A Viral Timeline Branching Process to study a Social Network

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***Abstract*—Bio-inspired paradigms are proving to be useful in analysing propagation and dissemination of information in networks. In this paper we explore the use of multi-type branching processes to analyse viral properties of content in a social network, with and without competition from other sources. We derive and compute various virality measures, e.g., probability of virality, expected number of shares, or the rate of growth of expected number of shares etc. They allow one to predict the emergence of global macro properties (e.g., viral spread of a post in the entire network) from the laws and parameters that determine local interactions. The local interactions, greatly depend upon the structure of the timelines holding the content and the number of friends (i.e., connections) of users of the network. We then formulate a non-cooperative game problem and study the Nash equilibria as a function of the parameters. The branching processes modelling the social network under competition turn out to be decomposable, multi-type and continuous time variants. For such processes types belonging to different sub-classes evolve at different rates and have different probabilities of extinction etc. We compute content provider wise extinction probability, rate of growth etc. We also conjecture the content-provider wise growth rate of expected shares.**

I. Introduction

In beginning of 2017, the average number of mobile YouTube videos exceeded 1,000,000,000 views, and Google's annual revenue generated from YouTube reached \$4,000,000,000.[1] The huge growth in the activity of the content industry on the internet has generated a wide interest in understanding content propagation in the Internet and on how the propagation can be accelerated. Previously peer to peer (P2P) networks (e.g., [11, 3]), have played an important role, while of late there has been a lot of interest in content propagation over social networks (e.g.,[1, 2, 4, 5] etc). The P2P networks pull the required information from their peers, while in viral marketing (e.g., [1, 2]) the information is pushed for marketing purposes. In P2P networks break files into chunks which adds to the complexity of modelling and analysing them, in social networks we have extra complexity due to the timelines. In viral marketing scenarios one needs to keep pushing information by passing on the information regularly to seed nodes to keep the flow going on. In this paper we consider content propagation over social networks like facebook, twitter etc., where the information (called post) is again pushed, but involuntarily. Here the post is forwarded to few initial seeds which propagate/get viral because of enormity of the network and based on the interest generated by the post. More importantly, these networks have extra complexity due to timelines (TLs), the inverse stacks, one dedicated for each user. A post (that we are interested in) on a TL is shifted down by one level with every new post (forwarded), the post can shift down to a considerably lower position and the user can miss it. One needs to consider these aspects for a realistic study. To the best of our knowledge such a study has not been considered in any viral marketing or other related literature. The goal of this paper is to develop a branching processing based modeling of social networks which will allow us to derive analytical performance measures and will provide insights for competition over visibility, popularity and influence.

Our approach and contribution: An important approach for modelling information diffusion in these networks has been branching processes (BPs), (see [1, 2, 3, 11]) as they are powerful enough to describe many phenomena that are characteristic to content diffusion (e.g., phase transition-epidemic threshold) and yet provide explicit expressions for many important performance measures. Authors in [4, 6, 5] etc., have shown that the branching process can well fit the content propagation trajectories collected from real data. In majority of Branching processes any parent produces IID (identically and independently distributed) offsprings. When one models a social network with a branching process, any parent should produce identical number of offsprings (identical to the previous parents) and independent of the offsprings produced previously. Further when it explodes, the parents keep producing offsprings. This is possible only when the social network has infinite population. One can assume the huge networks have infinite population (unbounded number of users), users have identically distributed number of friends, and then the BPs can model the content propagation over social networks. This simplifies modelling and analysis. Further we use Multi-Type Branching Processes (MTBPs) (e.g., [13, 12]) to model the influence of TL structure on content propagation in social networks. These processes can mimic most of the phenomenon that influence the content propagation. For example, one can model the effects of multiple posts being forwarded to the same friend, multiple forwards of the same post etc. A post on a higher level in TL has better chances of being read by the user. Influence of the quality of a post, influence of post-title (or content seen without opening the post) can be considered. Posts of similar nature appearing at lower levels on the TL have smaller chances of appreciation etc. To study these influences, one needs to differentiate the TLs that have the 'post' at different levels and this is possible only through multi-type BPs.

The IID friends can be a limitation, in reality friends of two neighbours can be correlated. But this can be inaccurate only for users close to each other, while majority of pair of users are 'faraway' and hence will mostly have independent friends. We aim to improve upon this aspect

[1] See Facts on youtube https://fortunelords.com/youtube-statistics/

in future work. Our approach has several advantages: a) Using the well known results of MTBPs, we obtain closed form expressions for some relevant performance measures. For others we either have approximate closed form expressions or simple fixed point equations, whose solutions provide the required performance. These provide insight for competing over visibility, popularity and influence. b) We study the influence of various network structural parameters on content propagation; c) We study the effects of posts sliding down the TLs due to network activity; d) We have a good approximation of the trajectories of the content propagation in almost sure sense, after sufficiently large time. We also establish that the content propagation has a certain dichotomy, either the post disappears or gets viral (i.e., number of forwards grows exponentially), as is the case for irreducible MTBPs ([14, 13] etc). Finally we consider relevant optimization problems.

We then study viral properties of competing content generated by various content providers (CPs). The resulting MTBPs (with competition) turn out to be decomposable (generator matrix is reducible, e.g., [9, 10]), and there is no dichotomy. Different types of population can have different viral chances, as well as different growth rates when viral. We obtain joint extinction probability of all types belonging to a 'super type', e.g., that of a CP. In [9], the authors show that the extinction probabilities of a super type are the minimal fixed point (FP) solution of a set of equations, obtained after appropriate constraints. While, we provide two sets of FP equations, which need to be solved one after the other and whose unique solution will be the vector of extinction probabilities corresponding to a super type. We also derive FP equations to obtain performance measures specific to one super type. This facilitates to derive interesting performance metrics (e.g., expected number of shares or growth rate of expected number of shares etc) related to *virality of content of a particular CP in a social network, influenced by competing content from other CPs.* We formulate non-cooperative games and analyse the Nash equilibrium (NE) as a function of system parameters. The CPs optimally chose the quality factor of their posts, which greatly influences the content-propagation. The trade-off being the cost incurred for ensuring the required quality.

