = 0 \; \forall \; u \in U_0 \; \forall \; \theta \in \Omega$
Here, T is UMVUE of θ and $u = T - T'$
So, $cov(u, T) = 0$ (by theorem)
$\implies cov(T - T', T) = 0$
$\implies V(T) - cov(T, T') = 0$
$\implies cov(T, T') = V(T)$

Solution 8

Let $t = s^2$ and $\theta = \sigma^2$. Because $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, we have

$$f(t|\theta) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{(n-1)/2}} \left(\frac{n-1}{\theta} t\right)^{[(n-1)/2]-1} e^{-(n-1)t/2\theta} \frac{n-1}{\theta}.$$

With $\pi(\theta)$ as given, we have (ignoring terms that do not depend on θ),

$$\begin{aligned}\pi(\theta) &\propto \left[\left(\frac{1}{\theta}\right)^{((n-1)/2)-1} e^{-(n-1)t/2\theta} \frac{1}{\theta}\right] \left[\frac{1}{\theta^{\alpha+1}} e^{-1/\beta\theta}\right] \\ &\propto \left(\frac{1}{\theta}\right)^{((n-1)/2)+\alpha+1} \exp\left\{-\frac{1}{\theta}\left[\frac{(n-1)t}{2} + \frac{1}{\beta}\right]\right\},\end{aligned}$$

which we recognize as the kernel of an inverted gamma pdf, $IG(a, b)$, with

$$a = \frac{n-1}{2} + \alpha, \text{ and } b = \left[\frac{(n-1)t}{2} + \frac{1}{\beta}\right]^{-1}.$$

Direct calculation shows that the mean of an $IG(a, b)$ is $\frac{1}{(a-1)b}$, so

$$\mathbb{E}\left[\theta|t\right] = \frac{\frac{n-1}{2}t + \frac{1}{\beta}}{\frac{n-1}{2} + \alpha - 1} = \frac{\frac{n-1}{2}s^2 + \frac{1}{\beta}}{\frac{n-1}{2} + \alpha - 1}$$

This is the Bayes Estimator of σ^2.

Solution 9

For n observations, $Y = \sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$.

1. The marginal pmf of Y is,

$$\begin{aligned}m(y) &= \int_0^\infty \frac{(n\lambda)^y e^{-n\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\ &= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{(y+\alpha)-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}} d\lambda \\ &= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\end{aligned}$$

Thus,

$$\pi(\lambda|y) = \frac{f(y|\lambda)\pi(\lambda)}{m(y)} = \frac{\lambda^{(y+\alpha)-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}}}{\Gamma(y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}} \sim Gamma\left(y+\alpha, \frac{\beta}{n\beta+1}\right)$$

2.
$$\mathbb{E}[\lambda|y] = (y+\alpha)\frac{\beta}{n\beta+1} = \frac{\beta}{n\beta+1}y + \frac{1}{n\beta+1}(\alpha\beta)$$

.

$$Var(\lambda|y) = (y+\alpha)\frac{\beta^2}{(n\beta+1)^2}.$$

Solution 10

Given: Let $X_1, X_2, \ldots, X_n$ be iid $Poisson(\lambda)$ RVs and suppose $\psi(\lambda) = \mathbb{P}_\lambda(X = 0) = e^{-\lambda}$.

1. **To find:** UMVUE of $\psi(\lambda)$.

We consider the estimator

$$\delta(X) = \begin{cases} 1, & \text{if } X_1 = 0 \\ 0, & \text{otherwise} \end{cases}$$

is unbiased for $\psi(\lambda)$ since

$$\mathbb{E}[\delta(X)] = \mathbb{E}[\delta(X)]^2 = \mathbb{P}\{X_1 = 0\} = e^{-\lambda}.$$
$$\text{Also, } var(\delta(X)) = e^{-\lambda}(1 - e^{-\lambda}).$$

We show next that $\delta(X)$ is the only unbiased estimator of θ and hence is the UMVUE. We know $T(X) = \sum_{i=1}^{n} X_i$ is sufficient and complete for $(\lambda) > 0$ and has $Poisson(n\lambda)$. Then by Rao-Blackwell Theorem, $\mathbb{E}[\delta(X)|T(X)]$ is UMVUE of $\psi(\lambda)$.

