A proof of conjecture arising from joint pricing and scheduling problem

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March 4, 2015

Abstract

In this paper, we settle the conjecture from "Pricing surplus server capacity for mean waiting time sensitive customers", by Sudhir K. Sinha, N. Rangaraj and N. Hemachandra, European Journal of Operational Research 205, Issue 1, (2010), 159-171. This conjecture arises from a joint pricing and scheduling problem introduced in Sinha et al. (2010). The overall objective of this problem is to optimally price the server's surplus capacity by introducing a new (secondary) class of customers that satisfies the predefined service level requirement of the existing (primary) class of customers, and simultaneously remains sensitive to their mean waiting times.

keywords Queueing, Admission control, Pricing of services, Dynamic priority schemes

1 Introduction

The pricing model proposed in Sinha et al. (2010) solves the class of problem where resource owner wants to optimally price the surplus server capacity of a stable $M/G/1$ queue for a new (secondary) class customers without affecting the service level of its existing (primary) customers. The objective of the model is to solve the joint pricing and scheduling problem such that resource owner's revenue will be maximized while maintaining the promised quality of service level for primary customers. Inclusion of secondary customers increases the load and affects the service level of primary customers. Hence, admission control of this new class is necessary. A non pre-emptive delay dependent priority scheme, introduced by Kleinrock (1964), is used to schedule customers across classes in Sinha et al. (2010). In this priority scheme, each class gains priority, dynamically, based on their delay in queue.

Sinha et al. (2010) decomposed this revenue maximization problem in two optimization problems arising from delay dependent priority queue discipline parameter being finite or infinite. In order to find the global optimal operating parameters, one needs to compare the optimal objectives of

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these two optimization problems. It was conjectured in Sinha et al. (2010) that optimal objective with finite scheduling parameter is better than that with infinite parameter for a particular range of primary class customer's service level. Assuming that the conjecture is true, a finite step algorithm is proposed in Sinha et al. (2010) to find the optimal revenue; the details of which are elaborated in Section 2. A sufficient condition for conjecture being true is given in Hemachandra and Raghav (2013) and Raghav (2011) which states that if a fifth order polynomial has no roots in a particular interval then this conjecture is true. In this paper, we give a complete proof of the conjecture by queueing and optimization based arguments. Hence, this completes the open question about the validity of finite step algorithm to find global optima of non convex revenue maximization problem.

This paper is organised as follows: Section 2 briefly discusses the conjecture and model introduced in Sinha et al. (2010). Section 3 presents proof of conjecture.

2 System Description

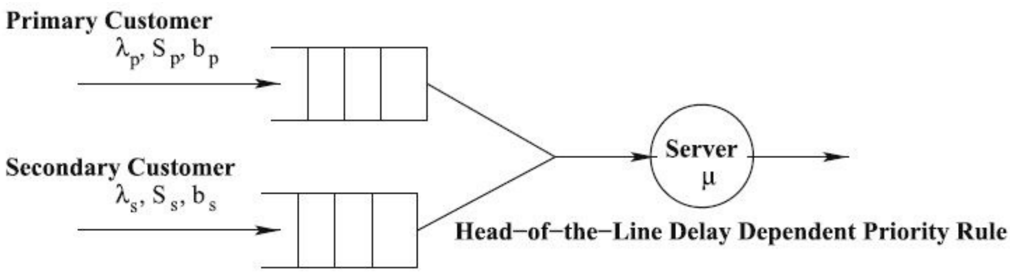

Figure 1: Schematic view of the model (Sinha et al., 2010)

This model addresses the question of pricing the surplus server capacity of a stable $M/G/1$ queue for a new class of customers that is sensitive to its mean waiting time. A schematic view of the model is shown in Figure 1. The primary class of customers arrive according to Poisson arrival process with rate λ_p. S_p, the desired limit on the mean waiting time of the primary class of customers, indicates the service level offered. The service time of customers is independent and identically distributed with mean $1/\mu$ and variance σ^2, irrespective of customer class. Idea of the problem is to determine the promised limit on the mean waiting time of a secondary class of customers, S_s and their unit admission price θ so as to maximize the revenue generated by the system, while constrained by primary class service levels. The secondary class of customers arrive according to an independent Poisson arrival process with rate λ_s, which is dependent on θ and S_s: $\lambda_s(\theta, S_s) = a - b\theta - cS_s$, where $a,\ b,\ c$ are positive constants driven by the market.

The mean waiting time of primary and secondary class customers depend on the queue scheduling rule. The scheduling discipline used in Sinha et al. (2010) was the non-preemptive delay dependent priority scheme, introduced by Kleinrock (1964). In such a scheme, the instantaneous priority at time t of class c customer that arrived at time T_c is calculated as $q_c(t) := (t - T_c)b_c$ for some positive number b_c. Let $c \in \{p, s\}$ so that b_p and b_s refer to the weights associated with primary and secondary classes, respectively. At each service completion, the server chooses the next job with the highest instantaneous priority $q_c(\cdot)$, $c \in \{p, s\}$. The steady state mean waiting times of each class of customers, derived by Kleinrock (1964), depends on the ratio of weights (b_i) given to each class. Let $\beta := b_s/b_p$. Note that $\beta = 0$ corresponds to static high priority to primary class customers, $\beta = 1$ is the global First Come First Serve (FCFS) queuing discipline across classes and $\beta = \infty$ corresponds to static high priority to secondary class customers. Let $W_p(\lambda_s, \beta)$ and $W_s(\lambda_s, \beta)$ be the mean waiting times of primary and secondary customers, when the arrival rate of secondary jobs is λ_s and queue management parameter is β.

