Autumn Semester 2021 **December 31, 2021**

IE 605: Engineering Statistics

Tutorial 10

Exercise 1. *Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, σ^2 known. An LRT of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is a test that rejects H_0 if $\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}} > c$. Find an expression, in terms of standard normal probabilities, for the power function of this test.*

$f : \mathcal{E} \rightarrow \mathcal{C} = \{1, 2, ...C\}$ *s.t.* $f(\textit{fusion embedding}) = class$

Let $f_1 \colon \mathcal{X}_1 \rightarrow \mathcal{E}_1$, $f_2 \colon \mathcal{X}_2 \rightarrow \mathcal{E}_2$ *and* $f_1 \colon \mathcal{X}_3 \rightarrow \mathcal{E}_3$
We want to learn: $g_{fusion} = f_1 \oplus f_2 \oplus f_3 \colon \mathcal{X} \rightarrow \mathcal{E}$
where $\mathcal{X} = \phi(\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3)$ *and* $\mathcal{E} = \phi^{'}(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3)$ *for some* $\phi(.)$ *and* $\phi^{'}(.)$

Exercise 2. *Suppose we observe m i.i.d. Bernoulli(θ) random variables, denoted by $Y_1, \ldots, Y_m$. Show that the LRT of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ will reject H_0 if $\sum\limits_{i=1}^{m} Y_i > b$.*

Exercise 3. *"If $f(x|\theta)$ is the pmf of a discrete random variable, then the numerator $\lambda(x)$, the LRT statistic, is the maximum probability of the observed sample when the maximum is computed over parameters in the null hypothesis. Furthermore, the denominator of $\lambda(x)$ is the maximum probability of the observed sample over all possible parameters."- Prove the above text.*

Exercise 4. *A random sample $X_1, \ldots, X_n$ is drawn from a Pareto population with pdf,*

$$f(x|\theta, \nu) = \frac{\theta\nu^\theta}{x^{\theta+1}} \mathbb{I}_{[\nu,\infty]}(x), \quad \theta > 0, \quad \nu > 0.$$

1. *Find the MLEs of θ and ν.*

2. *Show that the LRT of*

$$H_0 : \theta = 1, \nu \text{ unknown} \quad \text{versus} \quad H_1 : \theta \neq 1, \nu \text{ unknown},$$

has critical region of the form $\{x : T(x) \leq c_1 \text{ or } T(x) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log \left[\frac{\prod_{i=1}^{n} X_i}{(\min_i X_i)^n} \right].$$

3. *Show that, under H_0, $2T$ has a chi-squared distribution, and find the number of degrees of freedom. (Hint: Obtain the joint distribution of the $n - 1$*

non-trivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$, does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)

Exercise 5. *Suppose that we have two independent samples $X_1, \ldots, X_n$ are Exponential(θ), and $Y_1, \ldots, Y_m$ are Exponential(μ).*

1. *Find the LRT of*

$$H_0 : \theta = \mu \quad \textit{versus} \quad H_1 : \theta \neq \mu.$$

2. *Show that the test in part (1) can be based on the statistic*

$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}.$$

3. *Find the distribution of T when H_0 is true.*

Exercise 6. *Let $X_1, \ldots, X_n$ be i.i.d. Poisson(λ), and let λ have a Gamma(α, β) distribution, the conjugate family for the Poisson distribution.*

1. *Find the posterior distribution of λ, including the posterior mean and variance.*

2. *Consider the Bayesian test of*

$$H_0 : \lambda \leq \lambda_0 \quad \textit{versus} \quad H_1 : \lambda > \lambda_0.$$

   (a) *Calculate the expressions for the posterior probabilities of H_0 and H_1.*

   (b) *If $\alpha = \frac{1}{2}$ and $\beta = 2$, the prior distribution is a chi-squared distribution with 5 degrees of freedom. Explain how a chi-squared table could be used to perform a Bayesian test.*

Exercise 7. *For samples of size $n = 1, 4, 16, 64, 100$ from a normal population with mean μ and known variance σ^2, plot the power function of the following LRTs. Take $\alpha = 0.05$.*

1. $H_0 : \mu \leq 0$ *versus* $H_1 : \mu > 0$.

2. $H_0 : \mu = 0$ *versus* $H_1 : \mu \neq 0$.

Exercise 8. *The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0 : \theta = 1$ vs. $H_1 : \theta = 2$ needs to be constructed. Find the UMP level $\alpha = 0.10$ test and compute the Type II error probability.*

Exercise 9. *Let X be a random variable whose pmf under H_0 and H_1 is given by:*

Use the Neyman-Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$. Compute the probability of Type II error for this test.

Exercise 10. *Suppose X is one observation from a population with beta(θ,1) pdf. Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$. ? Is so, find it. If not, prove so.*

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79