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On the Facet Defining Inequalities of the Mixed-Integer Bilinear Covering Set

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Abstract We study the facet defining inequalities of the convex hull of a mixed-integer bilinear covering arising in trim-loss (or cutting stock) problem under the framework of disjunctive cuts. We show that all of them can be derived using a disjunctive procedure. Some of these are split cuts of rank one for a convex mixed-integer relaxation of the covering set, while others have rank at least two. For certain linear objective functions, the rank-one split cuts are shown to be sufficient for finding the optimal value over the convex hull of the covering set. A relaxation of the trim-loss problem has this property, and our computational results show that these rank-one inequalities find the lower bound quickly.

Keywords Mixed-Integer programming · global optimization · convex hull · disjunctive cut · split cut · split-rank.

1 Introduction

We study the facet defining inequalities of the convex hull of the mixed-integer bilinear covering set

$$S = \left\{ (x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^{n} x_i y_i \geq r \right\},$$

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where $r > 0$. This set appears in real life applications like trim loss (or cutting stock) problem [25,41]; see Section 9 for more details. The set S is a nonconvex set, even the continuous relaxation R of S defined as

$$R = \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i y_i \geq r \right\},$$

is nonconvex for $n \geq 2$. Tawarmalani et al. [39] noted that the bilinear constraint of R is composed of separable orthogonal terms that have the so-called "convex extension property". Based on this observation they obtained the convex hull of R with the help of their orthogonal disjunctive procedure. This convex hull, which we call $\hat{R}$, can be described using only one nonlinear constraint:

$$\hat{R} = conv(R) = \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^{n} \sqrt{\frac{x_i y_i}{r}} \geq 1 \right\}.$$

Tawarmalani et al. [39] also obtained the convex hull of S with the help of the orthogonal disjunctive procedure. However, unlike $\hat{R}$, the description of $conv(S)$ consists of countably infinite number of facet defining inequalities. We study these inequalities using the framework of split disjunctions when applied to $\hat{R}$ in an attempt to find those which might be computationally more useful and easy to obtain. Viewing these facet defining inequalities through the lens of split disjunctions, we see that some of them have split-rank one, and can be obtained easily. Further, when minimizing the objective function of trim-loss problems, these rank-one inequalities give the same bound as $conv(S)$. Some other facet defining inequalities are seen to have split-rank more than one, but can be obtained using other disjunctions. None of the remaining facet-defining inequalities can be obtained by applying any disjunctive procedure on $\hat{R}$, in fact each of them cuts off an integer point from $\hat{R}$. These inequalities cannot be derived from $\hat{R}$, but disjunctions can still be applied on the nonconvex set R to derive them. We observe that all facet generating disjunctions have a similar form.

The set S is a special case of a nonconvex set with integer constrained variables. The problem of minimizing an objective function over such sets is called Mixed-Integer Nonlinear Programming (MINLP). The decision version of MINLP, in general, is undecidable [26], and consequently there is no algorithm that can solve general MINLPs on current computer architectures. Even specific cases of MINLP are NP-Hard [27]. The problem of optimizing a linear function over the set R is solvable in polynomial time [36], but problems like trim loss where such constraints appear along with other constraints are NP-Hard [21]. In order to solve these problems, one can first obtain a convex relaxation of the nonconvex set, which is more tractable. The optimal value of the relaxation provides a lower bound on the optimal value of MINLP. The tighter the relaxation, the closer is the bound to the optimal value, and hence finding inequalities that describe the convex hull of the feasible region is important.

The problem of finding facet defining inequalities of a general nonconvex mixed-integer set is difficult, and there are no general algorithms for finding all facets of such sets. One has to exploit specific structures and properties of the given set in order to find facets, like is done in the orthogonal disjunctive procedure. A more common approach in these methods (see for example, [7, 29, 37]) is to first find a suitable disjunction, and then obtain an inequality that is valid for each subset of the disjunction.

The principle of obtaining disjunctive inequalities [4, 5, 15, 35], in particular split inequalities, has been quite useful for the case of integer linear optimization. Gomory Mixed Integer inequalities [23, 24], Mixed-Integer Rounding (MIR) [33, 34] inequalities, lift-and-project inequalities [6] and several others are all known to be special cases of split cuts [13]. While some of them are equivalent theoretically, they still provide their own computational advantages and insights.

The general approach of obtaining disjunctive inequalities for integer linear optimization has been extended to the class convex-MINLP consisting of MINLPs whose continuous relaxation is convex. Several studies on theoretical aspects of split inequalities [2, 3, 14, 31, 32] and on using them for solving convex MINLPs [8, 10, 28, 38] have been performed recently. Given the relatively well established foundations of convex integer sets, it is tempting to exploit it for nonconvex MINLPs as well. This is the main motivation for our work.

Unless otherwise mentioned, we use the following notations throughout this article. For a give set A, we use $conv(A)$ to denote the convex hull of the set A. We use $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ to denote the set of real numbers, the set of integers and the set of positive integers respectively. $\mathbb{R}^n_+ = [0, \infty)^n = \{x \in \mathbb{R}^n : x \geq 0\}, \mathbb{Z}^n_+ = \{x \in \mathbb{Z}^n : x \geq 0\}$. We use N for the set $\{1, 2, \ldots, n\}$. For a point $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, we write (x, y) in the form $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$. We use $\mathcal{L}(i, x_i, y_i)$ to denote the $2n$ dimensional point $(0, 0, \ldots, x_i, y_i, \ldots, 0, 0)$, i.e., $x_j = 0, y_j = 0, \forall j \in N, j \neq i$. The sign $\vee$ means "or" and $\wedge$ means "and". For an integer vector $\mu \in \mathbb{Z}^n$ and an integer μ_0, we use $gcd(\mu, \mu_0) = gcd(\mu_1, \ldots, \mu_n, \mu_0)$ to denote the greatest common divisor of $\mu_1, \ldots, \mu_n$ and μ_0.

2 Some definitions and basic terminologies

We start our discussion with definitions. Consider the following convex mixed-integer nonlinear program:

$$\begin{aligned} \min_{x,y}\ & f(x, y) \\ \text{s.t. } & g_i(x, y) \leq 0, i = 1, \ldots, m, \\ & x \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}. \end{aligned} \qquad (\text{MINLP}_{\text{CV}})$$

where $f, g_i, i = 1, \ldots, m$ are convex functions. Let $P \subset \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ be the feasible set of (MINLP_{CV}) and P_C be the continuous relaxation of P. Clearly P_C is convex, and suppose that P_C is closed.

Definition 1 (Disjunction [5]) Let $D_k = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : A^k x \le b^k\}$ for $k \in K$, where K is an index set (not necessarily finite). Define $D = \bigcup_{k \in K} D_k$. If $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \subseteq D$, then we call D a disjunction (or a valid disjunction) and each $D_k, k \in K$ is known as an atom of the disjunction D.

Definition 2 (Disjunctive Cut) A linear inequality is called a disjunctive cut for P obtained from the disjunction $D = \cup_{i \in K} D_k$, if it is valid for $P_C \cap D_k$ for all $k \in K$.

We say that a linear inequality is valid for the disjunction D if it is valid for $P_C \cap D_k$ for all $k \in K$. For some positive integer m, let us define the set notation:

$$\left[G^1 x \le h_1, \ldots, G^m x \le h_m\right] = \left\{(x, y) \in \mathbb{R}^{n_1+n_2} : G^1 x \le h_1, \ldots, G^m x \le h_m\right\},$$

where $G^i, i = 1, \ldots, m$ are rational matrices of suitable dimension and $h_i, i = 1, \ldots, m$ are rational numbers.

Split cuts are a special class of the disjunctive cuts which are obtained from a split disjunction, a special type of disjunction with only two atoms.

Definition 3 (Split Disjunction) Given a non-zero integer vector $\pi \in \mathbb{Z}^{n_1}$ and an integer π_0, the disjunction $\left[\pi^T x \le \pi_0\right] \vee \left[\pi^T x \ge \pi_0 + 1\right]$ is known as Split Disjunction. In a simpler way we write this disjunction as (π, π_0).

Note that any $(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ satisfies either $\pi^T x \le \pi_0$ or $\pi^T x \ge \pi_0 + 1$. Without loss of generality we can assume $gcd(\pi, \pi_0) = 1$. Let us define two sets.

$$P_L = P_C \cap \left[\pi^T x \le \pi_0\right], \text{ and } P_R = P_C \cap \left[\pi^T x \ge \pi_0 + 1\right].$$

Clearly $P \subseteq P_L \cup P_R$. Therefore, $P \subseteq P_L \cup P_R \subseteq conv\,(P_L \cup P_R)$.

Definition 4 (Split Cut) An inequality $c^T x + d^T y \ge b$ that is valid for both the sets P_L and P_R (or consequently valid for $conv\,(P_L \cup P_R)$) is known as a split cut.

Let us consider a linear inequality $c^T x + d^T y \ge b$. In order to check whether the inequality $c^T x + d^T y \ge b$ is valid for P_R and P_L, one can solve the following two optimization problems

$$\zeta_R = \min_{x,y} \; c^T x + d^T y \qquad \zeta_L = \min_{x,y} \; c^T x + d^T y$$
$$\text{s.t. } (x, y) \in P_R \qquad \text{s.t. } (x, y) \in P_L.$$

Clearly the inequality $c^T x + d^T y \ge b$ is valid for $conv\,(P_R \cap P_L)$ if and only if $\zeta_R \ge b$ and $\zeta_L \ge b$.

The subset of P_C obtained by adding all possible split cuts to P_C is known as the first split closure or the elementary split closure of P_C. Let us denote it by P_1. Clearly, P_1 is closed since P_C is closed. Similarly applying the split closure procedure to the set P_1 will give the second split closure P_2. Let P_t be the t^{th} split closure. Cook et al. [12] showed that, if P_C is a polyhedral set, then P_t is also polyhedral, for all $t \in \mathbb{N}$. More results on this can be found in [14,16,17].

Definition 5 (Split Rank) For a valid inequality $c^T x + d^T y \geq b$ for the set $conv(P)$, the split rank of the inequality is defined as the smallest integer t such that the inequality is valid for P_t but not for P_{t-1}.

In general, the problem of determining the split-rank for a given linear inequality is NP-hard. Even the simpler problem of checking whether a given linear inequality for a given MILP has rank one is NP-complete [9]. Determining bounds on the split-rank is relatively easy. Split-rank of an inequality can be finite or infinite [11]. A valid inequality having finite split-rank indicates that the inequality can be obtained by recursively applying the split cuts finite number of times.

3 Few properties of $\hat{R}$ and R

It is easy to see that the set $\hat{R}$ is a closed convex set in the positive orthant. In this section we analyze few more properties of the sets $\hat{R}$ and R that are necessary for our further discussion.