Now select a suitable pair of pricing parameters θ and S_s for the secondary class customers, a queue disciple management parameter β and an appropriate admission rate for the secondary class customers λ_s, that will maximize the expected revenue from their inclusion, while ensuring that the mean waiting time to the primary class customers does not exceed a given quantity S_p. Thus, the revenue maximization problem, P0, is (Sinha et al., 2010):

$$\textbf{P0:} \max_{\lambda_s,\beta,S_s,\theta} \theta\lambda_s \tag{1}$$

subject to:

$$W_p(\lambda_s, \beta) \leq S_p \tag{2}$$

$$W_s(\lambda_s, \beta) \leq S_s \tag{3}$$

$$\lambda_s \leq \mu - \lambda_p \tag{4}$$

$$\lambda_s \leq a - b\theta - cS_s \tag{5}$$

$$\lambda_s, \theta, S_s, \beta \geq 0 \tag{6}$$

Constraint (2) and (3) ensure the service level for primary and secondary class respectively. Constraint (4) is queue stability constraint. Constraint (5) ensures that the mean arrival rate of secondary class customers should not exceed the demand generated by charged price θ and offered service level S_s. This problem can be presented as a non-convex constrained optimization problem P1 (since constraints (3) and (5) are tight at optimality (Sinha et al., 2010))

$$\textbf{P1:} \max_{\lambda_s,\beta} \frac{1}{b}\left(a\lambda_s - \lambda_s^2 - c\lambda_s W_s(\lambda_s, \beta)\right) \tag{7}$$

subject to:

$$W_p(\lambda_s, \beta) \leq S_p \tag{8}$$

$$\lambda_s \leq \mu - \lambda_p \tag{9}$$

$$\lambda_s, \beta \geq 0 \tag{10}$$

Once the optimal secondary class mean arrival rate λ_s^* and optimal queue discipline management parameter β^* are calculated, the optimal admission price θ^* and optimal assured service level to secondary class S_s^* can be computed using $S_s^* = W_s(\lambda_s^*, \beta^*)$ and $\lambda_s^* = a - b\theta^* - cS_s^*$.

Note that above optimization problem P1 considers only finite values of β, though $\beta = \infty$ is also a valid decision variable as it corresponds to a static high priority to secondary class customers. Hence, the following one dimensional convex optimization problem, P2, wherein β is set to ∞ in problem

P1, is considered as:

$$\textbf{P2:} \max_{\lambda_s} \frac{1}{b}[a\lambda_s - \lambda_s^2 - c\lambda_s \tilde{W}_s(\lambda_s)] \tag{11}$$

subject to:

$$\tilde{W}_p(\lambda_s) \leq S_p, \tag{12}$$

$$\lambda_s \leq \mu - \lambda_p, \tag{13}$$

$$\lambda_s \geq 0. \tag{14}$$

where $\tilde{W}_p(\lambda_s) = W_p(\lambda_s, \beta = \infty)$ and $\tilde{W}_s(\lambda_s) = W_s(\lambda_s, \beta = \infty)$. These two optimization problems (P1 and P2) are analyzed for their global optima, and their optimal values are compared in Sinha et al. (2010) to give a solution to P0 via a finite step algorithm.

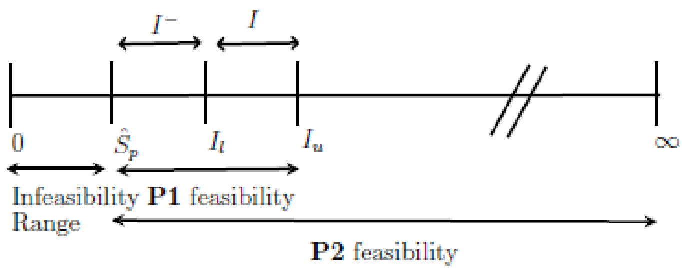

Figure 2: Illustration for range of S_p with optimal solutions coming from problem P1 and P2

Solution of these optimization problems are obtained in terms of different ranges of primary class service levels. For service level $S_p \leq \hat{S}_p = \dfrac{\lambda_p \psi}{\mu(\mu - \lambda_p)}$, the problems are infeasible, i.e., no secondary class customers can be accommodated. If $\dfrac{a}{c} > \dfrac{\lambda_p(2\mu - \lambda_p)}{\mu(\mu - \lambda_p)^2}\psi$ where $\psi = \dfrac{1 + \sigma^2\mu^2}{2}$ then both the optimization problems P1 and P2 have (global) optimal solutions for $S_p \in I^- \cup I$ (see Figure 2), for suitably identified finite intervals $I^- \equiv (\hat{S}_p, I_l)$ and $I \equiv [I_l, I_u)$ (see Theorem 1 and 2 in Sinha et al. (2010)). For $S_p \geq I_u$, the solution is given by problem P2 only, i.e. $\beta = \infty$. Hence, one has to compare the optimal objectives of problem P1 and P2 to obtain the global optima in service level range $I^- \cup I$ as both problems are feasible for given range. It was proved by Sinha et al. (2010) that for $S_p \in I$, the optimal solution of P0 is given by P1, i.e., optimal objective of P1 is more than that of P2 for this range. Further, based on computational results, Sinha et al. (2010) also *conjectured* the following result.

