Characterizations of semidefinite representable sets

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Abstract

The feasible regions of semidefinite programming problems are the semidefinite representable sets. Our aim is to characterize the semidefinite representable sets in a finite dimensional affine space, as convex sets, certain of whose projections and sections are semidefinite representable. This work continues the results of Klee [Kle59], which gives us some necessary and sufficient conditions for a convex set in affine space to be polyhedral. As, semidefinite representable sets are generalizations of polyhedra, our aim is to extend few results from [Kle59], which characterizes the polyhedral sets as convex sets, whose sections and projections are polyhedral.

Keywords: semidefinite programming, semidefinite representable sets, j-flat, j-section, j-projection, polyhedra.

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1 Introduction

Semidefinite programming has turned out to be very useful and valuable tool in polynomial optimization in recent years. It generalizes linear programming and is concerned to find the minimum objective value of a linear function over certain class of convex sets, which are called spectrahedra or, semidefinite representable sets. A large amount of problems from various branches of mathematics such as combinatorial optimization, control theory can be approached using semidefinite programming.

The importance of semidefinite programming comes from the fact that there are efficient algorithms to solve semidefinite programming problems [BTN01]. If a convex set is semidefinite representable, linear optimization over the convex set can be done in polynomial time, provided that the semidefinite representation is of polynomial size.

Semidefinite programming can be regarded as an extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities, or, equivalently, the first non-negative orthant is replaced by the cone of positive semidenite matrices.Therefore the theory of semidefinite programming closely parallels the theory of linear programming. The feasible regions of semidefinite programming are known as spectrahedra, more generally semidefinite representable sets.

The characterization of spectrahedra is contributed in [HV03], which states that spectrahedra are *rigidly convex sets*.

A set S is called *semidefinite representable set* (or, *projection of spectrahedra*) if it can be represented as

$$S = \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^N, (x, u) \in \bar{S}\}$$

where

$$\bar{S} = \{(x, u) \in \mathbb{R}^{n+N} : (A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{i=1}^{N} B_i u_i) \geq 0\}.$$

Here, A_0, A_i, B_i all are real symmetric matrices of same order and a matrix $A \geq 0$ denotes that A is positive semidefinite. Semidefinite programming can also be performed on semidefinite representable set. So, semidefinite representable sets are important to study. There are only two necessary conditions for a set to be semidefinite representable, being convex and semi-algebraic [JBR98]. A collection of sets are given in [BTN01], [Hen08], [Hen09] as examples of semidefinite representable sets.

It leads to a fundamental question: which sets are semidefinite representable? Helton and Nie [HN09b], [Nie12] give several sufficient conditions for a convex, semi-algebraic set S which is defined by polynomials $g_1, \ldots, g_m$ such as

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\},$$

to be semidefinite representable.

Other sufficient conditions on defining polynomials are also available in [HN09a]. The explicit construction is based on Lasserre's relaxation method [Las09] for semidefinite representation of a compact, convex, semi-algebraic sets with concave defining polynomials. Several sufficient conditions are contributed in [HN09b], involving the geometry i.e., the boundary of compact, convex, semi-algebraic sets. Even a new method is given in [Net09] to construct semidefinite representation for a class of non-closed sets. Recently, it is proved that all convex, semi-algebraic sets in $\mathbb{R}^2$ are semidefinite representable, [Sch12].

Motivation . As, semidefinite representable sets are generalization of polyhedral sets, our aim is to extend few results of polyhedra, contributed in [Kle59], considering semidefinite representable sets. This paper continues the work of V. Klee [Kle59]. The semidefinite representable set is convex and semi-algebraic but need not be closed or bounded. We will characterize semidefinite representable sets as convex sets, certain of whose projections and sections are semidefinite representable.

Organization . This paper deals with semidefinite representable sets of a finite-dimensional euclidean space E. In connection with the characterization of semidefinite representable sets, we will give some new definitions such as *boundedly semidefinite representable set*, a set which is *semidefinite representable at a point*. We will develop various properties of boundedly semidefinite representable sets and a set which is semidefinite representable at a point in it. We will define the smallest cone which contains a semidefinite representable set. We will establish some connections between a set which is semidefinite representable and a cone which contains a semidefinite representable set. As an indication of contents we list the headings of the remaining subsections :

- Some new definitions
- Cones and semidefinite representable sets
- Projections of semidefinite representable sets
- Sections of semidefinite representable sets

There are numerous questions for further research.

