NEW SUFFICIENT CONDITIONS FOR SEMIDEFINITE REPRESENTABILITY OF CONVEX SETS

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Abstract

If a convex set K is semidefinite representable, linear optimization over K can be done in polynomial time, provided that the semidefinite representation of K is of polynomial size. So, semidefinite representable sets play an important role in modern convex optimization. Our work is concerned with different sufficient conditions of a convex, semialgebraic set, say K to be semidefinite representable, depending on its j-projections and j-sections. We prove that the closure of a bounded convex set is semidefinite representable if all its j-projections are semidefinite representable where $2 \leq j \leq n$. We give another result which states that a closed convex set is semidefinite representable if all its j-sections are semidefinite representable. We work on affine space $\mathbb{E}^n$.

Introduction

A convex set $K \subseteq \mathbb{R}^n$ is semidefinite representable if there exists integers u, v such that $\{A_i\}_{i=0}^n$ and $\{B_j\}_{j=1}^u$ are all real entried $v \times v$ symmetric matrices and

$$K = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^u, A_0 + \sum_{i=1}^n A_i x_i + \sum_{j=1}^u B_j y_j \succeq 0 \right\},$$

where the matrix $A \succeq 0$ means that the matrix A is positive semidefinite. In another way, K is the linear projection of the set $K^{'}$ on $\mathbb{R}^n$ where

$$K^{'} = \left\{ (x, y) \in \mathbb{R}^{n+u} : A_0 + \sum_{i=1}^n A_i x_i + \sum_{j=1}^u B_j y_j \succeq 0 \right\} \subseteq \mathbb{R}^{(n+u)}.$$

The set $K^{'}$ is called the semidefinite representation of the convex set K. Lots of sufficient conditions for a convex, semialgebraic set to be semidefinite representable, are contributed in [Lasserre, 2009], [Helton and Nie, 2009], [Nie, 2012], [Helton and Nie, 2010].

Notation and terminology

A *j-flat* is a j-dimensional affine subspace in $\mathbb{E}^n$. A *j-projection* of K is the image of K under an affine projection, say π onto a j-flat. The *j-section* of K is the intersection of K with a j-flat. $cone(p, K)$ is the smallest cone containing K, with the vertex p. The origin in $\mathbb{E}^n$ is ϕ. N_i is a closed ball centred at x_i, with radius $\delta_i > 0$.

Definition 1. ***Boundedly semidefinite representable:*** *A subset K of $\mathbb{E}$ is said to be* boundedly semidefinite representable, *provided its intersection with each bounded semidefinite representable set in $\mathbb{E}$ is semidefinite representable.*

Definition 2. ***Semidefinite representable at a point:*** *A subset K of $\mathbb{E}$ is said to be* semidefinite representable at a point $p \in K$, *provided some neighbourhood of p relative to K is semidefinite representable.*

New sufficient conditions for convex sets to be semidefinite representable

Theorem 1. *Let K is a convex set in $\mathbb{E}^n$ and p is any point in K. If all j-projections of $cone(p, K)$ are semidefinite representable, $2 \leq j \leq n$ then, $clcone(p, K)$ is semidefinite representable.*

Theorem 2. *Suppose K is a convex set in $\mathbb{E}^n$ and p is any point in K. K is semidefinite representable at p iff πK is semidefinite representable at p whenever π is an affine projection of K onto a j-flat through p, where $2 \leq j \leq n$.*

Theorem 3. *With $2 \leq j \leq n$, the closure of a bounded convex subset of $\mathbb{E}^n$ is semidefinite representable if all its j-projections are semidefinite representable.*

Theorem 4. *Let K is a closed convex set in $\mathbb{E}^n$ and $p \in intK$. K is semidefinite representable iff all its j-sections of K through p are semidefinite representable, $2 \leq j \leq n$.*

Outline of the proof of Theorem 1

The following results play an important role in the proof:

Lemma 1. *Let, C is a convex cone in $\mathbb{E}^n$ all of whose j-projections are semidefinite representable, $2 \leq j \leq n$. Then, all $(n - j + 1)$-sections of clC are semidefinite representable.*

Lemma 2. *Suppose C is a convex cone with vertex ϕ, and L denotes the lineality space of clC. Let, C is not linear. Then there is a linear functional f such that $f = 0$ on L and $f > 0$ on $clC \sim L$. With $t > 0$, let $H_0 = f^{-1}0$ and $H_t = f^{-1}t$. Then C is closed and semidefinite representable iff $H_t \cap C$ is boundedly semidefinite representable and $L \subseteq C$.*

Lemma 1 implies that $(n - 1)$-sections of $clcone(p, K)$ are semidefinite representable. So, the $(n - 1)$-sections of $clcone(p, K)$ are trivially boundedly semidefinite representable. Also, by Lemma 2, we have $clcone(p, K)$ is semidefinite representable.