Conjecture (Sinha et al., 2010). *For $S_p \in I^-$, the optimal solution of P0 is given by optimal solution of P1.*

A sufficient condition for above conjecture is derived in Hemachandra and Raghav (2013), which states that the conjecture is true if a particular fifth order polynomial has no root in interval I^- of S_p.

We give proof for this conjecture in Section 3. We identified increasing concave structure of optimal objective of problem P2 in the range $(I^- \cup I)$ of interest. This property and previous results are exploited to prove the conjecture.

3 A Proof of conjecture of Sinha et al. (2010)

In this section, we present the proof of the conjecture described in previous section. We prove few claims and a theorem from which conjecture follows. Following claim is already proved (Sinha et al., 2008, page 23), we further write its proof and give complete details of all arguments.

Claim 1. *$\lambda_s^{(3)} > \lambda_s^{(1)}$, where $\lambda_s^{(3)}$ and $\lambda_s^{(1)}$ are the unique roots of cubics $\tilde{G}(\lambda_s)$ and $G(\lambda_s)$ respectively in the interval $(0, \mu - \lambda_p)$. $G(\lambda_s)$ and $\tilde{G}(\lambda_s)$ are given by*

$$\begin{aligned} G(\lambda_s) &= 2\mu\lambda_s^3 - [c\psi + \mu(a + 4\phi_0)]\lambda_s^2 + 2\phi_0[c\psi + \mu(a + \phi_0)]\lambda_s - a\mu\phi_0^2 + c\psi\lambda_p(\mu + \phi_0) \quad (15) \\ \tilde{G}(\lambda_s) &= 2\mu\lambda_s^3 - [a\mu + c\psi + 4\mu^2]\lambda_s^2 + 2\mu[a\mu + c\psi + \mu^2]\lambda_s - \mu[a\mu^2 - c\psi\lambda_p]. \quad (16) \end{aligned}$$

and $\phi_0 = \mu - \lambda_p$.

Proof. $\lambda_s^{(1)}$ is the unique root of cubic $G(\lambda_s)$ in the interval $(0, \mu - \lambda_p)$ whenever $\frac{a}{c} > \frac{\lambda_p(2\mu - \lambda_p)}{\mu(\mu - \lambda_p)^2}\psi$ (Sinha et al., 2010, Theorem 1). Hence, $\lambda_s^{(1)} \in (0, \mu - \lambda_p)$ for $a \in (a_l, \infty)$ where $a_l = \frac{\lambda_p(2\mu - \lambda_p)}{\mu(\mu - \lambda_p)^2}c\psi$. Similarly, $\lambda_s^{(3)}$ is the unique root of cubic $\tilde{G}(\lambda_s)$ in the interval $(0, \mu - \lambda_p)$ whenever $\frac{\mu - \lambda_p}{\mu\lambda_p} > \frac{a\lambda_p - c\psi}{2\mu\lambda_p^2 + c\psi(\mu + \lambda_p)}$ and $\frac{a}{c} > \frac{\lambda_p}{\mu^2}\psi$ (Sinha et al., 2010, Theorem 3). Hence, $\lambda_s^{(3)} \in (0, \mu - \lambda_p)$ for $a \in (\tilde{a}_l, \tilde{a}_u)$ where $\tilde{a}_l = \frac{\lambda_p}{\mu^2}c\psi$ and $\tilde{a}_u = 2(\mu - \lambda_p) + \frac{c\psi}{\lambda_p}\left[1 + \frac{\mu^2 - \lambda_p^2}{\mu\lambda_p}\right]$. If $\frac{\mu - \lambda_p}{\mu\lambda_p} \leq \frac{a\lambda_p - c\psi}{2\mu\lambda_p^2 + c\psi(\mu + \lambda_p)}$, i.e., $a \geq \tilde{a}_u$ then $\tilde{G}(\mu - \lambda_p) \leq 0$ and $\tilde{G}(0) < 0$ hold by the definition of cubic $\tilde{G}(.)$ and $a > \tilde{a}_l$. Hence, it follows from Claim 3 in Sinha et al. (2008) that $\mu - \lambda_p \leq \lambda_s^{(3)} < \mu$. It follows that $\tilde{a}_l = \frac{\lambda_p}{\mu^2}c\psi < \frac{\lambda_p(2\mu - \lambda_p)}{\mu(\mu - \lambda_p)^2}c\psi = a_l$. Note that $\lambda_s^{(1)} = 0$ at $a = a_l$, $\lambda_s^{(3)} = 0$ at $a = \tilde{a}_l$ and $\lambda_s^{(3)} = \mu - \lambda_p$ at $a = \tilde{a}_u$. These arguments follow as $G(0)$, $\tilde{G}(0)$ and $\tilde{G}(\mu - \lambda_p)$ is 0 at $a = a_l$, $\tilde{a}_l$ and $\tilde{a}_u$ respectively. a cannot be less than $\tilde{a}_l$, otherwise problem becomes infeasible. On the basis of relative values for a_l, $\tilde{a}_l$ and $\tilde{a}_u$ and using the fact that $\lambda_s^{(1)}$ and $\lambda_s^{(3)}$ are increasing functions of a (Sinha et al., 2008, claim 5, page 24), we have:

- If $a_l < \tilde{a}_u$, then,
  1. $\lambda_s^{(1)} \leq 0$ and $0 < \lambda_s^{(3)} < \mu - \lambda_p$ for $a \in (\tilde{a}_l, a_l]$
  2. $0 < \lambda_s^{(1)} < \mu - \lambda_p$ and $0 < \lambda_s^{(3)} < \mu - \lambda_p$ for $a \in (a_l, \tilde{a}_u)$
  3. $0 < \lambda_s^{(1)} < \mu - \lambda_p$ and $\mu - \lambda_p \leq \lambda_s^{(3)} < \mu$ for $a \geq \tilde{a}_u$
- If $a_l \geq \tilde{a}_u$, then,
  1. $\lambda_s^{(1)} < 0$ and $0 < \lambda_s^{(3)} < \mu - \lambda_p$ for $a \in (\tilde{a}_l, \tilde{a}_u)$
  2. $\lambda_s^{(1)} \leq 0$ and $\mu - \lambda_p \leq \lambda_s^{(3)} < \mu$ for $a \in [\tilde{a}_u, a_l]$
  3. $0 < \lambda_s^{(1)} < \mu - \lambda_p$ and $\mu - \lambda_p < \lambda_s^{(3)} < \mu$ for $a > a_l$

Hence, $\lambda_s^{(3)} > \lambda_s^{(1)}$ for all cases except the case when $a_l < \tilde{a}_u$ and $a \in (a_l, \tilde{a}_u)$. Note that $G(\lambda_s)$ and $\tilde{G}(\lambda_s)$ have exactly one root in interval $(0, \mu - \lambda_p)$. Root of $G(\lambda_s)$, $\lambda_s^{(1)}$, is 0 at $a = a_l$ and $G(\lambda_s)$ increases in the interval $(0, \mu - \lambda_p)$. Root of $\tilde{G}(\lambda_s)$, $\lambda_s^{(3)}$, is zero at $a = \tilde{a}_l < a_l$ and $\lambda_s^{(3)}$ is an increasing function of a. So $\lambda_s^{(3)} > \lambda_s^{(1)}$ at $a = a_l$. Plots of $G(\lambda_s)$ and $\tilde{G}(\lambda_s)$ are shown in Figure 3 and 4. As

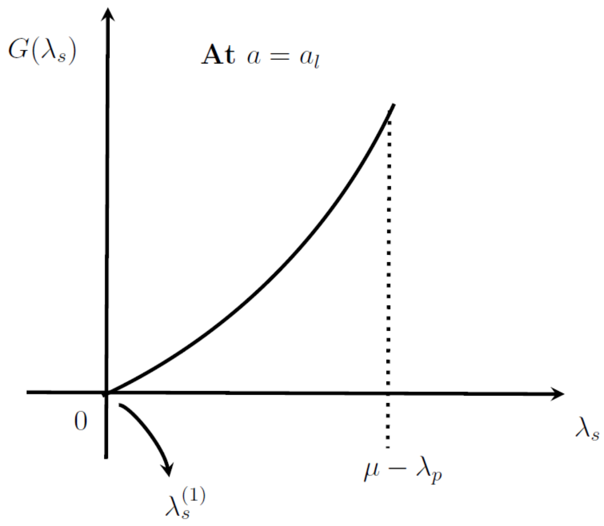

Figure 3: $G(\lambda_s)$ Vs λ_s in range $(0, \mu - \lambda_p)$ at $a = a_l$

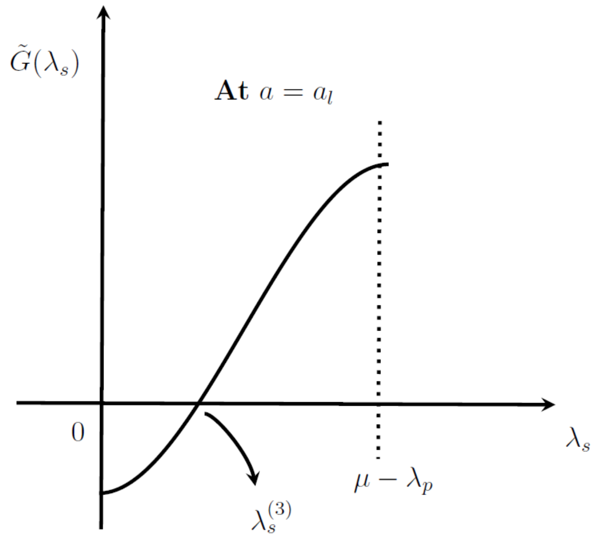

Figure 4: $\tilde{G}(\lambda_s)$ Vs λ_s in range $(0, \mu - \lambda_p)$ at $a = a_l$