Notation and terminology . The j-flat is a j-dimensional affine subspace of affine space E. The j-section of a set $S \subset E$ is the intersection of S with a j-flat. The j-projection of S is the image of S under an affine projection of E onto a j-flat. Set-theoretic union, intersection will be denoted by $\cup$, $\cap$. The Minkowski-sum will be denoted by $+$. The closure of set S is clS. An open ball will be denoted by $B(u, r)$ where u is the center of ball and $r > 0$ is the radius of ball. For a and b in E, we will denote $[a, b] = \{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}$ and $]a, b[= \{\lambda a + (1-\lambda)b : 0 < \lambda < 1\}$. For a real number x, we will denote $xS = \{xs : s \in S\}$ and $X + Y = \{x + y : x \in X, y \in Y\}$. The polar of a set K is denoted as K^0 and the inverse of a function f is f^{-1}.

2 Some characterizations of semidefinite representable sets

We already know, that semidefinite programming is the generalization of linear programming, as we are minimizing a linear function, on the cone of positive semidefinite matrices, instead of the cone of non-negative orthant. It is also stated before, that semidefinite representable sets (feasible regions of semidefinite programming) are the generalizations of polyhedra (feasible regions of linear programming). These facts motivate us to extend the results of convex polyhedra, [Kle59] considering a semidefinite representable set, K in E^n.

This work is an extension of the literature, 'Some characterization of convex polyhedra' by Victor Klee which contains some very strong and useful results of convex polyhedra in finite-dimensional affine space E. Some important results are :

Theorem 4.1., [Kle59]*Suppose K is a convex subset of E^n, $p \in K$, and $2 \leq j \leq n$. Then K is polyhedral at p iff πK is polyhedral at p whenever π is an affine projection of K onto a j-flat through p.*

This gives a corollary which is very interesting in characterization of polyhedra and its j-projections.

Corollary 4.5., [Kle59] *With $2 \leq j \leq n$, a bounded convex subset of E^n is polyhedral iff all its j-projections are polyhedral.*

This result gives a necessary and sufficient condition for a convex set of an affine space to be polyhedral. The restriction of boundedness is essential for $j = 2$, but this restriction can be removed when $j \geq 3$. Because there exists a nonpolyhedral convex set in E^3 all of whose 2-projections are polyhedral, Example 5.16., [Kle59].

Subsection 2.1 gives some new definitions which we will use in our work. We have defined new terms, *boundedly semidefinite representable* set and set which is *semidefinite representable at a point*. We will modify Proposition 2.17 of Klee literature [Kle59]. This is our Observation 3.

In Subsection 2.2, we will define a cone which contains a semidefinite representable set. This is the smallest cone containg a set. We will present a collection of results about convex cone proposition 3.1. from [Kle59]. We modify the part 3.1., (vii) of these results. This is our Observation 7. The generalized version of Observation 7 is Observation 8. Observation 9 is trivial, although we will need it to prove our main result. It states a trivial fact about semidefinite representability under affine mapping. Our aim is to extend Theorem 4.1., [Kle59]. Although, we could not do it till now. We extended some part of the main result. It is given in subsection 2.3 and in 2.4. We will extend the remaining parts in our future research.

2.1 Some new definitions

Section 2 of [Kle59] is concluded with two new definitions : *boundedly polyhedral* and a set which is *polyhedral at a point*. These are :

A subset K of E is said to be boundedly polyhedral provided its intersection with each bounded polyhedron in E is polyhedral,

A subset K of E is said to be polyhedral at a point $p \in K$ provided some neighbourhood of p relative to K is polyhedral.

In connection with these definitions regarding polyhedrality, we will give two new definitions in case of semidefinite representable sets in E. These new definitions are:

Definition 1. A subset K of E is said to be *boundedly semidefinite representable* provided its intersection with each bounded semidefinite representable set in E is semidefinite representable.