Proposition 1 *Consider the following optimization problem.*

$$\begin{aligned} \min\ & c\sum_{i=1}^{n} x_i + d\sum_{i=1}^{n} y_i \\ s.t.\ & \sum_{i=1}^{n} x_i \leq (\geq) k, \\ & \sum_{i=1}^{n} \sqrt{\frac{x_i y_i}{r}} \geq 1, \\ & x, y \geq 0, \end{aligned} \tag{P1}$$

where c, d and k are given scalars. Suppose (P1) has an optimal solution. Then there exists an optimal solution (x^*, y^*) to (P1) such that only one pair of its component is non zero, i.e., there exists $t \in N$ such that $x_i^* = 0, y_i^* = 0$ for all $i \in N \setminus \{t\}$.

Proof Since the proof can be easily generalized for any positive integer n, we prove our result for $n = 2$ only. Let $(\bar{x}, \bar{y}) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$ be an optimal solution to the optimization problem. Therefore, we have

$$\sum_{i=1}^{n} \bar{x}_i \leq (\geq) k, \text{ and} \tag{1}$$

$$\sqrt{\bar{x}_1 \bar{y}_1} + \sqrt{\bar{x}_2 \bar{y}_2} \geq \sqrt{r}. \tag{2}$$

The objective value at this point is $c\sum_{i=1}^{2} \bar{x}_i + d\sum_{i=1}^{2} \bar{y}_i$. Consider the point (x^*, y^*) such that

$$\begin{aligned} x_1^* &= \bar{x}_1 + \bar{x}_2, x_2^* = 0, \\ y_1^* &= \bar{y}_1 + \bar{y}_2, y_2^* = 0. \end{aligned}$$

At this point the objective value is same as the optimal value. Therefore, it is sufficient to show that (x^*, y^*) is feasible for (P1). Clearly, from (1) we see that, the point (x^*, y^*) satisfies the first constraint. Then

$$\begin{aligned} x_1^* y_1^* &= (\bar{x}_1 + \bar{x}_2)(\bar{y}_1 + \bar{y}_2) \\ &= \left(\sqrt{\bar{x}_1 \bar{y}_1} + \sqrt{\bar{x}_2 \bar{y}_2}\right)^2 + \left(\sqrt{\bar{x}_1 \bar{y}_2} - \sqrt{\bar{x}_2 \bar{y}_1}\right)^2 \\ &\geq r, \text{ using (2) and } \left(\sqrt{\bar{x}_1 \bar{y}_2} - \sqrt{\bar{x}_2 \bar{y}_1}\right)^2 \geq 0 \\ \Rightarrow \sqrt{x_1^* y_1^*} &\geq \sqrt{r} \end{aligned}$$

This implies that $(x^*, y^*) \in \hat{R}$, and thus, feasible for (P1). □

A result similar to the above be found in [18], where it is used to find tight conic relaxations of problems with multiple bilinear constraints. We, on the other hand use this result to study the ranks of facets (which are linear inequalities).

Proposition 2 *Consider the following optimization problem,*

$$\begin{aligned} z^* = \min\ & a_k x_1 + b_k y_1 \\ s.t.\ & \sqrt{x_1 y_1} \geq \sqrt{r}, \\ & x_1, y_1 \geq 0. \end{aligned}$$

where $a_k = \frac{1}{2k-1}, b_k = \frac{k(k-1)}{r(2k-1)}, k \in \mathbb{N} \setminus \{1\}$. *Then the unique optimal solution of the above problem is* $(\sqrt{k(k-1)}, r/\sqrt{k(k-1)}\ )$, *and* $\frac{1}{2} < z^* < 1$.

Proof Note that the above problem is a convex problem as the curve $y_1 = \frac{r}{x_1}$ is strictly convex. Therefore, by KKT conditions, we have that the unique optimal solution is $(\sqrt{k(k-1)}, r/\sqrt{k(k-1)})$. Therefore, the optimal value is

$$\frac{\sqrt{k(k-1)}}{2k-1} + \frac{k(k-1)}{r(2k-1)} \frac{r}{\sqrt{k(k-1)}} = \frac{2\sqrt{k(k-1)}}{2k-1} = \sqrt{\frac{4k^2-4k}{4k^2-4k+1}} < 1.$$

The function

$$f(v) = \frac{4v^2 - 4v}{4v^2 - 4v + 1}, v \geq 2$$

is continuously differentiable in the domain, and $f'(v) = \frac{8v-4}{(4v^2-4v+1)^2} > 0$ for $v \geq 2$. Therefore, f and hence $\sqrt{f}$ is strictly increasing for $v \geq 2$. Also we have $\sqrt{f(2)} = \sqrt{8/9} > 1/2$. Thus, the result follows. □

Corollary 1 *Let us consider the following optimization problem*

$$z^* = \min_{(x,y) \in \hat{R}} \sum_{i=1}^{n} (a_{k_i} x_i + b_{k_i} y_i)$$

where $n \in \mathbb{N}, a_{k_i} = \frac{1}{2k_i - 1}, b_{k_i} = \frac{k_i(k_i-1)}{r(2k_i-1)}, k_i \in \mathbb{N} \setminus \{1\}$. *Then* $z^* < 1$.

Proof Note that the point $(0, 0, \cdots, \sqrt{k_i(k_i - 1)}, r/\sqrt{k_i(k_i - 1)}, \cdots, 0, 0)$ is feasible for $\hat{R}$. The objective value at this point is $\frac{\sqrt{k_i(k_i-1)}}{2k_i-1} + \frac{k_i(k_i-1)}{r(2k_i-1)} \frac{r}{\sqrt{k_i(k_i-1)}}$ which is less than one from the proof of Proposition 2. This implies $z^* < 1$. □

Proposition 3 *Let $n \geq 2$. Consider the set R along with some additional linear constraints on the variables x. Let us call it R_X. Let $(\bar{x}, \bar{y})$ be an extreme point of the set $conv(R_X)$. Then there exists $t \in N$ such that $\bar{x}_t \bar{y}_t = r, y_i = 0, \forall i \in N, i \neq t$, i.e., only one pair of $(\bar{x}_i, \bar{y}_i), i = 1, \ldots, n$ can have both the non zero value.*

Proof Let $(\bar{x}, \bar{y})$ be an extreme point of $conv(R_X)$, then $(\bar{x}, \bar{y})$ must lie on the surface $\sum_{i=1}^n x_i y_i = r$. If this were not true, then since the additional linear constraints are on x only, we will get by varying y alone, two feasible points whose convex combination is the point $(\bar{x}, \bar{y})$.

If possible, let there exist two pairs of components of $(\bar{x}, \bar{y})$ that are strictly greater than zero. Without loss of generality let $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ have all their components greater than zero. Also let $\bar{x}_1\bar{y}_1 + \bar{x}_2\bar{y}_2 = \alpha$. Without loss of generality let us assume $\bar{x}_1\bar{y}_1 \geq \frac{\alpha}{2}$. Note that $(\bar{x}, \bar{y}) = \frac{1}{2}\chi_1 + \frac{1}{2}\chi_2$, where

$$\chi_1 = \left(\bar{x}_1, \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 0, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n\right) \text{ and}$$
$$\chi_2 = \left(\bar{x}_1, 2\bar{y}_1 - \frac{\alpha}{\bar{x}_1}, \bar{x}_2, 2\bar{y}_2, \bar{x}_3, \bar{y}_3, \ldots, \bar{x}_n, \bar{y}_n\right).$$

Clearly $\chi_1, \chi_2 \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$. Since the x components of the points $(\bar{x}, \bar{y}), \chi_1$ and χ_2 are the same, the points χ_1 and χ_2 satisfy the additional linear constraints on x that are present in R_X. It is easy to see that,

$$\begin{aligned}\bar{x}_1 \frac{\alpha}{\bar{x}_1} + \bar{x}_2 0 + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n &= \alpha + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n \\ &= \bar{x}_1\bar{y}_1 + \bar{x}_2\bar{y}_2 + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n \geq r.\end{aligned}$$

Again,

$$\begin{aligned}&\bar{x}_1\left(2\bar{y}_1 - \frac{\alpha}{\bar{x}_1}\right) + \bar{x}_2 2\bar{y}_2 + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n \\ &= 2(\bar{x}_1\bar{y}_1 + \bar{x}_2\bar{y}_2) - \alpha + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n \\ &= \bar{x}_1\bar{y}_1 + \bar{x}_2\bar{y}_2 + \bar{x}_3\bar{y}_3 + \ldots + \bar{x}_n\bar{y}_n \geq r.\end{aligned}$$

Thus, χ_1 and χ_2 lie in R_X. This shows that $(\bar{x}, \bar{y})$ cannot be an extreme point of $conv(R_X)$. Therefore, our assumption must be wrong which proves that $\bar{x}_i\bar{y}_i = 0$ for all $i \in N, i \neq t$. We still have to show that $\bar{y}_i = 0$ for all $i \in N, i \neq t$.

Let $\bar{x}_t\bar{y}_t = r$. If possible, let there exist $j \in N, j \neq t$ such that $\bar{y}_j > 0$. Therefore, using the above arguments, $\bar{x}_j = 0$. Let $\epsilon > 0$ be such that $\bar{y}_j - \epsilon > 0$. Then $(\bar{x}, \bar{y})$ lies in the middle of two points χ_3 and χ_4 such that χ_3 and χ_4 have the same components as $(\bar{x}, \bar{y})$ except the j^{th} component of the variable y, and the j^{th} component of χ_3 and χ_4 are $\bar{y}_j - \epsilon$ and $\bar{y}_j + \epsilon$ respectively. Since $\chi_3, \chi_4 \in S$, this contradicts the extremality of $(\bar{x}, \bar{y})$. □

4 The facet defining inequalities of *conv*(*S*)

The facet defining inequalities of the set $conv(S)$ have been obtained by Tawarmalani et al. [39] using its orthogonally restricted subsets (known as orthogonal disjunctive subsets) S_i of S that are defined as

$$S_i = \{\mathcal{L}(i, x_i, y_i) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : x_i y_i \geq r\}.$$

Since the function $y_i = \frac{r}{x_i}$ is convex in the non-negative orthant, the continuous relaxation of the set S_i is convex. The description of $conv(S_i)$ can be given as following.

$$conv\,(S_i) = \{\mathcal{L}(i, x_i, y_i) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : a_j x_i + b_j y_i \geq 1, \forall j \in \mathbb{N}\}$$

where $a_j = \frac{1}{2j-1}$ and $b_j = \frac{j(j-1)}{r(2j-1)}, j \in \mathbb{N}$. Note that the coefficients $a_j, b_j, j \in \mathbb{N}$ are independent of $i \in N$.

Unless otherwise mentioned, in our further discussion, without loss of generality we assume that the right hand side of each facet defining inequality of $conv(S_i)$ is scaled to one. Note that there are countably infinite number of facet defining inequalities in the description of $conv(S_i)$, and consequently $conv(S_i)$ is not a polyhedral set. The following figure illustrates the facets geometrically within a bounded region.