$\dfrac{\partial G(\lambda_s)}{\partial a} = -\mu(\mu - \lambda_p - \lambda_s)^2$ and $\dfrac{\partial \tilde{G}(\lambda_s)}{\partial a} = -\mu(\mu - \lambda_s)^2$, $G(\lambda_s)$ and $\tilde{G}(\lambda_s)$ are decreasing functions of a. Since $\tilde{G}(\lambda_s)$ decreases with higher rate than $G(\lambda_s)$, $\lambda_s^{(3)}$ will increase with higher rate than $\lambda_s^{(1)}$. Hence, $\lambda_s^{(3)} > \lambda_s^{(1)}$ holds for $a \in (a_l, \tilde{a}_u)$ also and the claim follows.

□

Now, Sinha et al. (2010) has described, in their Theorem 1 and 2, the optimal solution of problem P1 in terms of primary class service level range I and I^-. While optimal solution of problem P2 is given by Theorem 3 and 4 in terms of service level range J and J^-. We present the following claim that relates I and I^- with the range J and J^-. Claim 1 is useful in proving following Claim 2.

Claim 2. *Range $I^- \cup I$ is contained in J^-, i.e., $I^- \cup I \subset J^-$.*

Proof. The solution of optimization problem P2 is given by Theorem 3 and 4 in Sinha et al. (2010) and by service level range J and J^-. So, the entire feasible range of service level $(\hat{S}_p, \infty)$ is divided in interval $J^- \cup J$ as shown in Figure 5.

From Theorem 3 and 4, it is clear that if $\dfrac{\mu - \lambda_p}{\mu \lambda_p} \leq \dfrac{a\lambda_p - c\psi}{2\mu\lambda_p^2 + c\psi(\mu + \lambda_p)}$ then $J^- = (\hat{S}_p, \infty)$ and $J = \phi$ otherwise $J^- = (\hat{S}_p, J_l]$ and $J = (J_l, \infty)$ where $J_l = \dfrac{\psi\lambda_3}{(\mu - \lambda_s^{(3)})(\mu - \lambda_3)}$ and $\hat{S}_p = \dfrac{\psi\lambda_p}{\mu(\mu - \lambda_p)}$. Now, consider the following two cases to prove the claim:

Case 1: When $\dfrac{\mu - \lambda_p}{\mu \lambda_p} \leq \dfrac{a\lambda_p - c\psi}{2\mu\lambda_p^2 + c\psi(\mu + \lambda_p)}$

Interval J^- becomes $(\hat{S}_p, \infty)$ under the condition of this case as discussed above. Since lower and

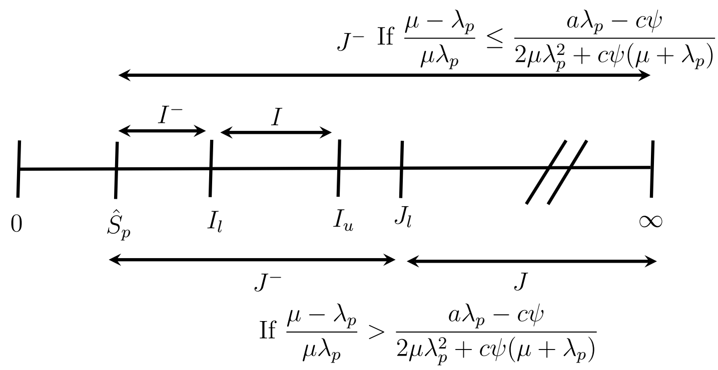

Figure 5: Relation among intervals of S_p

upper limit of $I^- \cup I$ are $\hat{S}_p$ and I_u respectively. These limits are finite (see Sinha et al. (2010)). Hence, $I^- \cup I \subset J^-$ holds.

Case 2: When $\dfrac{\mu-\lambda_p}{\mu\lambda_p} > \dfrac{a\lambda_p - c\psi}{2\mu\lambda_p^2 + c\psi(\mu+\lambda_p)}$