This definition implies a subset K of E is boundedly semidefinite representable if $K \cap B$ is semidefinite representable for each bounded semidefinite representable set B in E.

Definition 2. A subset K of E is said to be *semidefinite representable at a point $p \in K$* provided some neighbourhood of p relative to K is semidefinite representable.

Hence, a subset K of E is semidefinite representable at a point $p \in K$ if T is a semidefinite representable neighbourhood of p such that $T \subset K$ and $B(p, r) \cap K \subset T$ for some $r > 0$.

Observation 3. *A set is boundedly semidefinite representable iff it is closed,convex and semidefinite representable at all its points.*

Proof. Let, a closed, convex set K of E is boundedly semidefinite representable.
So, for each bounded semidefinite representable set B in E, $K \cap B$ is semidefinite representable, and hence, $cl(K \cap B)$ i.e., $K \cap clB$ is semidefinite representable. By compactness of $K \cap clB$,

$$K \cap clB = conv\Bigg(\cup_{i=1}^{l} (N_i \cap K \cap clB)\Bigg) \tag{1}$$

where $N_i = clB(x_i, r_i)$ is a closed ball with centre $x_i \in K \cap clB$ and radius $r_i > 0$, a very small quantity. It is well known that N_i is semidefinite representable. Now, we have

$$K \cap clB \subset K \text{ and } (N_i \cap clB) \cap K \subset K \cap clB$$

and $K \cap clB$ is semidefinite representable. Hence, $K \cap clB$ is semidefinite representable neighbourhood of x_i relative to K. So, K is semidefinite representable at x_i. As, x_i is an arbitrary point in $K \cap clB$ and B is any bounded semidefinite representable set in E, it is always possible to find out a semidefinite representable neighbourhood of any point in K.

Conversely, suppose K is closed, convex and semidefinite rpresentable at all its points.
Consider a bounded semidefinite representable set B. Each point $x \in K$ admits a bounded semidefinite representable neighbourhood N_x relative to K, and by compactness of $clB \cap K$ there must be a finite set $X \subset K$ with $clB \cap K \subset \cup_{x \in X} N_x$. Let, $Z = conv(\cup_{x \in X} N_x)$. Then, Z is semidefinite representable, whence so is $clB \cap Z$, and we have

$$\begin{aligned} clB \cap K &\subset Z \subset K \\ \implies clB \cap K &\subset clB \cap Z \subset clB \cap K \\ \implies clB \cap K &= clB \cap Z \end{aligned}$$

Hence, $clB \cap K$ is semidefinite representable. Hence, K must be boundedly semidefinite representable. □

Observation 4. *Let, K is a subset in E and K is boundedly semidefinite representable. If K is bounded, then K is semidefinite representable.*

Proof. K is a compact set and hence, we will get a finite number of balls to cover K. Then,

$$K = conv\Bigg(\cup_{i=1}^{l} (N_i \cap K)\Bigg), \tag{2}$$

where N_i is a closed ball centred at $x_i \in K$ and with radius $r_i > 0$, a very small quantity. $N_i \cap K$ is a semidefinite representable neighbourhood of x_i relative to K. Thus, $conv\Bigg(\cup_{i=1}^{l}(N_i \cap K)\Bigg)$ is semidefinite representable and so is K. □

Remark 5. Suppose, K is a semidefinite representable set in E. It is trivial, that K is boundedly semidefinite representable, as the intersection of K with any bounded semidefinite representable set in E is semidefinite representable. By Observation 3, as K is boundedly semidefinite representable, then K is closed, convex and semidefinite representable at all its points.

2.2 Cones and semidefinite representable sets

Suppose, K is a subset in E and p is any point in E. The $cone(p, K)$ is the smallest cone which contains K and has vertex p. It is denoted as :

$$cone(p, K) = p +]0, \infty[(K - p).$$

It is very trivial to state, that the point p belongs to $cone(p, K)$ iff p belongs to K. It is clear from the definition of $cone(p, K)$, that if X is semidefinite representable, then $cone(p, K)$ is also semidefinite representable. But the converse may not be true as the example shows in the following:

Example 6. Let, K is the convex hull of the integer points in the set $\{(x, y) : y \geq x^2\}$ where $cone(p, K) = \{(x, y) : y \geq x^2\}$ is semidefinite representable. K is the convex hull of infinite number of discrete points, which is not semi-algebraic, and hence, is not semidefinite representable.