$$\begin{aligned}
\mathbb{E}[\delta(X)|T(X) = t] &= 1.\mathbb{P}\{X_1 = 0|T(X) = t\} \\
&= \frac{\mathbb{P}\{X_1 = 0, T(X) = t\}}{\mathbb{P}\{T(X) = t\}} \\
&= \frac{\mathbb{P}\left\{X_1 = 0, \sum_{i=2}^{n} X_i = t\right\}}{\mathbb{P}\left\{\sum_{i=1}^{n} X_i = t\right\}} \\
&= \frac{\mathbb{P}\{X_1 = 0\}.\mathbb{P}\left\{\sum_{i=2}^{n} X_i = t\right\}}{\mathbb{P}\left\{\sum_{i=1}^{n} X_i = t\right\}} \\
&= \frac{e^{-\lambda}.e^{-(n-1)\lambda}\frac{((n-1)\lambda)^t}{t!}}{e^{-n\lambda}\frac{(n\lambda)^t}{t!}} \\
&= \left(\frac{n-1}{n}\right)^t
\end{aligned}$$

Hence, the UMVUE of $\psi(\lambda)$ is given by $T_0 = \left(\frac{n-1}{n}\right)^{\sum\limits_{i=1}^{n} X_i}$.

2. **To find:** The variance of T_0

$$\begin{aligned}
\mathbb{E}\left[T_0\right] &= \sum_{t=0}^{\infty} \left(\frac{n-1}{n}\right)^t e^{-n\lambda} \frac{(n\lambda)^t}{t!} \\
&= e^{-\lambda} \sum_{t=0}^{\infty} e^{-(n-1)\lambda} \frac{((n-1)\lambda)^t}{t!} \\
&= e^{-\lambda} \\
\mathbb{E}\left[T_0^2\right] &= \sum_{t=0}^{\infty} \left(\frac{n-1}{n}\right)^2 t e^{-n\lambda} \frac{(n\lambda)^t}{t!} \\
&= \sum_{t=0}^{\infty} e^{-n\lambda} \frac{\left(\left(\frac{(n-1)^2}{n}\right)\lambda\right)^t}{t!} \\
&= e^{-\lambda[n - \frac{(n-1)^2}{n}]} \\
&= e^{-\lambda\left(\frac{2n-1}{n}\right)} \\
var(T_0) &= \mathbb{E}\left[T_0^2\right] - \left(\mathbb{E}\left[T_0\right]\right)^2 \\
&= e^{-2\lambda}\left(e^{\frac{\lambda}{n}} - 1\right)
\end{aligned}$$

3. **To find:** Fisher Cramer-Rao lower bound for any unibiased estimator of $\psi(\lambda)$.

$$L_X(x|\lambda) = \frac{e^{-n\lambda}\lambda^{\sum\limits_{i=1}^{n} x_i}}{\prod\limits_{i=1}^{n} x_i!}$$

$$ln(L_X(x|\lambda)) = -n\lambda + \sum_{i=1}^{n} x_i ln(\lambda) - ln(\prod_{i=1}^{n} x_i!)$$

Differentiating the log of likelihood function wrt λ we get

$$\frac{dln(L)}{d\lambda} = -n + \frac{\sum\limits_{i=1}^{n} x_i}{\lambda}$$

$$\frac{d^2ln(L)}{d\lambda^2} = -\frac{\sum\limits_{i=1}^{n} x_i}{\lambda^2} \implies I(\lambda) = \mathbb{E}\left[-\frac{d^2ln(L)}{d\lambda^2}\right] = \frac{\mathbb{E}\left[\sum\limits_{i=1}^{n} x_i\right]}{\lambda^2} = \frac{n}{\lambda}$$

Therefore, the Cramer-Rao lower bound is given by

$$CRLB = \frac{[\psi'(\lambda)]^2}{I(\lambda)} = e^{-2\lambda}\frac{\lambda}{n}$$

4. Is T_0 the most efficient estimator?

   We know that $e^{\frac{\lambda}{n}} - 1 \geq \frac{\lambda}{n}, \forall \lambda > 0$ since $e^x \geq 1 + x, \forall x > 0$.

$$\implies e^{-2\lambda}\left(e^{\frac{\lambda}{n}} - 1\right) \geq e^{-2\lambda}\frac{\lambda}{n}$$

$$\implies var(T_0) \geq e^{-2\lambda}\frac{\lambda}{n}.$$

Therefore, T_0 is not most efficient.