In this case, $J^- = (\hat{S}_p, J_l]$ where $J_l = \dfrac{\psi\lambda_3}{(\mu-\lambda_s^{(3)})(\mu-\lambda_3)}$ and $I^- \cup I = (\hat{S}_p, I_u)$ where $I_u = \frac{\psi\lambda_1}{(\mu-\lambda_s^{(1)})(\mu-\lambda_1)}$ and by definition $\lambda_1 = \lambda_p + \lambda_s^{(1)}$, $\lambda_3 = \lambda_p + \lambda_s^{(3)}$. Note that, $I_u = \xi(\lambda_s^{(1)})$ and $J_l = \xi(\lambda_s^{(3)})$ where $\xi(\lambda_s) = \dfrac{\psi\lambda}{(\mu-\lambda_s)(\mu-\lambda)}$ and $\lambda = \lambda_p + \lambda_s$. On computing partial derivative of $\xi(\lambda_s)$ with respect to λ_s, we get $\dfrac{\partial\xi(\lambda_s)}{\partial\lambda_s} = \dfrac{\psi(\mu(\mu-\lambda_s)+\lambda(\mu-\lambda))}{(\mu-\lambda_s)^2(\mu-\lambda)^2} > 0$, i.e., $\xi(\lambda_s)$ is an increasing function of λ_s. So, $\xi(\lambda_s^{(3)}) > \xi(\lambda_s^{(1)})$ iff $\lambda_s^{(3)} > \lambda_s^{(1)}$, But $\lambda_s^{(3)} > \lambda_s^{(1)}$ follows from claim 1. Thus, $\xi(\lambda_s^{(3)}) > \xi(\lambda_s^{(1)})$ follows and equivalently $J_l > I_u$ holds. Hence $I^- \cup I \subset J^-$ holds in this case too, and the claim follows. □

It follows from Claim 2 that the structure of optimal objective of problem P2 in service level range $I \cup I^-$ can be identified by finding the same in service level range J^-. Such a structure is identified in following theorem and will be useful in proving the conjecture.

Theorem 1. *The optimal objective function for problem P2, i.e.,* O_2^* *is increasing concave in service level range* $I^- \cup I$*, while the optimal objective function for problem P1,* O_1^**, is increasing concave in* I^- *and linearly increasing in* I*.*

Proof. It is shown in Sinha et al. (2008) that O_1^* is increasing concave in I^- and linearly increasing in I, we give details here for its completeness. The optimal objective of problem P1 and P2 are given by O_1^* and O_2^*, and the corresponding optimal solutions are given by (λ_s^f, β^f) and (λ_s^i, ∞), respectively. In case of finite β, solution is given by Theorem 1 and 2 in Sinha et al. (2010), for $S_p \in I^- \cup I$. Hence, the waiting time constraint is binding ($W_p(\lambda_s, \beta) \leq S_p$) from these theorems. In case of infinite β, the solution is given by Theorem 4 in Sinha et al. (2010), for $S_p \in I^- \cup I$ as $I^- \cup I \subset J^-$ (Claim 2). It follows from Theorem 4 that the primary class customer's waiting time constraint is binding. Therefore, waiting time constraint $W_p(\lambda_s, \beta) \leq S_p$ is always binding for $S_p \in I^- \cup I$. By using the interpretation of Lagrange multiplier (Proposition 3.3.3 in (Bertsekas, 1999, page 315) and (Sinha et al., 2008, page 25)), we have

$$\frac{\partial O_1^*}{\partial S_p} = -u_1^f \text{ and } \frac{\partial O_2^*}{\partial S_p} = -v_1^i \tag{17}$$

where u_1^f and v_1^i are the corresponding values of the Lagrangian multipliers associated with the constraint $W_p(\lambda_s, \beta) = S_p$ of the optimization problems P1 and P2 respectively. As defined in (Sinha et al., 2008, page 25):

$$u_1^f = \frac{(\mu - \lambda_p)G(\lambda_s^f)}{b\psi(\mu - \lambda_p - \lambda_s^f)^2} - \frac{c\lambda_p}{b} \text{ and} \tag{18}$$

$$v_1^i = \frac{(\mu - \lambda_p - \lambda_s^i)^2 \tilde{G}(\lambda_s^i)}{b\psi\mu[\mu(\mu + \lambda_p) - (\lambda_p + \lambda_s^i)^2]} \tag{19}$$

The optimal objective of optimization problem P2, i.e., O_2^* is given by Theorem 4 of Sinha et al. (2010) for service level range $S_p \in I^- \cup I$ as $I^- \cup I \subset J^-$. $\lambda_s^{(4)}$ is the optimal admission rate for secondary class customers in Theorem 4. Hence, $\lambda_s^i = \lambda_s^{(4)}$. Sign of v_1^i is decided by $\tilde{G}(\lambda_s^i)$. $\lambda_s^{(3)}$ is the root of cubic $\tilde{G}(\lambda_s)$ as discussed in Claim 1. It follows that $\lambda_s^{(4)} < \lambda_s^{(3)}$ (see proof of Theorem 4 in Sinha et al. (2008)). We note that $\tilde{G}(\lambda_s)$ is negative and increasing in interval $[0, \lambda_s^{(3)}]$. So $\tilde{G}(\lambda_s^{(4)}) = \tilde{G}(\lambda_s^i) \leq 0$ and hence $v_1^i \leq 0$ for $S_p \in I^- \cup I$. Now, it follows from Equation (17) that $\frac{\partial O_2^*}{\partial S_p} \geq 0$.