The present subsection consists of developing several connections between a semidefinite representable set K and the cone containing K which is $cone(p, K)$. We need these tools in our proofs of main results. First we will start stating some elementary and very useful results about convex cones which are contributed in Section 3, [Kle59].

The Proposition 3.1. states

Proposition 3.1., [Kle59] *Suppose C is a convex cone in E with vertex ϕ, and let L denote the lineality space of $clC(L = clC \cap -clC)$. Then C is linear iff $C \subset L$. Suppose now that C is not linear, so $C \not\subset L$. Then there is a linear functional f on E such that $f = 0$ on L and $f > 0$ on $clC \sim L$. With $t > 0$, let $H_0 = f^{-1}0$ and $H_t = f^{-1}t$. Then the following statements are true:*

(i) $C = (H_0 \cap C) \cup]0, \infty[(H_t \cap C)$;

(ii) $H_t \cap C \supset (H_0 \cap C) + (H_t \cap C)$ *and* $C \supset (H_0 \cap C) +]0, \infty[(H_t \cap C)$, *with equality when* $\phi \in C$*(and also under certain conditions);*

(iii) if $p \in H_t \cap C$, *then* $cone(p, H_t \cap C) = \pi C + p$, *where* π *is the projection of* E *onto* H_0 *which is the identity on* H_0 *and maps* p *onto* ϕ.

(iv) if S *is a subspace supplimentary to* L *in* H_0 *and* S_t *is a translate of* S *to* H_t *then* $S_t \cap C$ *is bounded;*

(v) if $L \subset C$, *then* $H_t \cap C = L + (S_t \cap C)$ *and* $C = L + [0, \infty[(S_t \cap C)$;

(vi) C *is closed iff* $H_t \cap C$ *is closed and* $L \subset C$;

(vii) C *is polyhedral iff* $H_t \cap C$ *is boundedly polyhedral and* $L \subset C$.

The above stated results $(i), (ii), (iii), (iv), (v), (vi)$ are essentially true for positive semidefinite cone, as positive semidefinite cone is a convex cone. It is very interesting to verify the result 3.1, (vii) taking a convex cone C which is semidefinite representable. This interesting fact gives our next Observation.

Observation 7. *Suppose C is a convex cone in E with vertex ϕ, and let L denote the lineality space of clC. Then C is linear iff $C \subset L$. Suppose now that C is not linear, so $C \not\subset L$. Then there is a linear functional f on E such that $f = 0$ on L and $f > 0$ on $clC \sim L$. With $t > 0$, let $H_0 = f^{-1}0$ and $H_t = f^{-1}t$. Then the following is true :*
C is closed and semidefinite representable iff $H_t \cap C$ is boundedly semidefinite representable and $L \subset C$.

Proof. Suppose, C is closed and semidefinite representable convex cone. By, 3.1., (vi), [Kle59] $H_t \cap C$ is closed and $L \subset C$. We know, S is the subspace supplementary to L and so, S is semidefinite representable. S_t is the translate of S to H_t and hence, S_t is semidefinite representable. The intersection of S_t and C is semidefinite representable. As, $L \subset C$, by 3.1.(v), [Kle59] $H_t \cap C = L + (S_t \cap C)$. L and $S_t \cap C$ are both semidefinite representable. Thus, the sum of L and $S_t \cap C$ is semidefinite representable. Hence, $H_t \cap C$ is semidefinite representable. So, the intersection of $H_t \cap C$ with a bounded semidefinite representable set is semidefinite representable. Thus, $H_t \cap C$ is boundedly semidefinite representable.