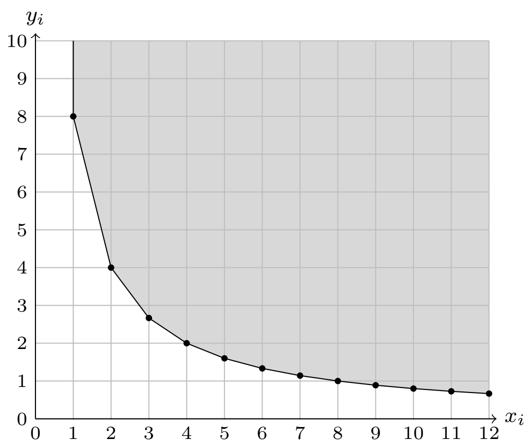

Fig. 1: Convex hull of S_i for $r = 8$ (in the restricted domain $x \leq 12, y \leq 10$)

Let us consider the following collection of columns (M). The entries in column i of (M) are the linear functions that can be used to define the facet defining inequalities of $conv(S_i)$.

$$\begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 & \dots & a_2x_n+b_2y_n \\ \dots & \dots & \dots & \dots & \dots \\ a_kx_1+b_ky_1 & a_kx_2+b_ky_2 & a_kx_3+b_ky_3 & \dots & a_kx_n+b_ky_n \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (M)$$

In the above (M), $a_j = \frac{1}{2j-1}$ and $b_j = \frac{j(j-1)}{r(2j-1)}, j \in \mathbb{N}$ as defined earlier. Note that (M) is not a matrix, but just an illustration useful for expressing all facets of $conv(S)$. To construct the facet defining inequalities for $conv(S)$, we select n terms from (M) taking one from each column and constrain their sum to be greater than or equal to one [39]. For example, one may take the first term of each column to get the facet

$$\sum_{i=1}^{n} x_i \geq 1.$$

The general form of the facet defining inequalities of $conv(S)$ is

$$\sum_{i\in J_1}(a_{j_1}x_i + b_{j_1}y_i) + \sum_{i\in J_2}(a_{j_2}x_i + b_{j_2}y_i) + \ldots + \sum_{i\in J_p}(a_{j_p}x_i + b_{j_p}y_i) \geq 1, \quad (\mathrm{I_G})$$

where $j_1, j_2, \ldots, j_p$ are different row numbers of (M) for some $p \in \mathbb{N}$. Without loss of generality we can assume $j_1 < j_2 < \ldots < j_p$. The index sets $J_1, J_2, \ldots, J_p$ define a partition on the set N. Note that, for any such partition, we get one facet defining inequality of $conv(S)$ and vice versa. Since there are countably infinite terms in each column of (M), there are infinitely many (but countable) facet defining inequalities. Also known [39] is the fact that these facet defining inequalities along with the nonnegativity constraints $x \geq 0, y \geq 0$ describe $conv(S)$ completely.

5 Split-rank of the facet defining inequalities of $conv(S)$

In this section, we derive the ranks of the facet defining inequalities of $conv(S)$. We first analyze the simpler cases for $n = 1$, and then generalize it for any positive integer n.

5.1 The case $n = 1$

For $n = 1$, we have the set $S = \{(x_1, y_1) \in \mathbb{Z}_+ \times \mathbb{R}_+ : x_1y_1 \geq r\}$. In this case the convex hull of S can be written as

$$conv(S) = \{(x_1, y_1) \in \mathbb{R}_+ \times \mathbb{R}_+ : x_1 \geq 1, a_jx_1 + b_jy_1 \geq 1, \forall j \in \mathbb{N} \setminus \{1\}\},$$

where, $a_jx_i + b_jy_i = 1$ is the straight line joining the two points $(x_i, y_i) = (j-1, r/(j-1))$ and $(j, r/j), \forall j \in \mathbb{N} \setminus \{1\}$. Moreover, in this case we have $\hat{R} = R$.

Lemma 1 *Let $n = 1$. Consider a point (u, v) on the boundary of $\hat{R}(= R)$, i.e., $uv = r$. Then the point (u, v) is cut off by the facet defining inequality $a_j x_1 + b_j y_1 = \frac{x_1}{2j-1} + \frac{y_1 j(j-1)}{r(2j-1)} \geq 1$ of conv(S) if and only if $u \in (j-1, j)$. In other words, the optimal value of the optimization problem*

$$\min_{(x_1,y_1)\in R(=\hat{R})} a_j x_1 + b_j y_1$$

is less than one if and only if $u \in (j-1, j)$.

Proof For $j = 1$, we have the facet defining inequality $x_1 \geq 1$ and therefore, the proof is straightforward. For $j \geq 2$, since the facet defining inequality $\frac{x_1}{2j-1} + \frac{y_1 j(j-1)}{r(2j-1)} \geq 1$ is constructed by joining the points $(j-1, r/(j-1))$ and $(j, r/j)$, and since the curve $y_1 = \frac{r}{x_1}$ is strictly convex in the positive orthant, the result follows. □

In our further discussion, we consider the following convex mixed-integer relaxation $\hat{S}$ of S obtained by adding integer constraints to $\hat{R}$.

$$\hat{S} = \left\{ (x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : \sum_{i=1}^n \sqrt{\frac{x_i y_i}{r}} \geq 1 \right\}.$$

We study the facet defining inequalities of *conv*(S) as split cuts for $\hat{S}$ and determine their split-ranks.

Theorem 1 *For $n = 1$, every facet defining inequality of conv(S) is a rank-one split inequality for $\hat{S}$.*

Proof Consider the facet defining inequality $x_1 \geq 1$ of *conv*(S). The point $(\frac{1}{2}, 2r)$ lies in $\hat{R}$ and it is cut off by this facet defining inequality of *conv*(S). Therefore, it cannot have split-rank zero. Clearly, the inequality $x_1 \geq 1$ is valid for both the sets $\hat{R} \cap [x_1 \leq 0](= \phi)$ and $\hat{R} \cap [x_1 \geq 1]$, i.e., the inequality $x \geq 1$ is valid for the disjunction $[x_1 \leq 0] \vee [x_1 \geq 1]$. Therefore, the split-rank of this inequality is one.

Next consider a facet defining inequality $a_j x_1 + b_j y_1 = \frac{x_1}{2j-1} + \frac{y_1}{r(2j-1)} \geq 1, j \in \mathbb{N}, j \neq 1$ of *conv*(S). Since $\frac{1}{2}(2j-1) \in (j-1, j)$, by Lemma 1, the point $((2j-1)/2, 2r/(2j-1)) \in \hat{R}$ is cut off by this inequality. Therefore, it has split-rank at least one. Since the facet $a_j x_1 + b_j y_1 = 1$ is constructed by joining the two points $(j-1, r/(j-1))$ and $(j, r/j)$ and the curve $y_1 = \frac{r}{x_1}$ is concave, the inequality $a_j x_1 + b_j y_1 \geq 1$ is valid for both the sets $\hat{R} \cap [x_1 \leq j-1]$ and $\hat{R} \cap [x_1 \geq j]$, and consequently it is valid for the disjucntion $[x_1 \leq j-1] \vee [x_1 \geq j]$, and its split-rank is one. The following figure illustrates this geometrically. □

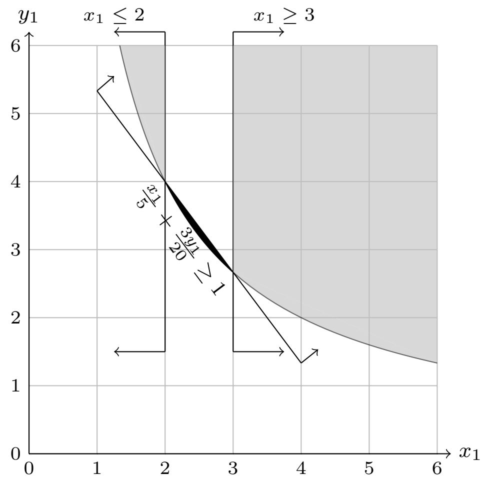

Fig. 2: The split disjunction $[x \leq 2] \vee [x_1 \geq 3]$ and the split cut $\frac{x_1}{5} + \frac{3y_1}{20} \geq 1, r = 8$

5.2 Split-ranks for higher dimension

In this section we discuss about the split ranks for general positive integer n.

Proposition 4 *Any facet defining inequality of conv(S) that is constructed using exactly one row of (M), i.e., of the form $a_j \sum_{i=1}^n x_i + b_j \sum_{i=1}^n y_i \geq 1$ for any $j \in \mathbb{N}$ is a rank-one split inequality for $\hat{S}$.*

Proof The point $(x, y) = ((2j-1)/2, 2r/(2j-1), 0, 0, \ldots, 0, 0)$ lies in $\hat{R}$ but violates the given inequality for any $j \in \mathbb{N}$. Therefore, this inequality has split-rank at least one. Consider the disjunction $[\sum_{i=1}^n x_i \leq j-1] \vee [\sum_{i=1}^n x_i \geq j]$ and the following two optimization problems.

$$\begin{aligned} \min_{(x,y)\in\hat{R}} \ & a_j \sum_{i=1}^n x_i + b_j \sum_{i=1}^n y_i \\ \text{s.t.} \ & \sum_{i=1}^n x_i \geq j, \end{aligned} \qquad \begin{aligned} \min_{(x,y)\in\hat{R}} \ & a_j \sum_{i=1}^n x_i + b_j \sum_{i=1}^n y_i \\ \text{s.t.} \ & \sum_{i=1}^n x_i \leq j-1. \end{aligned}$$

Consider the first optimization problem. From Proposition 1, there exists an optimal solution say $(\bar{x}, \bar{y})$ and an index $t \in N$ such that $x_i = 0, y_i = 0$ for all $i \in N \setminus \{t\}$. Because of symmetry, we assume $t = 1$. Therefore, the problem reduces to the following optimization problem

$$\min_{(x_1,y_1)\in\mathbb{R}^2_+} \quad a_j x_1 + b_j y_1$$
$$\sqrt{\frac{x_1 y_1}{r}} \geq 1,$$
$$\text{s.t. } x_1 \geq j.$$

From Lemma 1 the optimal value of this optimization problem is at least one. In fact the optimal value is exactly one as $(j, \frac{r}{j})$ is a feasible point with objective value one. Thus, the inequality $a_j \sum_{i=1}^n x_i + b_j \sum_{i=1}^n y_i \geq 1$ is valid for $\hat{R} \cap [\sum_{i=1}^n x_i \geq j]$.

Similarly we can show that the inequality $a_j \sum_{i=1}^n x_i + b_j \sum_{i=1}^n y_i \geq 1$ is valid for $\hat{R} \cap [\sum_{i=1}^n x_i \leq j-1]$. Consequently it is rank-one split inequality for $\hat{S}$. □

The following results give us a lower bound on the split-ranks for rest of the facet defining inequalities.

