Autumn Semester 2021 **December 31, 2021**

IE 605: Engineering Statistics

Tutorial 9

Exercise 1. *1. Show that* $\left[\frac{\sum_{i=1}^{n} X_i\left(\sum_{i=1}^{n} X_i-1\right)}{n(n-1)}\right]$ *is the unbiased estimator of* θ^2*, for the sample* $x_1, x_2, \ldots, x_n$ *drawn on* X *which takes values* 1 *or* 0 *with respective probabilities* θ *and* $(1-\theta)$*.*

2. Let X *be distributed in the Poisson form with parameter* θ*. Find the unbiased estimator of* $\exp[-(k+1)\theta], k > 0$*.*

Exercise 2. *1. Given the probability density function*

$$f(x|\theta) = \frac{1}{\pi\{1+(x-\theta)^2\}}; -\infty < x < \infty, -\infty < \theta < \infty.$$

Show that the Cramer-Rao lower bound of variance of an unbiased estimator of θ *is* $\frac{2}{n}$*, where* n *is the size of the random sample drawn from this distribution.*

2. Prove that under certain general conditions of regularity to be stated clearly, the mean square deviation $\mathbb{E}\left[\hat{\theta}-\theta\right]^2$ *of an estimator* $\hat{\theta}$ *of the parameter* θ*, can never fall below a positive limit depending only on the density function* $f(x,\theta)$*, the size of the sample and the bias of the estimate.*

Exercise 3. *The independent random variables* $X_1, X_2, \ldots, X_n$ *have the common distribution,*

$$P(X_i \le x|\alpha,\beta) = \begin{cases} 0, & \text{if } x < 0 \\ \left(\frac{x}{\beta}\right)^{\alpha}, & \text{if } 0 \le x \le \beta \\ 1, & \text{if } x > \beta \end{cases}$$

where the parameters α *and* β *are positive.*

1. *Find a two-dimensional sufficient statistic for* (α, β)*.*
2. *Find the MLEs of* α *and* β*.*
3. *The length (in mm) of cuckoos' eggs found in hedge sparrow nets can be modeled with this distribution. For the data* $22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7, 23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0$*, find the MLEs of* α *and* β*.*

x	$f(x|1)$	$f(x|2)$	$f(x|3)$
0	1/2	1/4	0
1	1/3	1/4	0
2	0	1/4	1/4
3	1/6	1/4	1/2
4	1/6	0	1/4

Table 1: The Probability distribution of $f(x|\theta), \theta \in \{1, 2, 3\}$

Exercise 4. *Let $X_1, X_2, \dots, X_n$ be iid with pmf,*

$$f(x|\theta) = \theta x^{(\theta-1)}, 0 \le x \le 1, 0 < \theta < \infty.$$

1. *Find the MLE of θ, and show that its variance $\to 0$ an $n \to \infty$.*

2. *Find the method of moments estimator of θ.*

Exercise 5. 1. *One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ. The table is given in Table 1.*

2. *Given a random sample $X_1, X_2, \dots, X_n$ from a population with pdf $f(x|\theta)$, show that maximizing the likelihood function, $L(\theta|x)$, as a function of θ is equivalent to maximizing* $\log L(\theta|x)$.

Exercise 6. 1. *Let $X_1, X_2, \dots, X_n (n \ge 2)$ be a sample from $N(\mu, \sigma^2)$. Find an unbiased estimator for σ^p, where $p + n > 1$. Find a minimum MSE estimator of σ^p.*

2. *Let $X_1, X_2, \dots, X_n$ be iid $N(\mu, \sigma^2)$ RVs. Find a minimum MSE estimator of the form αS^2 for the parameter σ^2. Compare the variances of the minimum MSE estimator and the obvious estimator σ^2.*

Exercise 7. *Let $X_1, X_2, \dots, X_n$ be a sample from a population with mean θ and finite variance, and T be an estimator of θ of the form $T(X_1, X_2, \dots, X_n) = \sum_{i=1}^{n} \alpha_i X_i$. If T is an unbiased estimator of θ that has minimum variance and T' is another linear unbiased estimator of θ, then*

$$cov_\theta(T, T') = var_\theta(T).$$

Exercise 8. *If S^2 is the sample variance based on a sample of size n from a normal population, we know that $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2_{n-1} distribution. The conjugate prior for σ^2 is the inverted gamma pdf, $IG(\alpha, \beta)$, given by,*

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{(\alpha+1)}} e^{-\frac{1}{(\beta\sigma^2)}}, 0 < \sigma^2 < \infty,$$

where α and β are positive constants. Show that the posterior distribution of σ^2 is $IG\left(\alpha + \frac{n-1}{2}, \left[\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right]^{-1}\right)$. Find the mean of this distribution, the Bayes estimator of σ^2.

Exercise 9. *Let $X_1, X_2, \ldots, X_n$ be iid Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for Poisson.*

1. *Find the posterior distribution of λ.*
2. *Calculate the posterior mean and variance.*

Exercise 10. *Let $X_1, X_2, \ldots, X_n$ be iid $Poisson(\lambda)$ RVs and suppose $\psi(\lambda) = \mathbb{P}_\lambda(X = 0) = e^{-\lambda}$.*

1. *Find the UMVUE of $\psi(\lambda)$. Denote the estimator as T_0.*
2. *Find the variance of T_0.*
3. *Find the Fisher Cramer-Rao lower bound for any unibiased estimator of $\psi(\lambda)$.*
4. *Is T_0 the most efficient estimator?*