Conversely, $H_t \cap C$ is boundedly semidefinite representable and $L \subset C$. We know, S_t is translate of S to H_t. So, $S_t \subset H_t$. So, $(S_t \cap C) \subset (H_t \cap C)$. Obviously, $(S_t \cap C) \subset S_t$. We know, by 3.1.(iv), [Kle59],$(S_t \cap C)$ is bounded. So, there exists a bounded set B in S_t such that $(S_t \cap C) \subset B \subset S_t$. Let us consider the set $(H_t \cap C) \cap B$. Then,

$$(H_t \cap C) \cap B \tag{3}$$
$$= (L + (S_t \cap C)) \cap B, \text{ as } L \subset C \tag{4}$$
$$= (S_t \cap C) \cap B \tag{5}$$
$$= S_t \cap C, \text{ as } S_t \cap C \subset B \tag{6}$$

The equivalence between the sets (3) and (4) is an application of Proposition $3.1.$, (v), [Kle59]. We will give the equivalence of the two sets (4) and (5) in the following :

First, we will show $(S_t \cap C) \cap B \subset (L + (S_t \cap C)) \cap B$.

$$\text{We know, } S_t \cap C \subset H_t \cap C$$
$$\implies (S_t \cap C) \cap B \subset (H_t \cap C) \cap B$$
$$\implies (S_t \cap C) \cap B \subset (L + (S_t \cap C)) \cap B, \text{ by } 3.1., (v)$$

So, our first part is completed. Now, we will prove,$(L + (S_t \cap C)) \cap B \subset (S_t \cap C) \cap B$. This is our second part. It is in the following :

$$\text{Let, a point } x \in (L + (S_t \cap C)) \cap B$$
$$\implies x \in (L + (S_t \cap C)) \text{ and } x \in B$$
$$\implies x = u + v \text{ such that } u \in L \text{ and } v \in (S_t \cap C), x \in B$$
$$\text{So, } u \in C, \text{ as } L \subset C; v \in S_t \text{ and } v \in C; x \in B, \text{ thus } x \in S_t$$
$$\text{Thus, we are getting } u, v \in C \text{ and } x \in B, x \in S_t$$
$$\text{As } C \text{ is closed cone }, u + v \in C \text{ and } x \in S_t \cap B$$
$$\text{Hence, } x \in C \text{ and } x \in S_t \cap B$$
$$\implies x \in (S_t \cap C) \cap B$$

Hence, we can say $(L + (S_t \cap C)) \cap B \subset (S_t \cap C) \cap B$. Combining the first part and the second part, we can say $(L + (S_t \cap C)) \cap B = (S_t \cap C) \cap B$. This proves the equivalence of the sets in (4) and (5).

Hence, $(H_t \cap C) \cap B$ is the set $S_t \cap C$. Suppose, B is semidefinite representable. As, $(H_t \cap C)$ is boundedly semidefinite representable, then $(H_t \cap C) \cap B$ is semidefinite representable. Hence, $S_t \cap C$ is semidefinite representable. As $L \subset C$, then by $3.1.$, (v), [Kle59] $C = L + [0, \infty[(S_t \cap C)$. Now, L is lineality space of clC, so it is semidefinite representable. Hence, the sum of L and $[0, \infty[(S_t \cap C)$ is semidefinite representable. Thus, C is semidefinite representable. □

A slight modification of Observation 7 gives the next Observation.

Observation 8. *Suppose C is a convex cone in E with vertex ϕ, and let L denote the lineality space of clC. Then C is linear iff $C \subset L$. Suppose now that C is not linear, so $C \not\subset L$. Then there is a linear functional f on E such that $f = 0$ on L and $f > 0$ on $clC \sim L$. With $t > 0$, let $H_0 = f^{-1}0$ and $H_t = f^{-1}t$. Then the following is true :*
C is closed and semidefinite representable iff $H_t \cap C$ is closed and semidefinite representable and $L \subset C$.

Proof. Suppose, C is closed and semidefinite representable convex cone. By, 3.1., (vi), [Kle59], $H_t \cap C$ is closed and $L \subset C$. We know, S is the subspace supplementary to L and so, S is semidefinite representable. S_t is the translate of S to H_t and hence, S_t is semidefinite representable. The intersection of S_t and C is semidefinite representable. As, $L \subset C$, by 3.1.(v), $H_t \cap C = L + (S_t \cap C)$. L and $S_t \cap C$ are both semidefinite representable. Thus, the sum of L and $S_t \cap C$ is semidefinite representable. Hence, $H_t \cap C$ is semidefinite representable.