Theorem 2 *Consider a facet defining inequality of conv(S) that is constructed using two or more rows of (M). For any such inequality, there does not exist any split disjunction* $(\pi, \pi_0) \in \mathbb{Z}^{n+1}, \pi \neq 0$ *of* $\hat{S}$ *for which it is valid.*

Proof Without loss of generality we assume that two different rows of (M) are used for the variables with the first two indices, i.e., we consider the facet defining inequality

$$a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 + \sum_{i=3}^{n} (a_{p_i} x_i + b_{p_i} y_i) \geq 1 \qquad (\mathrm{I_S})$$

of *conv*(S) where $j \neq k$. Since the set $\hat{R}$ lies entirely in the positive orthant, it is sufficient to consider $\pi_0 \geq 0$ and those π that have at least one positive component. Consider the two optimization problems:

$$\min_{(x,y)\in\hat{R}} \left[a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 + \sum_{i=3}^{n} (a_{k_i} x_i + b_{k_i} y_i) \right]$$
$$\text{s.t. } \pi^T x \leq \pi_0, \qquad (Q_{\leq})$$

and

$$\min_{(x,y)\in\hat{R}} \left[a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 + \sum_{i=3}^{n} (a_{k_i} x_i + b_{k_i} y_i) \right]$$
$$\text{s.t. } \pi^T x \geq \pi_0 + 1. \qquad (Q_{\geq})$$

We show that at least one of the above problems has optimal value strictly less than one. We consider the following cases.

CASE A: When $\pi_1 \leq 0$, the point $((2j-1)/2, 2r/(2j-1), 0, 0, \ldots, 0, 0)$ is feasible for ($Q_{\leq}$) with objective value $a_j \frac{2j-1}{2} + b_j \frac{2r}{2j-1}$. Since $\frac{2j-1}{2} \in (j-1, j)$, by Lemma 1, the objective value is strictly less than one, and so is the optimal value of ($Q_{\leq}$).

CASE B: When $\pi_2 \leq 0$, we can similarly show that ($Q_{\leq}$) has optimal value less than one.

Therefore, the inequality ($\mathrm{I_S}$) is not valid for ($Q_{\leq}$) for both the above cases.

CASE C: The remaining case is when π_1 and π_2 are both positive integers. Suppose that one of the following relations holds true.

$$\pi_1\sqrt{j(j-1)} \leq \pi_0, \tag{3}$$

$$\pi_2\sqrt{k(k-1)} \leq \pi_0, \tag{4}$$

$$\pi_1\sqrt{j(j-1)} \geq \pi_0 + 1, \tag{5}$$

$$\pi_2\sqrt{k(k-1)} \geq \pi_0 + 1. \tag{6}$$

If (3) is true then clearly the point $(\sqrt{j(j-1)}, r/\sqrt{j(j-1)}, 0, 0, \ldots, 0, 0)$ is feasible for ($Q_{\leq}$) and by Proposition 2 and its corollary, the objective value at this point is strictly less than one. Again, if (4) is true then by the same arguments, ($Q_{\leq}$) has optimal value less than one. Similarly, using the same arguments, the optimal value of ($Q_{\geq}$) is less than one when (5) or (6) hold.

Finally, suppose none of the above four relations hold. Therefore we have,

$$\pi_0 < \pi_1\sqrt{j(j-1)} < \pi_0 + 1, \text{ and}$$
$$\pi_0 < \pi_2\sqrt{k(k-1)} < \pi_0 + 1.$$

Therefore both the values of π_1 and π_2 cannot be one from Proposition 10 in Appendix. By Proposition 11 in Appendix we have,

$$j - 1 < \frac{\pi_0 + 1}{\pi_1} \leq j \tag{7}$$

$$k - 1 < \frac{\pi_0 + 1}{\pi_2} \leq k \tag{8}$$

$$j - 1 \leq \frac{\pi_0}{\pi_1} < j \tag{9}$$

$$k - 1 \leq \frac{\pi_0}{\pi_2} < k \tag{10}$$

Since, $\pi_1, \pi_2 \in \mathbb{N}$, not both equal to one, at least one of the values of $\frac{\pi_0+1}{\pi_1}$, $\frac{\pi_0+1}{\pi_2}$, $\frac{\pi_0}{\pi_1}$ and $\frac{\pi_0}{\pi_2}$ must be non-integral.

If $\frac{\pi_0}{\pi_1}$ is non-integral, then we have $j-1 < \frac{\pi_0}{\pi_1} < j$ form (9). Clearly, the point $\mathcal{L}(1, \frac{\pi_0}{\pi_1}, \frac{r\pi_1}{\pi_0}) = (\frac{\pi_0}{\pi_1}, \frac{r\pi_1}{\pi_0}, 0, 0, \ldots, 0, 0)$ is feasible for the optimization problem ($Q_{\leq}$). Since at this point exactly one pair of components is positive, Lemma 1 is applicable, and the objective value at this point is strictly less than one. Similarly, the point $\mathcal{L}(2, \frac{\pi_0}{\pi_2}, \frac{r\pi_2}{\pi_0}) = (0, 0, \frac{\pi_0}{\pi_2}, \frac{r\pi_2}{\pi_0}, 0, 0, \ldots, 0, 0)$ is feasible for the optimization problem ($Q_{\leq}$) with objective value strictly less than one if $\frac{\pi_0}{\pi_2}$ is non-integral.

Similarly, if $\frac{\pi_0+1}{\pi_1}$(or $\frac{\pi_0+1}{\pi_2}$) is non-integral, from (7) (or (8)) and Lemma 1, we can say that the optimization problem ($Q_{\geq}$) has optimal value strictly less than one.

Therefore, for all the possible cases there does not exist any split disjunction (π, π_0) for which the facet defining inequality ($\mathrm{I_S}$) is valid. □

Corollary 2 *Any facet defining inequality of conv(S) that is constructed using two or more number of rows of (M) is not a rank-one split cut for $\hat{S}$.*

Proof Since a facet defining inequality for *conv*(S) cannot be expressed as a linear combination of any other valid inequalities for *conv*(S), it follows from Theorem 2 that its split-rank is at least two. □

6 Disjunctions for the facet defining inequalities

In the proof of Theorem 2, we showed that there does not exist any split disjunction for a class of inequalities that are valid. In this section we show that there exist facet defining inequalities of *conv*(S) that can be derived using some other more general disjunctions on the set $\hat{S}$. Furthermore, no other facet defining inequality, besides the above two types can be derived by disjunctive procedure on $\hat{S}$. Then we derive a closed convex relaxation from which any given facet defining inequality of *conv*(S) can be derived by the disjunctive procedure.

Proposition 5 *A facet defining inequality of conv(S) that is constructed using two rows of (M), one of which is the first row, is a disjunctive cut for $\hat{S}$.*

Proof Such a facet defining inequalities of *conv*(S) is of the following form:

$$\sum_{i\in J} x_i + \sum_{i\in K}(a_k x_i + b_k y_i) \geq 1 \tag{$\mathrm{I_D}$}$$

for some $k \in \mathbb{N}, k \neq 1$, where $J \cup K = N, J \cap K = \phi, J \neq \phi$ and $K \neq \phi$. We show that ($\mathrm{I_D}$) is valid for the disjunction $\left[\sum_{i\in J} x_i \geq 1\right] \vee \left[\sum_{i\in J} x_i \leq 0, \sum_{i\in K} x_i \geq k\right] \vee \left[\sum_{i\in J} x_i \leq 0, \sum_{i\in K} x_i \leq k-1\right]$ applied to $\hat{S}$.

Clearly the disjunction is valid. Like Proposition 4, we consider each atom separately. Consider the optimization problem:

$$\begin{aligned} \min_{(x,y)\in\hat{R}} \ & \sum_{i\in J} x_i + \sum_{i\in K}(a_k x_i + b_k y_i) \\ \text{s.t.} \ & \sum_{i\in J} x_i \geq 1. \end{aligned}$$

Since $\sum_{i\in J} x_i \geq 1$ is a constraint, the optimal value has to be at least one. Therefore, the inequality ($\mathrm{I_D}$) is valid for $\hat{R} \cap \left[\sum_{i\in J} x_i \geq 1\right]$. Consider the following optimization problem:

$$\begin{aligned} \min_{(x,y)\in\hat{R}} \quad & \sum_{i\in J} x_i + \sum_{i\in K} (a_k x_i + b_k y_i) \\ \text{s.t.} \quad & \sum_{i\in J} x_i \le 0, \\ & \sum_{i\in K} x_i \ge k. \end{aligned}$$

Clearly $x_i = 0$ for all $i \in J$. Using the same logic as in the proof of Proposition 4 (treating it as $n = |K|$) it is clear that for any $i \in K, \mathcal{L}(i, k, \frac{r}{k})$ is an optimal solution with optimal value one, and consequently ($\mathrm{I_D}$) is valid for $\hat{R} \cap \left[\sum_{i\in J} x_i \le 0, \sum_{i\in K} x_i \ge k\right]$.

Finally, using the proof of Proposition 4 again, we can show that ($\mathrm{I_D}$) is valid for $\hat{R} \cap \left[\sum_{i\in J} x_i \le 0, \sum_{i\in K} x_i \le k-1\right]$. □

The following result shows that there exist facet defining inequalities of *conv*(S) that cannot be derived by any disjunctive procedure on $\hat{S}$. In fact we show that many of the facet defining inequalities of *conv*(S) are not valid for $\hat{S}$.