The solution of optimization problem P1 is given by Theorem 1 of Sinha et al. (2010) for $S_p \in I$ with $\lambda_s^{(1)}$ as the optimal admission rate for secondary class customers. Hence, $\lambda_s^f = \lambda_s^{(1)}$. $\lambda_s^{(1)}$ is the root of cubic $G(\lambda_s)$. From Equation (18), $u_1^f = \frac{-c\lambda_p}{b} \leq 0$ for $S_p \in I$.

Solution of problem P1 is given by Theorem 2 of Sinha et al. (2010) for $S_p \in I^-$ with $\lambda_s^{(2)} = \frac{\mu(\mu - \lambda_p)S_p}{\psi} - \lambda_p$ as the optimal admission rate for secondary class customers. Hence, $\lambda_s^f = \lambda_s^{(2)}$. Note that $\lambda_s^{(2)}$ linearly increases with S_p and $\lambda_s^{(2)} = \lambda_s^{(1)}$ at $S_p = \frac{\psi(\lambda_p + \lambda_s^{(1)})}{\mu(\mu - \lambda_p)} = I_l$, the upper limit of interval I^-. Thus, $\lambda_s^{(2)} < \lambda_s^{(1)}$ for $S_p \in I^-$. $G(\lambda_s)$ is an increasing function of $\lambda_s \in (0, \lambda_s^{(1)})$ (see proof of claim 1 in Sinha et al. (2008)). This implies that $G(\lambda_s^{(2)}) \leq G(\lambda_s^{(1)}) = 0$. From equation (18), $u_1^f \leq 0$ for $S_p \in I^-$. Thus $u_1^f \leq 0$ for $S_p \in I^- \cup I$ and we get the following from Equation (17)

$$\frac{\partial O_1^*}{\partial S_p} \geq 0 \text{ and } \frac{\partial O_2^*}{\partial S_p} \geq 0 \tag{20}$$

O_1^* and O_2^* are increasing functions of S_p, in the interval $I^- \cup I$. Partial derivatives of Lagrangian multipliers with respect to λ_s^f and λ_s^i are shown to be positive (Sinha et al., 2008, page 25):

$$\frac{\partial u_1^f}{\partial \lambda_s^f} \geq 0 \text{ and } \frac{\partial v_1^i}{\partial \lambda_s^i} \geq 0 \tag{21}$$

From Equation (17):

$$\frac{\partial^2 O_1^*}{\partial S_p^2} = -\frac{\partial u_1^f}{\partial S_p} = -\frac{\partial u_1^f}{\partial \lambda_s^f}\frac{\partial \lambda_s^f}{\partial S_p} \tag{22}$$

Recall Corollary 1 in Sinha et al. (2010) which states that the mean arrival rate of secondary class customers $\lambda_s^{(1)}$ is independent of S_p in interval I, so, $\frac{\partial \lambda_s^{(1)}}{\partial S_p} = 0$. But for $S_p \in I$, $\lambda_s^f = \lambda_s^{(1)}$ and hence we have

$$\frac{\partial^2 O_1^*}{\partial S_p^2} = 0 \tag{23}$$

Consider Corollary 2 in Sinha et al. (2010) which states that the mean arrival rate of secondary class customers $\lambda_s^{(2)}$ is linearly increasing function of S_p in interval I^-, i.e., $\frac{\partial \lambda_s^{(2)}}{\partial S_p} > 0$. But for $S_p \in I^-$, $\lambda_s^f = \lambda_s^{(2)}$ and we get the following from Equation (22),

$$\frac{\partial^2 O_1^*}{\partial S_p^2} \leq 0 \tag{24}$$

By using equations (20), (23) and (24), we can say that O_1^* is a linearly increasing function of S_p, in the interval I, while it is an increasing concave function of S_p in the interval I^-. For $S_p \in I^- \cup I$ with β as infinity, solution of problem P2 is given by Theorem 4 of Sinha et al. (2010) with $\lambda_s^{(4)}$ as the optimal admission rate for secondary class customers. Hence, $\lambda_s^i = \lambda_s^{(4)}$. From Equation (17), we have

$$\frac{\partial^2 O_2^*}{\partial S_p^2} = -\frac{\partial v_1^i}{\partial S_p} = -\frac{\partial v_1^i}{\partial \lambda_s^i}\frac{\partial \lambda_s^i}{\partial S_p} \tag{25}$$

Consider Corollary 3 in Sinha et al. (2010) which states that $\lambda_s^{(4)}$ is an increasing function of S_p in the interval J^-. Since $I^- \cup I \subset J^-$, $\frac{\partial \lambda_s^i}{\partial S_p} \geq 0$ for $S_p \in I^- \cup I$. So from equation (21) and (25), we have

$$\frac{\partial^2 O_2^*}{\partial S_p^2} \leq 0 \text{ for } S_p \in I^- \cup I \tag{26}$$

O_2^* is, thus, an increasing function of S_p as $\frac{\partial O_2^*}{\partial S_p} \geq 0$ and concave as $\frac{\partial^2 O_2^*}{\partial S_p^2} \leq 0$ for $S_p \in I^- \cup I$. Hence, the theorem follows. □

Conjecture is proved via following theorem using the increasing concave nature of optimal objective function of problem P2 as identified in above theorem.