Outline of the proof of Theorem 2

($\Rightarrow$) Suppose that K is semidefinite representable at p. The following Lemma

Lemma 3. *A convex set K is semidefinite representable at p iff $cone(p, K)$ is semidefinite representable.*

states that $cone(p, K)$ is semidefinite representable. Thus, $\pi cone(p, K)$ is semidefinite representable where π is an affine projection of K onto j-flat through p. So, $cone(p, \pi K)$ is semidefinite representable. Lemma 3 implies that πK is semidefinite representable at p.

($\Leftarrow$) Suppose that πK is semidefinite representable at p. The $cone(p, \pi K)$ is semidefinite representable by Lemma 3. So, $\pi cone(p, K)$ is semidefinite representable. The Theorem 1 implies that $clcone(p, K)$ is semidefinite representable. A slight modification of Lemma 3 gives the following result:

Lemma 4. *If $clcone(p, K)$ is semidefinite representable, then clK is semidefinite representable at $p \in K$.*

Lemma 4 states that clK is semidefinite representable at p. The following result

Proposition 1. *clK is semidefinite representable at $p \in K$ iff K is semidefinite representable at p.*

implies that K is semidefinite representable at p.

Outline of the proof of Theorem 3

Let K is a bounded convex set in $\mathbb{E}^n$ and we have πK is semidefinite representable. So, πK is boundedly semidefinite representable. The following Lemma

Lemma 5. *With $2 \leq j \leq n$, if all j-projections of a convex subset of $\mathbb{E}^n$ are boundedly semidefinite representable, the closure of the set itself must be semidefinite representable.*

states that clK is boundedly semidefinite representable. So, clK is compact, convex and semidefinite representable at all its points. So, we get a finite number of balls N_i to cover clK:

$$clK = conv[\cup_{i=1}^t (N_i \cap clK)]$$

$(N_i \cap clK)$ is semidefinite representable neighbourhood of x_i in clK. So, clK is semidefinite representable.

Outline of the proof of Theorem 4

($\Rightarrow$) If K is semidefinite representable, then all its j-sections are semidefinite representable.

($\Leftarrow$) Suppose $p = \phi$. Then K^0 is bounded and $(K^0)^0 = K$. Consider a linear projection π of $(\mathbb{E}^n)^{'}$ onto one of its j-subspaces and M be the kernel of π and $L = M^0$. We know, from [Klee, 1959],

$$(L \cap K)^0 = M + cl(\pi K^0)$$

As K^0 is compact, πK^0 is closed. Hence,

$$(L \cap K)^0 = M + \pi K^0 \tag{1}$$

Let us fix a j. We have, $L \cap K$ is semidefinite representable. So, $(L \cap K)^0$ is semidefinite representable. From equation 1, we say $M + \pi K^0$ is semidefinite representable and so is πK^0. Also, by Theorem 1, we say K^0 is semidefinite representable. Hence, K is semidefinite representable.

Conclusions

The main contribution of this work is to give new sufficient conditions for a convex set K to be semidefinite representable in an affine space. The sufficient conditions are related to the j-projections and j-sections of K. The Theorem 1 states that $clcone(p, K)$ is semidefinite representable if all the j-projections of $cone(p, K)$ are semidefinite representable. The Theorem 3 gives another sufficient condition: the closure of a bounded convex set is semidefinite representable if all its j-projections are semidefinite representable. The Theorem 4 states that a closed convex set is semidefinite representable if all its j-sections are semidefinite representable. The explicit construction of semidefinite representation of K, given the semidefinite representations of all its j-projections or, j-sections is an interesting future research problem.

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