Proposition 6 *Let (I) be a facet defining inequality of conv(S) constructed form a set of rows Γ from (M). If there exist two distinct $j, k \in \Gamma$ and $j \ne 1, k \ne 1$, then (I) is not valid for $\hat{S}$.*

Proof We show there are integer points in $\hat{S}$ that are not in *conv*(S). Thus, some of the valid inequalities for *conv*(S) are not valid for $\hat{S}$. The proof considers a valid inequality of *conv*(S) of the above form and constructs a point in $\hat{S}$ that is violated by this inequality. We consider the case when the inequality is constructed taking the j^{th} entry from the first column and the k^{th} entry from the second column with $j, k \ge 2, j \ne k$. Without loss of generality we assume $j < k$. The proof for the general case is similar. Now we have the inequality:

$$a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 + \sum_{i=3}^{n} (a_{p_i} x_i + b_{p_i} y_i) \ge 1, \tag{I}$$

where $a_j = \frac{1}{2j-1}, b_j = \frac{j(j-1)}{r(2j-1)}, a_k = \frac{1}{2k-1}, b_k = \frac{k(k-1)}{r(2k-1)}, a_{p_i} = \frac{1}{2p_i-1}, b_{p_i} = \frac{p_i(p_i-1)}{r(2p_i-1)}$. Consider the following point:

$$(\bar{x}, \bar{y}) = \left(j-1, \frac{r(j-1)b_k^2}{((j-1)b_k + b_j)^2}, 1, \frac{r b_j^2}{((j-1)b_k + b_j)^2}, 0, 0, \dots, 0, 0\right)$$

Clearly the point $(\bar{x}, \bar{y})$ lies in $\hat{S}$. Therefore it cannot be cut off by applying any disjunctive procedure on $\hat{S}$. We will be done if we can show that the inequality (I) cuts off $(\bar{x}, \bar{y})$, i.e., if

$$\frac{j-1}{2j-1} + \frac{r b_j b_k^2 (j-1)}{((j-1)b_k + b_j)^2} + \frac{1}{2k-1} + \frac{r b_k b_j^2}{((j-1)b_k + b_j)^2} < 1.$$

$$\begin{aligned}
\text{LHS} &= \frac{j-1}{2j-1} + \frac{rb_k b_j\left((j-1)b_k + b_j^2\right)}{\left((j-1)b_k + b_j\right)^2} + \frac{1}{2k-1} \\
&= \frac{j-1}{2j-1} + \frac{r}{\frac{j-1}{b_j} + \frac{1}{b_k}} + \frac{1}{2k-1} \\
&= \frac{j-1}{2j-1} + \frac{1}{\frac{2j-1}{j} + \frac{2k-1}{k(k-1)}} + \frac{1}{2k-1} \\
&= 1 - \frac{1}{2-\frac{1}{j}} + \frac{1}{2-\frac{1}{j} + \frac{2k-1}{k(k-1)}} + \frac{1}{2k-1} \\
&= 1 - \frac{\frac{2k-1}{k(k-1)}}{\left(2-\frac{1}{j}\right)\left(2-\frac{1}{j}+\frac{2k-1}{k(k-1)}\right)} + \frac{1}{2k-1} \\
&< 1 + \frac{1}{2k-1} - \frac{\frac{2k-1}{k(k-1)}}{\left(2-\frac{1}{k}\right)\left(2-\frac{1}{k}+\frac{2k-1}{k(k-1)}\right)}, \text{ since } j < k \\
&= 1.
\end{aligned}$$

□

Thus, applying disjunctive inequalities (or any other valid inequalities) of $\hat{S}$ is not sufficient to obtain all the facet defining inequalities of $conv(S)$. In order to obtain the facet defining inequalities as disjunctive inequalities, we use the following approach. Recall the form (I_G) of the facet defining inequalities of $conv(S)$.

Theorem 3 *The facet defining inequality (I_G) of $conv(S)$ is valid for the following disjunction on the nonconvex set R.*
$\left[\sum_{i\in J_1} x_i \geq j_1\right] \vee \left[\sum_{i\in J_1} x_i \leq j_1 - 1, \sum_{i\in J_2} x_i \geq j_2\right] \vee \cdots \vee$
$\left[\sum_{i\in J_1} x_i \leq j_1 - 1, \sum_{i\in J_p} x_i \geq j_p\right] \vee$
$\left[\sum_{i\in J_1} x_i \leq j_1 - 1, \sum_{i\in J_2} x_i \leq j_2 - 1, \ldots, \sum_{i\in J_p} x_i \leq j_p - 1\right].$

Proof Clearly the disjunction in the statement of the theorem is valid. We prove our result for $n = 2$. It can be easily generalized to any $n \geq 2$. For $n = 2$, the inequality (I_G) can be given as:

$$a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 \geq 1 \tag{11}$$

where $j, k \in \mathbb{N}, j < k$ (assuming $j_1 = j$ and $j_2 = k$). We have to show that the inequality is valid for the following disjunction:

$$[x_1 \geq j] \vee [x_1 \leq j-1, x_2 \geq k] \vee [x_1 \leq j-1, x_2 \leq k-1].$$

Case 1: Suppose $j = 1$. Therefore $a_j = 1$ and $b_j = 0$. Consider the global optimization problem:

$$\begin{aligned}\min_{x,y}\ & x_1 + a_k x_2 + b_k y_2 \\ \text{s.t. } & x_1 y_1 + x_2 y_2 \geq r, \\ & x_1 \geq 1, x_i, y_i \geq 0, i = 1, 2.\end{aligned}$$

Since we have $x_1 \geq 1, (1, r, 0, 0)$ is an optimal solution with optimal value 1. Next consider the global optimization problem:

$$\begin{aligned}\min_{x,y}\ & x_1 + a_k x_2 + b_k y_2 \\ \text{s.t. } & x_1 y_1 + x_2 y_2 \geq r, \\ & x_1 \leq 0, \\ & x_2 \geq k, \\ & x_i, y_i \geq 0, i = 1, 2.\end{aligned}$$

Since $x_1 = 0$, the problem reduces the $n = 1$ case. It is clear to see that $(0, 0, k, \frac{r}{k})$ is an optimal solution with optimal value 1. In an exactly similar way we can show that the optimal value is 1 for $R \cap [x_1 \leq 0, x_2 \leq k-1]$ also. Thus, the inequality (11) is valid for all the three atoms.

Case 2: When $j \geq 2$. We consider the global optimization problem:

$$\begin{aligned}\min_{x,y}\ & a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 \\ \text{s.t. } & x_1 y_1 + x_2 y_2 \geq r, \\ & x_1 \geq j, \\ & x_i, y_i \geq 0, i = 1, 2.\end{aligned}$$

The point $(j, \frac{r}{j}, 0, 0)$ is feasible with objective value one. Since the objective function is linear, it is equivalent to optimize over the convex hull of the feasible region of the above problem. Let $(\bar{x}, \bar{y})$ be an an extreme point optimal solution. Therefore either $\bar{x}_1\bar{y}_1 = r$ or $\bar{x}_2\bar{y}_2 = r$ by Proposition 3. Suppose $\bar{x}_1\bar{y}_1 = r$. Then $\bar{x}_1$ cannot be more than j because the value of $a_j\bar{x}_1 + b_j\bar{y}_1$ will be strictly greater than one (Lemma 1).

If $\bar{x}_2\bar{y}_2 = r$, then by Theorem 3 we have $y_1 = 0$. At this point the objective value is $\frac{t}{2t-1} + a_k\bar{x}_2 + b_k\bar{y}_2$ for some $t \geq j$. Since we are minimizing the objective function, from Proposition 2 and its corollary, the minimum value of $a_k x_2 + b_k y_2$ subject to the given constraints will be more than $\frac{1}{2}$ as $k \geq 2$. Also $\frac{t}{2t-1} > \frac{1}{2}$, since $t \geq j \geq 2$. Therefore, the objective value is more than one. Therefore, the optimal value of the above optimization problem is one and the inequality (11) is valid for $R \cap [x_1 \geq j]$.

We can show similarly that the inequality is valid for $R \cap [x_2 \geq k]$ and consequently for its subsets. Since $[x_1 \leq j-1, x_2 \geq k]$ is a subset of $[x_2 \geq k]$, the inequality is valid for the set $R \cap [x_1 \leq j-1, x_2 \geq k]$.

Finally, consider the global optimization problem:

$$\begin{aligned}\min_{x,y}\ & a_j x_1 + b_j y_1 + a_k x_2 + b_k y_2 \\ \text{s.t. } & x_1 y_1 + x_2 y_2 \geq r, \\ & x_1 \leq j-1, \\ & x_2 \leq k-1, \\ & x_i, y_i \geq 0, i = 1, 2.\end{aligned}$$

Let $(\bar{x}, \bar{y})$ be an extreme point optimal solution of the convex hull of the feasible region. Therefore, either $\bar{x}_1\bar{y}_1 = r$ or $\bar{x}_2\bar{y}_2 = r$ (by Proposition 3). Suppose $\bar{x}_1\bar{y}_1 = r$. If $\bar{x}_1 < j-1$, then by Lemma 1, the value of $a_j\bar{x}_1 + b_j\bar{y}_1$ is strictly greater than one, and therefore, the point $(j-1, \frac{r}{j-1}, 0, 0)$ gives the least objective value with objective value one. If $\bar{x}_2\bar{y}_2 = r$, then by the same logic the point $(0, 0, k-1, \frac{r}{k-1})$ gives the least objective value with objective value one. Therefore, the optimal solution of the above optimization problem is one. Thus, the inequality (11) is valid for all the nonconvex atoms. □

Let $P = \{J_1, \ldots, J_p\}$ be any partition of $N = \{1, \ldots, n\}$, with $p = |P|$, and let $j_1, \ldots, j_p$ be distinct positive integers (not necessarily sorted) associated with each element of P. Then we have a facet defining inequality ($\mathrm{I_G}$) of $conv(S)$ corresponding to each $(P, j_1, \ldots, j_p)$. Let us define the set $S(P, j_1, \ldots, j_p)$ to be the closure of the convex hull of the unions of the atoms of Theorem 3, i.e., let

$$S(P, j_1, \ldots, j_p) = cl\left(conv\left(\bigcup_{q=1}^{p+1} S_{J_q}\right)\right),$$

$$\text{where } S_{J_q} = \begin{cases} R \cap \left[\sum_{i\in J_t} x_i \geq j_t\right], \text{if } q = t, \\ R \cap \left[\sum_{i\in J_q} x_i \geq j_q, \sum_{i\in J_t} x_i \leq j_t - 1\right], q = 1, \ldots, p, q \neq t, \\ R \cap \left[\sum_{i\in J_1} x_i \leq j_1 - 1, \ldots, \sum_{i\in J_p} x_i \leq j_p - 1\right], q = p+1, \end{cases}$$

where $t \in \{1, \ldots, p\} : j_t \leq j_q, q = 1, \ldots, p$. Then, we have the following results.

Corollary 3 *The facet defining inequality (I_G) of $conv(S)$ can be constructed using disjunctive procedure on the closed convex set $S(P, j_1, \ldots, j_p)$.*

Corollary 4 *Let a set S_{CC} be defined as:*

$$S_{CC} = \bigcap_{P, j_1, \ldots, j_p} S(P, j_1, \ldots, j_p).$$

Then $S_{CC} = conv(S)$.