Conversely, Suppose, $H_t \cap C$ is closed and semidefinite representable and $L \subset C$. As, $H_t \cap C$ is closed and $L \subset C$, by 3.1., (vi), [Kle59] we can say C is closed. Now, we know $S_t \cap C$ is bounded. So, we can get a semidefinite representable set B in S_t, such that $S_t \cap C \subset B \subset S_t$. We have proved in Observation 7, that $(H_t \cap C) \cap B$ is the set $S_t \cap C$. Now, $H_t \cap C$ and B both are semidefinite representable. So, $(H_t \cap C) \cap B$ is semidefinite representable. Hence, $S_t \cap C$ is semidefinite representable. As, $L \subset C$, by 3.1., (v), [Kle59] we know, $C = L + [0, \infty[(S_t \cap C)$. L is lineality space of clC. So, L is trivially semidefinite representable. Hence, the sum of L and $[0, \infty[(S_t \cap C)$ is semidefinite representable. Thus, C is semidefinite representable. □

Suppose, K is a convex subset of $E^n, p \in K$ and $2 \leq j \leq n$. π is an affine projection of K onto a j-flat through p. Let us consider $cone(p, K)$. We will derive one result connecting the semidefinite representation of $cone(p, K)$ and its projections i.e., $\pi cone(p, K)$. The result is proved in the following :

Observation 9. *Suppose, K is a convex subset of $E^n, p \in K$ and $2 \leq j \leq n$. π is an affine projection of K onto a j-flat through p. If $cone(p, K)$ is semidefinite representable, then $\pi cone(p, K)$ is semidefinite representable.*

Proof. Semidefinite representation is closed under linear mapping. Affine projection π is obtained as a composition of a translation with a linear projection. Hence, $\pi cone(p, K)$ is semidefinite representable. □

A very important result is proposition 3.2, proved by Klee, [Kle59]. This result states that : *Suppose X and Y are convex subsets of E, $p \in X \cap Y$, and $Y \subset cone(p, X)$. Then if Y is polyhedral, the set $X \cap Y$ is a neighbourhood of p relative to Y.*

We want to extend this proposition 3.2 considering a semidefinite representable set, Y. But, we will give some important and elementary facts about semidefinite representable set and a positive semidefinite cone. These are :

- Suppose, K is a semidefinite representable set in E. Then, K is the intersection of a positive semidefinite cone and an affine-linear subspace. Let, the positive semidefinite cone is S and the affine-linear subspace is L_a.
- A positive semidefinite cone S is the intersection of infinite number of closed halfspaces vectorized by matrix variable.
- The positive semidefinite cone has vertex ϕ, the origin in E and it is a pointed, closed, convex cone.
- The vertex ϕ belongs to the relative boundary of all the closed halfspaces.

Let us take a finite number of halfspaces, say k from the set of all halfspaces of the positive semidefinite cone. Consider, $C = H_1 \cap \ldots \cap H_k$. Then the following statements are true :

- C is nonempty and it is a cone.
  We know, $\phi \in rbdH_i$ for all $i = 1, \ldots, k$. Hence, $\phi \in C$ and C is nonempty. Let, $x, y \in C$. Then, $\lambda x + \mu y \in C$ for all $\lambda, \mu \geq 0$. So, C is a cone.
- C is polyhedral cone with vertex ϕ.
  C is generated by a finite number of halfspaces and the halfspaces are intersecting at point ϕ.
- C is closed, convex cone.

We know, positive semidefinite cone is an intersection of infinite number of halfspaces, we can not use any results related to polyhedrality to extend proposition 3.2., [Kle59].

Now, Corollary 3.3, [Kle59] states :
A convex set K is polyhedral at a point $p \in K$ iff $cone(p, K)$ is polyhedral. The extended version is in the following :

Observation 10. *If a convex set K is semidefinite representable at a point $p \in K$, then $cone(p, K)$ is semidefinite representable.*

Proof. Let, K is semidefinite representable at a point $p \in K$. Hence, there exists a bounded semidefinite representable neighbourhood J of p relative to K. From convexity of K, we can say

$$\begin{aligned} cone(p, K) &= cone(p, J) \\ &= p+]0, \infty[(J - p) \end{aligned}$$

So, $cone(p, K)$ is semidefinite representable. □

We have proved only one part of the Corollary 3.3, [Kle59]. The other part gives us our future research scope.