Theorem 2. *The optimal solution of P0 is given by optimal solution of P1 for $S_p \in I^- \cup I$.*

Proof. The optimal objective of problem P1 is more than that of P2 for $S_p \in I$, i.e., $O_1^*(\lambda_s^f, \beta^f) > O_2^*(\lambda_s^i, \beta^i)$ follows for service level range I (see (Sinha et al., 2008, page 26)). Thus, the optimal solution of P0 is given by that of P1 for $S_p \in I$.

In service level range I^-, four possible scenarios can arise as shown in Figures 6, 7, 8, and 9. Figure 7 is not possible as it contradicts above Theorem 1, which states that O_2^* is concave. Figure 8 is not possible as it contradicts feasibility requirements (i.e. $S_p \geq \hat{S}_p$), and the fact that $O_2^*(\lambda_s^i, \infty) < O_1^*(\lambda_s^f, \beta^f)$ at $\hat{S}_p + \epsilon$, where ϵ is a small positive number (Sinha et al., 2008, page 26). Figure 9 is also not possible as it contradicts the statement that $O_1^*(\lambda_s^f, \beta^f) > O_2^*(\lambda_s^i, \beta^i)$ in interval I (Sinha et al., 2010). Thus, the only possible scenario is Figure 6, where $O_2^*(\lambda_s^i, \infty) < O_1^*(\lambda_s^f, \beta^f)$ for $S_p \in I^-$. Hence the theorem follows.

□

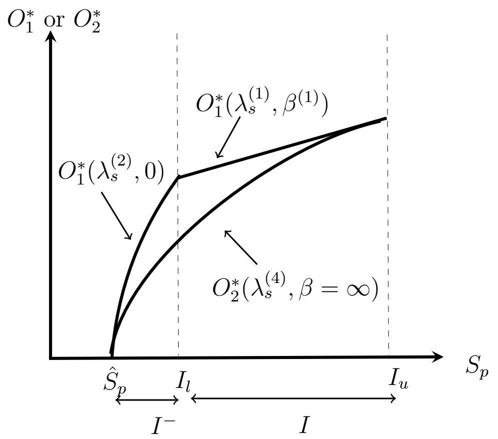

Figure 6: No contradiction

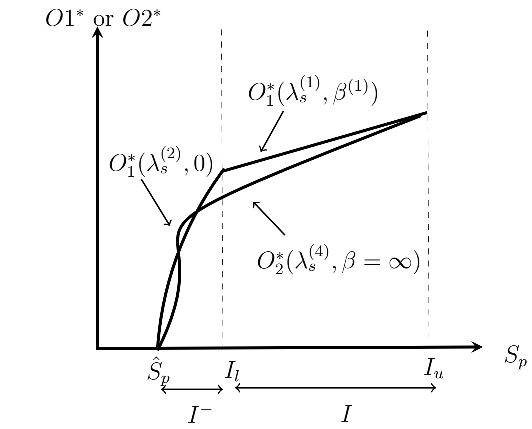

Figure 7: Contradiction from concavity of O_2^*

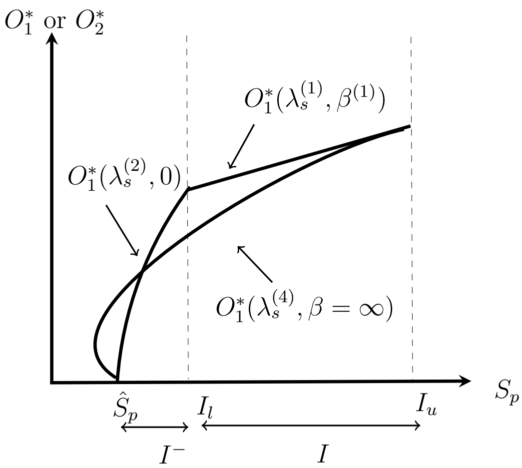

Figure 8: Contradiction by infeasibility and $O_2^*(\lambda_s^i, \infty) < O_1^*(\lambda_s^f, \beta^f)$ at $\hat{S}_p + \epsilon$

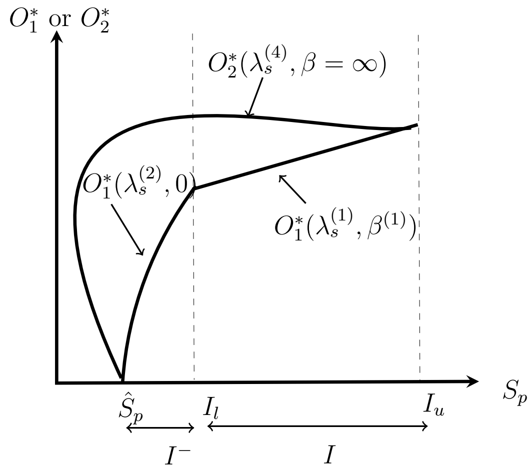

Figure 9: Contradiction by the fact that $O_2^*(\lambda_s^i, \infty) < O_1^*(\lambda_s^f, \beta^f)$ for $S_p \in I$

The conjecture follows from above theorem.

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