Proof Since each set $S(P, j_1, \ldots, j_p)$ is a convex relaxation of S, therefore S_{CC} is also a convex relaxation of S. The set S_{CC} is constructed by intersecting $S(P, j_1, \ldots, j_p)$ over all possible partitions of the index set N and distinct $(j_1, \ldots, j_p)$. Thus, every facet defining inequality of $conv(S)$ is valid for the set S_{CC}, and therefore, $conv(S) = S_{CC}$. □

7 The gap between rank-one facet defining inequalities of $\hat{S}$ and $conv(S)$

Let S^1 be the set of points that satisfy all the facet defining inequalities of $conv(S)$ that have split-rank one for $\hat{S}$. Therefore,

$$S^1 = \left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \frac{1}{2k-1}\sum_{i=1}^n x_i + \frac{k(k-1)}{r(2k-1)}\sum_{i=1}^n y_i \geq 1, k \in \mathbb{N}\right\}.$$

In this section we study the "gap" between the set S^1 and $conv(S)$. Here, by the "gap" we mean the difference between the optimal objective values of a linear objective function $c^T x + d^T y$ over S^1 and $conv(S)$. Since both the sets S^1 and $conv(S)$ are unbounded, we cannot compare them in terms of their volumes. Let

$$Z = \min_{(x,y)\in conv(S)} c^T x + d^T y, \text{ and}$$
$$Z^1 = \min_{(x,y)\in S^1} c^T x + d^T y.$$

Note that S^1 may not be the first split closure of $\hat{S}$ because S^1 is constructed from a few facets of $conv(S)$. First split closure of $\hat{S}$ may have other facets that are valid inequalities for $conv(S)$, but they may not be facets of $conv(S)$. We leave this subject for future research, and derive some conditions for which the gap between S^1 and $conv(S)$ is zero. We also give an example with an arbitrarily large gap.

Proposition 7 *Consider the optimization problem* $\min_{(x,y)\in S^1} c^T x + d^T y$. *Let* $\lambda, \mu \in N$ *be such that* $c_\lambda \leq c_i$ *for all* $i \in N$ *and* $d_\mu \leq d_i$ *for all* $i \in N$. *Then this optimization problem has the same optimal value as the optimization problem* $\min_{(x_\lambda,y_\mu)\in Q} c_\lambda x_\lambda + d_\mu y_\mu$, *where*

$$Q = \left\{(x_\lambda, y_\mu) \in \mathbb{R}_+ \times \mathbb{R}_+ : \frac{x_\lambda}{2k-1} + \frac{k(k-1)y_\mu}{r(2k-1)} \geq 1, k \in \mathbb{N}\right\}.$$

Proof We see that if $(x_\lambda, y_\mu) \in Q$ then $\mathcal{L}(1, x_\lambda, y_\mu) = (x_\lambda, y_\mu, 0, 0, \ldots, 0, 0) \in S^1$. Again, if $(x,y) \in S^1$ then we have $(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i) \in Q$. Therefore, the two sets S^1 and Q are feasibility wise equivalent in the sense that if one has a feasible solution, we can construct a feasible solution to the other with the same objective value and vice versa. Note that the set Q is the convex hull of the two dimensional mixed-integer bilinear covering set $\{(x_\lambda, y_\mu) \in \mathbb{Z}_+ \times \mathbb{R}_+ : x_\lambda y_\mu \geq r\}$. We consider the following cases.

Case 1: One of the values of c_λ and d_μ is negative. Then clearly both the optimization problems are unbounded.

Case 2: When $c_\lambda = 0$. Then for both the optimization problems, the optimal value of the objective function is c_λ if $d_\mu = 0$, and infimum is zero if $d_\mu > 0$ [36].

CASE 3: When $c_\lambda > 0$ and $d_\mu = 0$, then for both the optimization problems the optimal value is c_λ [36].

CASE 4: $c_\lambda > 0$ and $d_\mu > 0$. We see that the constraints in the descriptions of S^1 are symmetric about the indices of the variables x and y, i.e., interchanging the variables x_i with x_j (or y_i with y_j) for any $i, j \in N$ is not going to affect the feasibility of S^1. Again since we are minimizing $c^T x + d^T y$ over S^1 which lies in the positive orthant, we can always chose an optimal solution $(\bar{x}, \bar{y})$ such that $\bar{x}_\lambda > 0, \bar{x}_i = 0$ for all in $N, i \neq \lambda$ and $\bar{y}_\mu > 0, \bar{y}_i = 0$ for all in $N, i \neq \mu$. It is now clear that $(\bar{x}_\lambda, \bar{y}_\mu)$ is an optimal solution to the problem $\min_{(x_\lambda, y_\mu) \in Q} c_\lambda x_\lambda + d_\mu y_\mu$. □

Note that $Z^1 = \min_{(x_\lambda, y_\mu) \in Q} c_\lambda x_\lambda + d_\mu y_\mu$ from the above proposition. Z is also unbounded when $\min_{i \in N}\{c_i, d_i\} < 0$, just like Z^1. Therefore, we consider $c \geq 0, d \geq 0$. In our previous work [36], the algorithm to derive the values of Z and Z^1 are described with closed form solutions for both and they are as follows. It is also described in [36] that there exist $q \in N$ such that $Z = \min_{\mathcal{L}(q, x_q, y_q) \in S_q} c_q x_q + d_q y_q$, where S_q is an orthogonal disjunctive subset of S which is defined in the earlier section of this article.

$$Z^1 = \begin{cases} \min\left\{ c_\lambda \left\lfloor \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rfloor + \frac{rd_\mu}{\left\lfloor \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rfloor}, c_\lambda \left\lceil \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rceil + \frac{rd_\mu}{\left\lceil \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rceil} \right\} \text{when } c_\lambda > 0, d_\mu > 0, \\ c_\lambda, \text{when } c_\lambda \geq 0, d_\mu = 0, \\ 0 \ \ \text{(actually infimum value), when } c_\lambda = 0, d_\mu > 0. \end{cases}$$

$$Z = \min_{i \in N} Z_i, \text{ where}$$

$$Z_i = \begin{cases} \min\left\{ c_i \left\lfloor \sqrt{\frac{rd_i}{c_i}} \right\rfloor + \frac{rd_i}{\left\lfloor \sqrt{\frac{rd_i}{c_i}} \right\rfloor}, c_i \left\lceil \sqrt{\frac{rd_i}{c_i}} \right\rceil + \frac{rd_i}{\left\lceil \sqrt{\frac{rd_i}{c_i}} \right\rceil} \right\} \text{when } c_i > 0, d_i > 0, \\ c_i, \text{when } c_i \geq 0, d_i = 0, \\ 0 \ \ \text{(actually infimum value), when } c_i = 0, d_i > 0. \end{cases}$$

We assume $\frac{1}{\left\lfloor \sqrt{\frac{rd_i}{c_i}} \right\rfloor}$ and $\frac{1}{\left\lfloor \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rfloor}$ are infinite if $\left\lfloor \sqrt{\frac{rd_i}{c_i}} \right\rfloor$ and $\left\lfloor \sqrt{\frac{rd_\mu}{c_\lambda}} \right\rfloor$ are zeros respectively with $c_i > 0, d_i > 0, c_\lambda > 0$ and $d_\mu > 0$.

7.1 When the gap is zero

It can be seen clearly that $Z^1 \leq Z$ as S^1 is a relaxation of $conv(S)$. The following result gives us a characterization for $Z^1 = Z$.

Proposition 8 *Let Λ, Δ be two subsets of N such that $c_\lambda = c_i$ for all $i \in \Lambda$ and $d_\mu = d_i$ for all $i \in \Delta$. Then $Z^1 = Z$ if and only if $\Lambda \cap \Delta$ is non empty.*

Proof Let $p \in \Lambda \cap \Delta$. Then by Proposition 7, there exists an optimal solution $(\bar{x}, \bar{y})$ of the form $\mathcal{L}(p, \bar{x}_p, \bar{y}_p)$ to the problem $\min_{(x,y)\in S^1} c^T x + d^T y$. Therefore, $(\bar{x}_p, \bar{y}_p)$ satisfies

$$\frac{\bar{x}_p}{2k-1} + \frac{k(k-1)\bar{y}_p}{r(2k-1)} \geq 1, k \in \mathbb{N}.$$

If we show $(\bar{x}, \bar{y}) \in conv(S)$, we will have $Z = Z^1$. We know that

$$conv\,(S_p) = \left\{ \mathcal{L}(p, x_p, y_p) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \frac{x_p}{2k-1} + \frac{k(k-1)y_p}{r(2k-1)} \geq 1, k \in \mathbb{N} \right\}$$

where, $S_p = \{\mathcal{L}(p, x_p, y_p) \in \mathbb{Z}^n_+ \times \mathbb{R}^n_+ : x_p y_p \geq r\}$ is an orthogonal disjunctive subset of S. This implies that $\mathcal{L}(p, \bar{x}_p, \bar{y}_p) \in conv(S_p)$. Again, since $conv(S)$ is closed and satisfies the convex extension property, i.e., $conv(S) = conv(\bigcup_{i=1}^n S_i)$ [39], where $S_i, i \in N$ are orthogonal disjunctive subsets of S, the point $\mathcal{L}(p, \bar{x}_p, \bar{y}_p) \in conv(S)$. Therefore, we have a point in $conv(S)$ with objective value Z^1, consequently $Z^1 = Z$.

Conversely, let $\Lambda \cap \Delta$ be empty. We know that there exists $q \in N$ such that $Z_{CV} = \min_{\mathcal{L}(q,x_q,y_q)\in S_q} c_q x_q + d_q y_q = \min_{\mathcal{L}(q,x_q,y_q)\in conv(S_q)} c_q x_q + d_q y_q$ [36]. Since $\Lambda \cap \Delta = \phi$, $c_\lambda \leq c_q, d_\mu \leq d_q$ with $c_\lambda < c_q$ or $d_\mu < d_q$. Again, since the two sets $conv(S_q)$ and Q (defined in Proposition 7) are feasibility wise equivalent, we have $Z^1 < Z$. □

7.2 When the gap is arbitrary large : An example

Let $n = 2, r = 16$ and consider the objective function $x_1 + \eta^2 y_1 + \eta^2 x_2 + y_2$ where η is a positive integer.

We see that $c_\lambda = c_1 = 1$ and $d_\mu = d_2 = 1$. Therefore, $Z^1 = 8$ with optimal solution $(4, 0, 0, 4)$ which is not feasible for $conv(S)$. Clearly, if we increase the value of η, the value of Z^1 is not going to change.

On the other hand, we see that $Z = \eta Z^1$. Since for any value of $\eta \geq 1$, the value of Z^1 is constant, the value of Z increases by a factor of η with Z^1, therefore, the gap between the values of Z and Z^1 can be arbitrary large.