2.3 Projections of semidefinite representable sets

The *j-projection* of a set $X \subset E$ is the image of X under an affine projection of E onto a j-flat. In this subsection, we will discuss about a semidefinite representable set and its j-projections. Suppose K is a convex subset of E^n and $p \in K$. Then, $cone(p, \pi K) = \pi cone(p, K)$ whenever π is an affine projection of K onto a j-flat through p. It is very trivial to verify which is described in the following :

$$\begin{aligned} \pi cone(p, K) &= \pi[p + (0, \infty)(K - p)] \\ &= \pi p + (0, \infty)(\pi K - \pi p) \\ &= p + (0, \infty)(\pi K - p) \\ &= cone(p, \pi K) \end{aligned} \tag{7}$$

We will use this fact in our next Observation 11 which extends a fundamental result, Theorem 4.1 by Klee. The statement of the Theorem 4.1 is : *Suppose K is a convex subset of $E^n, p \in K$ and $2 \leq j \leq n$. Then K is polyhedral at p iff πK is polyhedral at p whenever π is an affine projection of K onto a j-flat through p.* Our modified result is as follows :

Observation 11. *Suppose K is a convex subset of $E^n, p \in K$ and $2 \leq j \leq n$. Then if K is semidefinite representable at p, πK is semidefinite representable at p whenever π is an affine projection of K onto a j-flat through p.*

Proof. Suppose, K is semidefinite representable at p. By Observation 10, $cone(p, K)$ is semidefinite representable. By, Observation 9, $\pi cone(p, K)$ is semidefinite representable. So, $cone(p, \pi K)$ is semidefinite representable. As, we could not extend the corollary 3.3, [Kle59], so we can not say that πK is semidefinite representable at p or not. □

This will be included in our future research problem.

This extends only some parts of the main result, Theorem 4.1., [Kle59].

The converse of the Observation is more interesting to verify. Let, πK is semidefinite representable at p. By using Observation 10, we can say $cone(p, \pi K)$ is semidefinite representable. This implies that $\pi cone(p, K)$ is semidefinite representable which provides that, all j-projections of $cone(p, K)$ are semidefinite representable for $2 \leq j \leq n - 1$.

Now, we already know from Observation 7 that, a convex cone is closed and semidefinite representable iff the $(n-1)$-sections of the convex cone are closed and semidefinite representable and if $L \subset C$. If we can connect all j-projections of $cone(p, K)$ and $(n-1)$-sections of $cone(p, K)$, then we will be able to prove, $cone(p, K)$ is semidefinite representable. If we can prove the converse part of Observation 10,(which says that a convex set K is semidefinite representable at a point p, if $cone(p, K)$ is semidefinite representable), then we can say, K is semidefinite representable at p. This gives our future research topic.

The Observation 11 provides a several useful corollaries which are as follows :

Observation 12. *With $2 \leq j \leq n$, if a convex subset of E^n is semidefinite representable at all its points, then all its j-projections have this property.*

Proof. Let, a convex subset K of E^n is semidefinite representable at all its points. So, K is semidefinite representable at p for $p \in K$. By the converse part of Observation 11(which is still remained to be proved), πK is semidefinite representable at p for $p \in \pi K$. □

Observation 13. *With $2 \leq j \leq n$, if a closed, convex subset of E^n is boundedly semidefinite representable, then all its j-projections are semidefinite representable at each point.*

Proof. Let, K is a closed, convex subset of E^n which is boundedly semidefinite representable. So, by Remark 5, K is semidefinite representable at p for any point p in K. By the converse part of Observation 11(which is still remained to be proved), πK is semidefinite representable at p for any point p in πK. □

As we could not extend corollary 3.3., [Kle59], we are not able to prove Theorem 4.1., [Kle59]. Hence, we are able to derive only some parts of the important corollaries $4.2., 4.3.$ of [Kle59], which are given in Observation $12, 13$. If we can prove, that if $cone(p, K)$ is semidefinite representable, then the convex set K is semidefinite representable at a point p in K. Then, we can complete all our Observations $11, 12, 13$.