7.3 Empirical study of gap for general n

We perform an empirical study of the gap between Z and Z^1 when n is larger than two. For a selected value of n, we randomly generate instances of the form

$$\begin{aligned} &\min_{x,y}\; c^T x + d^T y \\ &\text{s.t.} \; \sum_{i=1}^n x_i y_i \geq r, \\ &\qquad x, y \geq 0, \\ &\qquad x \in \mathbb{Z}^n, \end{aligned}$$

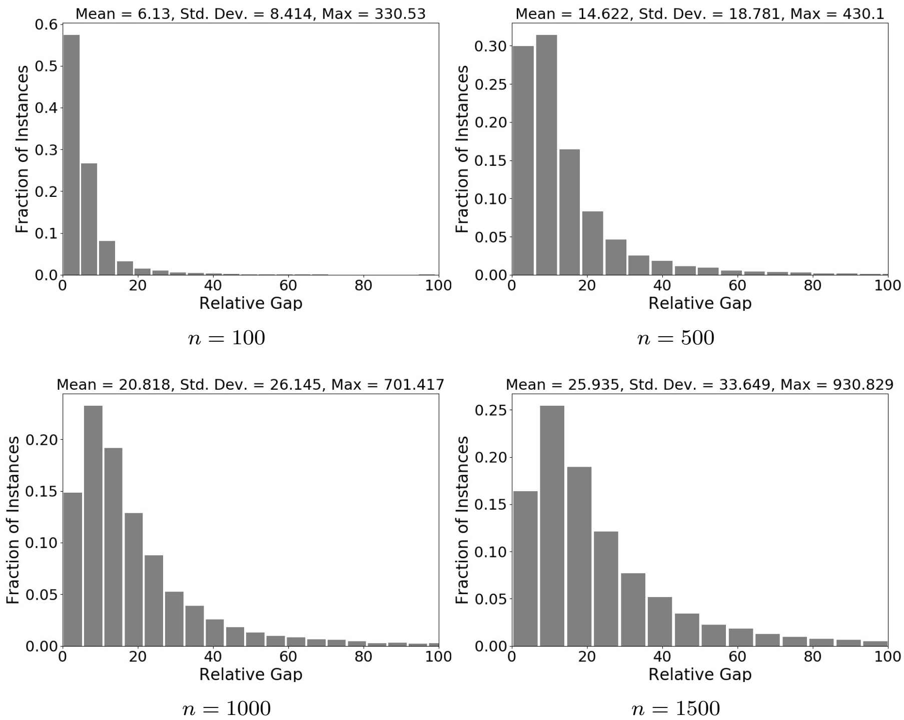

Fig. 3: Histograms of relative gap observed in randomly generated instances.

with r, c, d as parameters. If $r \leq 0$, then the constraint is redundant and the problem is trivial, so we assume $r > 0$. Further, if any component of c, d is negative, the optimal value is unbounded, so we assume $c, d \geq 0$. Also, the bilinear constraint can be scaled so that we get an equivalent problem with right-hand side one. This scaling can be done by letting a new variable, say $w = y/r$ and letting $\hat{d} = rd$. This transformation yields: $\min c^T x + \hat{d}^T w$ subject to $\sum_{i=1}^{n} x_i w_i \geq 1$. Thus, we can focus only on changing c, d.

We performed computational experiments in which components of vectors c, d are generated uniformly randomly in the range $[0, 100]$ while r is fixed to 20. 20,000 random instances are generated in this way and the values of Z, Z^1 evaluated for each of them in the manner described above. The relative gap $(Z - Z^1)/Z^1$ is measured for each of the random instance and a histogram is produced as in Figure 3. The experiment is repeated for $n = 100$, 500, 1000 and 1500.

The X-axis in Figure 3 represents the relative gap as defined above, and the Y-axis represents the fraction of instances on which a certain relative gap was observed. We observe that both, mean relative gap and the variance of the relative gap increase as we increase n. Smaller the value of n, better is the approximation by rank-one facets to the convex hull. For a fixed n, the fraction of instances with very high gaps is relatively small, but this fraction keeps increasing with n.

8 Separation of facet defining inequalities of $conv(S)$ and S^1

A linear-time separation algorithm for the facet defining inequalities for $conv(S)$ is described in [36]. Broadly speaking, the algorithm works as follows. Let $(\bar{x}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ be a given point that we want to separate. For each column of (M) the term that gives the minimum value at $(\bar{x}, \bar{y})$ is found. If the sum of these n values (one from each column) is greater than or equal to one, then $(\bar{x}, \bar{y})$ is feasible to $conv(S)$. Otherwise, the inequality constructed using these n terms cuts off $(\bar{x}, \bar{y})$. The minimum term from each column is obtained in the following manner. We find

$$\xi_i = \min\left\{\frac{\bar{x}_i}{2w-1} + \frac{\bar{y}_i w(w-1)}{r(2w-1)}, w \in \mathbb{N}\right\}, i = 1, \ldots, n.$$

Clearly, $\sum_{i=1}^n \xi_i \geq 1$ if and only if $(\bar{x}, \bar{y}) \in conv(S)$. Let us now define the following functions

$$f_i(w) = \frac{\bar{x}_i}{2w-1} + \frac{\bar{y}_i w(w-1)}{r(2w-1)}, w \geq 1, i = 1, \ldots, n. \tag{12}$$

Our goal is to find a $\hat{w}_i \in \mathbb{N}$ that minimizes f_i, for each $i \in N$. The following three cases arise:
Case 1 When $\bar{x}_i = 0$, then clearly $\xi_i = 0, \hat{w}_i = 1$.
Case 2 When $\bar{y}_i = 0$, then $\inf f_i(w) = 0$. So, ξ_i can be taken as 0. A sufficiently large $\hat{w}_i$ in this case can be found. See [36] for details.
Case 3 When $\bar{x}_i, \bar{y}_i > 0$, then, clearly f_i is strictly convex if $4\bar{x}_i r - \bar{y}_i > 0$. Then $\frac{1}{2} + \frac{1}{2}\sqrt{4\bar{x}_i r/\bar{y}_i - 1} = \bar{w}$ (say) is the unique minimizer of f_i. If $4\bar{x}_i r - \bar{y}_i \leq 0$, f_i is concave. A boundary point must then be a minimizer, and 1 is the only boundary point of the domain. Therefore, we have the following

$$\hat{w}_i = \begin{cases} 1, \text{ when } 4\bar{x}_i r - \bar{y}_i > 0 \text{ and } \bar{w} \leq 1, \\ \lceil \bar{w} \rceil, \text{ when } 4\bar{x}_i r - \bar{y}_i > 0, \bar{w} > 1 \text{ and } f(\lceil \bar{w} \rceil) \leq f(\lfloor \bar{w} \rfloor), \\ \lfloor \bar{w} \rfloor, \text{ when } 4\bar{x}_i r - \bar{y}_i > 0, \bar{w} > 1 \text{ and } f(\lceil \bar{w} \rceil) \geq f(\lfloor \bar{w} \rfloor), \\ 1, \text{ when } 4\bar{x}_i r - \bar{y}_i \leq 0. \end{cases}$$

Separating inequalities for S^1 can be found by modifying the above approach. In order to separate a given point $(\bar{x}, \bar{y})$ we consider the separation problem

$$\min_{k \in \mathbb{N}} \frac{1}{2k-1} \sum_{i=1}^n \bar{x}_i + \frac{k(k-1)}{r(2k-1)} \sum_{i=1}^n \bar{y}_i.$$

If the optimal value to the above problem is one or more, then $(\bar{x}, \bar{y}) \in S^1$. Any $k \in \mathbb{N}$ for which the above function has value less than one gives us a separating inequality. This function is similar to $f_i(k)$ (12), only $(\bar{x}_i, \bar{y}_i)$ are replaced by $(\sum_{i=1}^n \bar{x}_i, \sum_{i=1}^n \bar{y}_i)$. Thus the function is independent of i in this case. The approach used for minimizing f_i above gives us the optimal value of k, and thus the separating inequality.

9 Rank-one facets and the cutting stock problem

We now study the gap between the bounds that are obtained using the inequalities of S^1 and the inequalities for $conv(S)$ performing computational experiments on some cutting stock problem instances of the following form.

$$\begin{aligned} \min & \sum_{i=1}^{n} y_i \\ & \sum_{i \in N} x_{ij} y_i \geq d_j,\ j \in F, \\ & \sum_{j \in F} l_j x_{ij} \leq L, i \in N, \\ & x_{ij} \in \mathbb{Z}_+, y_i \in \mathbb{Z}_+, \forall i \in N, j \in F, \end{aligned} \tag{CS}$$

where $N = \{1, \ldots, n\}$ is the index set for different cutting patterns that are used, and F is the index set of different sizes of the finals that are to be cut. Here, L is the size of each large roll and $l_j, j \in F$ are the lengths of the finals with respective demands $d_j, j \in F$ which are known. Let x_{ij} be the number of final j cut according in the pattern $i, i \in N, j \in F$, and y_i be the number of rolls cut with cutting pattern $i, i \in N$. The first set of constraints are demand satisfaction constraints which are $|F|$ in number and bilinear. The second set of constraints are knapsack constraints which are for the feasibility of the patterns. These knapsack constraints imply $x_{ij} \leq \left\lfloor \frac{L}{l_j} \right\rfloor, \forall i \in N, j \in F$. The problem naturally has $|F|$ bilinear covering sets whose facets can be used as valid inequalities. These inequalities may not define facets for the cutting stock formulation as each bilinear set is considered separately and variables y are relaxed to allow continuous values. Extending the facets of bilinear covering sets to those for the cutting stock problem is a direction for future work.

Proposition 9 *Consider the problem (CS) without the knapsack constraints. Then the lower bound obtained by considering all the facet defining inequalities of each bilinear constraint is equal to the lower bound obtained by considering only the rank-one facet defining inequalities for each bilinear constraint.*

Proof Let $(\bar{x}, \bar{y})$ be an optimal solution when we consider only the rank-one facet defining inequalities of each bilinear constraint. Therefore, we have

$$\sum_{i \in N} \frac{\bar{x}_{ij}}{2k-1} + \sum_{i \in N} \frac{\bar{y}_i k(k-1)}{d_j(2k-1)} \geq 1, \text{ for all } j \in F, k \in \mathbb{N}, \tag{13}$$

and the optimal objective value at this point is $\sum_{i\in N}\bar{y}_i$. If we can show that there exists a point satisfying all the facet defining inequalities of each bilinear constraint with objective value $\sum_{i\in N}\bar{y}_i$, we will be done. Consider the point (x^*, y^*) defined as:

$$x^*_{ij} = \begin{cases} \sum_{i\in N}\bar{x}_{ij}, \text{ if } i = 1 \\ 0, \text{ if } i \neq 1, i \in N \end{cases} \quad \text{for } j \in F$$
$$y^*_i = \begin{cases} \sum_{i\in N}\bar{y}_i, \text{ if } i = 1 \\ 0, \text{ if } i \neq 1, i \in N. \end{cases}$$

Therefore, from relation (13) we see that for each bilinear inequality of (CS), at the point (x^*, y^*), the value of each entry in the first column of (M) is greater than or equal to 1. Again, since each facet defining inequality of a given bilinear constraint of (CS) is constructed by adding n elements from (M) taken one from each column. This implies that the point (x^*, y^*) satisfies all the facet defining inequalities of each bilinear constraint, and the objective value at this point is $\sum_{i\in N}\bar{y}_i$. □