2.4 Sections of semidefinite representable sets

We know that *j-section* of a set $K \subset E$ is the intersection of K with a j-flat. A j-flat is a j-dimensional affine subspace of E and so, it is semidefinite representable. Hence, if a set K in E is semidefinite representable, then all its j-sections i.e., the intersection of K with j-flats are semidefinite representable. It is obvious.

The main interesting part is to show : if all the j-sections of a convex set K in n-dimensional affine space are semidefinite representable, then K is semidefinite representable or not, for $2 \leq j \leq n-1$. We can introduce a very useful remark, 4.6. of Klee in this case. This is given in the following :
Let, K is a closed convex set in E with $\phi \in E$, L is a subspace of E, and π is a linear projection of $E^{'}$ whose kernel is L^0. L^0 is a subspace of $E^{'}$ and πK^0 lies in the supplementary subspace $\pi E^{'}$. One important connection between the section of a set K and the projection of K^0 is :

$$(L \cap K)^0 = L^0 + cl\pi K^0 = \pi^{-1}(cl\pi K^0). \tag{8}$$

Now, we will discuss the importance of the above connection between the section of K and the projection of K. Suppose, any j-section of K is semidefinite representable. Let, $L \cap K$ is semidefinite representable. Then, the polar of $L \cap K$, i.e., $(L \cap K)^0$ is obviously semidefinite representable. So, the equation (8) implies that $\pi^{-1}(cl\pi K^0)$ is semidefinite representable. Now, we know semidefinite representability is preserved under linear mapping. Hence, if $\pi^{-1}(cl\pi K^0)$ is semidefinite representable, then $cl\pi K^0$ is semidefinite representable. Now, let us assume that ϕ is in K. Then, K^0 is bounded. K^0 is closed and hence, K^0 is a compact set. So, the linear projection of K^0 is obviously closed. Hence, $cl\pi K^0 = \pi K^0$. Thus, we can say πK^0 is semidefinite representable.

Hence, if we can prove that a convex set in n-dimensional affine space is semidefinite representable iff all its j-projections are semidefinite representable, for all $2 \leq j \leq n-1$. Then, we can say, K^0 is semidefinite representable and, the polar of K^0 is K. Hence, K is semidefinite representable. This gives us future research scope.

3 Conclusion and future research scope

This paper deals with the characterizations of semidefinite representable sets, as convex sets certain of whose projections and sections are semidefinite representable. This is an extension of the paper [Kle59], where some sufficient and necessary conditions have been derived for convex polyhedra. We tried to extend some of the results from [Kle59], in Section 2. Although, we could not extend all of the results about convex polyhedra [Kle59]. We defined two new terms in semidefinite programming : *boundedly semidefinite representable set* and, a set, which is *semidefinite representable at a point*. We generalized proposition 2.17., [Kle59]. This is our Observation 3. We defined the smallest cone containg a semidefinite representable set. An important result is given in [Kle59], proposition 3.1.. We generalized 3.1., (vii). This is our Observation 7.

The main results of Klee's paper [Kle59] give an interesting characterization of convex polyhedra in affine space. The results state that, a convex set in n -dimensional affine space is polyhedral iff all its j-sections are polyhedral, where $2 \leq j \leq n-1$. Another important characterization says that, a convex set in n -dimensional affine space is polyhedral iff all its j-projections are polyhedral, where $2 \leq j \leq n-1$. As, semidefinite representable sets generalize polyhedra, we are trying to extend the results from [Kle59] considering a semidefinite representable set. Till now, we were able to extend only some of the results(some propositions, some corollaries) from [Kle59], which work as basic tools in proving the main results. We extended proposition $2.17.$, Proposition $3.1.$, (vii), some parts of Corollary$3.3.$ from [Kle59]. Our aim is to derive the results 3.6, $3.7.$, and mainly $4.1.$ from [Kle59] in case of a semidefinite representable sets. These give us our future research topic which are discussed in Subsection 2.3 and Subsection 2.4

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