In order to check whether the result also holds when the knapsack constraints are present, we performed a computational experiment on benchmark problems selected from [40] (Fiber-xx-xxx), CUTGEN [22] generated instances (CutGen-xx-xx), nine instances are taken from [1] and five randomly generated instances (Rand-xx). The random instances (Rand-xx) were generated by fixing L to 1030 and selecting specifc problem size n (denoted as 'xx' in the name). The final lengths l_j were generated randomly between 75 and 600, and d_j between 300 and 5000. We compare the bounds obtained using the above two approaches and the number of steps taken. For each instance, in either case we start the iterations with the facet defining inequalities $\sum_{i\in N} x_{ij} \geq 1, j \in F$, the bound constraints $x_{ij} \leq \left\lfloor \frac{L}{l_j} \right\rfloor, \forall i \in N, j \in F$ and the knapsack constraints, i.e., the following LP.

$$\begin{aligned} \min \ & \sum_{i=1}^{n} y_i \\ \text{s.t.} \ & \sum_{i=1}^{n} x_{ij} \geq 1, \forall j \in F, \\ & 0 \leq x_{ij} \leq \left\lfloor \frac{L}{l_j} \right\rfloor, \forall i \in N, j \in F \\ & \sum_{j\in F} l_j x_{ij} \leq L, i \in N, \\ & y \geq 0. \end{aligned} \qquad \text{(ILP)}$$

We use separation algorithms of Section 8 to separate the facet defining inequalities for $conv(S)$ and the inequalities of S^1. Therefore, we add at most

$|F|$ cuts in each iteration for both the cases (i.e., at the k^{th} iteration we solve an LP with $k|F|$ number of linear inequalities in addition to those in the starting LP). We stop when either of two conditions hold: (a) we cannot find any more violated inequalities, or (b) the time limit of 3600 seconds exceeds (we write "3600*" for such cases in Table 1).

We have used Python based PuLP 1.6.9 [30] to model the problem and CBC 2.10 [19] (which uses CLP 1.17 [20] LP solver) to solve the linear programs. The experiment was performed on a system having Intel(R) Xeon(R) CPU E5-2643 0 @ 3.30GHz processor, 12 GB of RAM and Linux (Ubuntu 18.04) operating system. We have used a single core for each run. The results are compiled in Table 1. The results show that the optimization over S^1 does not give a worse bound than the convex hull, even though it is a relaxation of the convex hull. We also observe that rank-one facets improve the bounds in much fewer iterations and in less time for all input problems. One possible explanation for the somewhat unexpected results is that the separation procedure only finds the 'most violated' inequality. It does not find the most distant inequality or the one that pushes the bound by the most. Thus many higher ranked inequalities are added when we run the algorithm for all facets, but they do not improve the bound as much.

To further explore the matter, we performed one more experiment on the 14 instances that hit the time limit for $conv(S)$. In this experiment, we first generated only rank-one facets. Once we had optimized over S^1, we switched to separation algorithm for $conv(S)$ and let it run for 3600s. In this experiment also all instances hit time limit without any improvement in bounds. Further investigation is required to ascertain whether the two bounds are always equal for this class of problems or there are some instances where they can be different.

10 Concluding Remarks

We showed that all facet defining inequalities of $conv(S)$ can be viewed as disjunctive cuts derived from disjunctions specified in the discussion above. Some of them have split-rank one for a convex mixed-integer relaxation of S. These cuts are sufficient to find the optimal value over $conv(S)$ for certain objective functions like those in trimloss problems. Finding strong valid inequalities for the convex hull of the feasible region of trimloss problems is still open and can be taken up in the future.

Appendix : Additional Proofs

Proposition 10 *Let $j, k \in \mathbb{N}$ with $j \neq k$, then the following two relations*

$$\pi_0 < \sqrt{j(j-1)} < \pi_0 + 1 \textit{ and}$$
$$\pi_0 < \sqrt{k(k-1)} < \pi_0 + 1$$

Instances	n	Using inequalities for $conv(S)$				Using inequalities for S^1 only			
		Iter	Cuts	LB	Time	Iter	Cuts	LB	Time
Fiber-10-5180	10	226	1917	6.88	47.33	5	37	6.88	0.21
Fiber-10-9080	10	223	2045	3.85	45.57	6	42	3.85	0.26
Fiber-11-5180	11	288	2673	6.10	93.52	4	35	6.10	0.19
Fiber-11-9080	11	335	2946	3.40	130.58	5	45	3.40	0.25
Fiber-14-5180	14	473	5417	3.34	529.18	5	57	3.34	0.31
Fiber-14-9080	14	476	6211	1.90	537.12	5	57	1.90	0.33
Fiber-15-5180	15	560	7219	3.74	1321.66	6	65	3.74	0.42
Fiber-15-9080	15	1052	11311	2.09	3600*	6	72	2.11	0.44
Fiber-16-5180	16	756	9763	5.17	2797.61	5	63	5.17	0.41
Fiber-16-9080	16	723	10330	2.93	2268.56	6	78	2.93	0.48
CutGen-01-01	10	252	2244	1.24	61.00	5	43	1.24	0.21
CutGen-01-02	10	270	2443	0.97	70.32	5	40	0.97	0.22
CutGen-01-25	10	246	2157	0.99	55.32	5	39	0.99	0.20
CutGen-01-100	10	244	2131	1.25	56.31	5	41	1.25	0.21
CutGen-02-40	10	262	2272	10.41	64.23	4	33	10.41	0.17
CutGen-02-60	10	275	2480	10.10	78.57	5	40	10.10	0.22
Rand-10	10	185	1601	697.22	31.87	4	33	697.22	0.17
Rand-15	15	728	8242	576.69	3600*	4	53	576.69	0.31
Rand-16	16	719	8395	686.27	3600*	4	55	686.27	0.34
Rand-20	20	611	11637	631.02	3600*	5	75	631.02	0.56
Rand-25	25	485	12150	459.75	3600*	6	100	529.74	0.93
gau3	50	199	10000	0.82	3600*	5	176	22.31	3.53
m50-100-01	50	215	10800	0.40	3600*	5	169	17.24	3.09
m50-100-02	50	217	10900	0.58	3600*	5	176	18.58	3.19
10-20-01	40	371	14880	5.45	3600*	4	117	28.25	1.57
10-20-02	40	385	15440	4.36	3600*	4	111	25.70	1.54
250-80	40	207	8320	37.90	3600*	6	177	144.6	2.39
Falkenauer-t60-00	50	238	11950	0.02	3600*	5	167	0.413	3.23
Falkenauer-t60-01	56	236	13272	0.01	3600*	5	228	0.366	4.94
Falkenauer-u120-00	58	207	12064	0.01	3600*	4	190	0.846	4.18

Table 1: Comparison of iterations and time taken to optimize using the inequalities for S^1 only and the convex hull.

cannot hold simultaneously for any non-negative integer π_0.

Proof Without loss of generality let $k > j$. Note that, it is equivalent to prove that $\sqrt{k(k-1)} - \sqrt{j(j-1)} \geq 1$. This is because, if $\sqrt{k(k-1)} - \sqrt{j(j-1)} \geq 1$ holds, then both the values $\sqrt{k(k-1)}$ and $\sqrt{j(j-1)}$ cannot lie between two consecutive integers.

Also note that, since the function $f(j) = \sqrt{j(j-1)}$ is strictly increasing for $j \in \mathbb{N}$, it is sufficient to prove the result when $k = j+1$, i.e., we show that $\sqrt{j(j+1)} - \sqrt{j(j-1)} \geq 1$. Since we are dealing with positive numbers only, in our following steps of proof, we consider only the positive roots. For any

$j \in \mathbb{N}$,

$$\begin{aligned}
&4j^2 - 4j + 1 > 4j^2 - 4j \\
\Rightarrow\ &(2j-1)^2 > 4j(j-1) \\
\Rightarrow\ &2j - 1 > 2\sqrt{j(j-1)} \\
\Rightarrow\ &j^2 + j > 1 + 2\sqrt{j(j-1)} + j^2 - j \\
\Rightarrow\ &j(j+1) > 1 + 2\sqrt{j(j-1)} + j(j-1) \\
\Rightarrow\ &j(j+1) > \left(1 + \sqrt{j(j-1)}\right)^2 \\
\Rightarrow\ &\sqrt{j(j+1)} - \sqrt{j(j-1)} > 1.
\end{aligned}$$

This completes the proof. □

Proposition 11 *Let $k \in \mathbb{N}$ with $k \geq 2$. Consider the positive integers μ_0, μ with $\mu \geq 1$. If $\mu_0 < \mu\sqrt{k(k-1)} < \mu_0 + 1$ then $k - 1 < \frac{\mu_0+1}{\mu} \leq k$ and $k - 1 \leq \frac{\mu_0}{\mu} < k$.*

Proof Since $\mu_0 < \mu\sqrt{k(k-1)} < \mu_0 + 1$, then $\frac{\mu_0+1}{\mu\sqrt{k(k-1)}} > 1$. Again since $\sqrt{\frac{k-1}{k}} < 1$, we have

$$\begin{aligned}
&\frac{\mu_0 + 1}{\mu\sqrt{k(k-1)}} > \sqrt{\frac{k-1}{k}} \\
\Rightarrow &\frac{\mu_0 + 1}{\mu} > k - 1
\end{aligned}$$

We show the other side of the desired inequality next. From the given condition we have,

$$\begin{aligned}
&\mu\sqrt{k(k-1)} > \mu_0 \\
\Rightarrow &\mu\left\lceil\sqrt{k(k-1)}\right\rceil > \mu_0 \\
\Rightarrow &\mu\left\lceil\sqrt{k(k-1)}\right\rceil \geq \mu_0 + 1 \text{ (since the left hand side is integral)} \\
\Rightarrow &\mu k \geq \mu_0 + 1 \ \left(\text{since } k \geq \left\lceil\sqrt{k(k-1)}\right\rceil\right) \\
\Rightarrow &k \geq \frac{\mu_0 + 1}{\mu}
\end{aligned}$$

Therefore, we have $k - 1 < \frac{\mu_0+1}{\mu} \leq k$.

Since $\frac{\mu_0+1}{\mu} \leq k$, we have $\frac{\mu_0}{\mu} < k$. It remains to show $k-1 \leq \frac{\mu_0}{\mu}$. From the given relation we have

$$\begin{aligned}
&\mu_0 + 1 > \mu\sqrt{k(k-1)} \\
\Rightarrow &\mu_0 + 1 > \mu(k-1) \ \left(\text{since } \sqrt{k(k-1)} > k-1\right) \\
\Rightarrow &\mu_0 \geq \mu(k-1) \ \text{(since both sides are integral)} \\
\Rightarrow &\frac{\mu_0}{\mu} \geq k-1
\end{aligned}$$

Therefore, we get $k-1 \leq \frac{\mu_0}{\mu} < k$. □

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