We prove that the expected value of the total shares (is different from total progeny) corresponding to a single CP grows exponentially fast with time in viral scenarios, with the same rate as that of the unread posts. We conjecture using partial theoretical arguments that the expected shares corresponding to one CP grow exponentially fast with time (if viral), even in the presence of competition (i.e., for decomposable MTBPs). We verify the same numerically. *In fact the virality chances of a content is greatly influenced by the competition, however once viral, the growth rate of expected shares remains the same as that without competition.* This is true for many scenarios, while for other decomposable MTBPs we have non-standard growth patterns for the expected shares (sum of two exponential curves).

We observe that TL structure has significant impact on the analysis and conclusions. The analysis without considering TL structure indicates an inflated virality chances and inflated virality rate. More importantly we observe that *a more active network can deteriorate the performance.* Basically the post shifts down the TL fast and is lost more often. As the mean number of friends increases, the network activity increases, and the virality chances improve initially as anticipated. However we find an optimal mean number of friends, beyond which the virality chances actually deteriorate.

Related work: There is a vast literature that studies the propagation of content over social networks. Many models discretize the time and study content propagation across the discrete time slots (e.g., [4]). As argued in [6] and references therein, a continuous time version (events occurs at continuously distributed random time instances) is a better model ([1, 2] etc) and we consider the same. In majority of the works, that primarily use graph theoretic models, information is spread at maximum to one user at any message forward event (e.g., [4, 6, 5] etc.). However when a user visits a social network (e.g., facebook, twitter etc.), it typically forwards multiple posts and typically (each post) to multiple friends. Authors in ([1, 2] etc) study viral marketing problem, where the marketing message is pushed continuously via emails, banner advertisements, or search engines etc. This scenario allows multiple forwards of the same post, and is analysed using BPs. However they do not consider the influence of other posts using the same medium, and the other effects of TLs. As already discussed these aspects majorly influence the analysis.

Branching processes have been used in analysing many types of networks, such as, polling systems ([7]) which have been used to model local area networks and P2P networks ([8]) etc. We use branching processes not only to study time evolution of content (extinction and viral growth) but also to provide a spatio-temporal description of the timeline process. We model the evolution of the number of timelines that have a given content at a given level of the timeline (e.g. top of the timeline). The analysis of this type of an object can be used to study the influence of content, since the level of a content in a timeline has an influence on its visibility.

II. System description and Background

We consider an online social network with a large number of users, like facebook, twitter etc. We represent each user by a Timeline (TL), which is an inverse stack containing posts. The TLs are assumed to be of finite size N either because they are really of finite size, or because majority of the users do not scroll much (e.g., facebook, twitter etc). When a user visits its TL, it views posts on its TL and shares 'interesting' post(s) to some/all of its friends. Whenever it shares a post, the post appears on the top of the TLs of the friends with whom the content is shared. And every share shifts down the existing content of the TL by one level. It can share as many posts as it wants. The number of shares of a particular post to friends depends upon the extent to which the user likes the post. We assume that the number of friends $\mathbb{F}$ of a typical user is random, and this number is independently and identically distributed (IID) across various users.

In facebook, a user can 'like/share/comment' a post. In case of like/comment operations some/all friends of that user get notified, but the TLs of the friends are not affected.

Facebook also allows to share content to a list of friends, and then the content appears on the TLs of the recipients. In twitter the same is done via the retweet button. In all, *we study those operations, which affect the TLs of the users which in turn have major impact on the propagation of the commercial content.* We assume only the user can share the contents of its TL.

Content Propagation and Branching process

The propagation of content in such a social network is as follows. A user visits[2] its TL and views the posts. If some title attracts the attention, it reads the post and then may share the same to a random number of users (amongst its friends). Now the post, call it post-**P**, would be placed on the top of the TLs of the shared users. And this post shifts below if some other post(s) are also shared with those shared users. For instance when one more post is shared after the post-**P** to the same shared user then the post-**P** resides on the second level of the corresponding TL. When any one of the shared users visits its TL, it follows the same protocol of viewing, reading and sharing the posts. In particular if they read post-**P**, they may share it to their friends. And this continues. Majority of dynamics related to the propagation process can be well captured by a continuous time branching process (CTBP). Below we describe a typical CTBP (e.g., [13]) and also discuss the aspects of the CTBP that well fit the post-propagation process:

- A CTBP starts say at time 0, with initial population $X(0)$. Say we start observing the propagation of post-**P** at time 0, $X(0)$ be the number of unread[3] TLs with post-**P** at time 0.
- Any user among the initial population of the CTBP can 'die' after a time period that is exponentially distributed with parameter say λ. The 'death' times of the particles are independent of the others and hence the first death occurs after a time period that is exponentially distributed with parameter $X(0)\lambda$. Just before the death, the user gives birth to a random non-negative number (say ζ) of new offsprings, which join the existing population. Thus the size of population immediately after the death of the first particle changes to $X(0)-1+\zeta$.

In a similar way the number of unread TLs (NU-TLs), with post-**P**, may change when one of the initial $X(0)$ users visits its TL. The visit times of different users are independent of the others. If we assume memoryless visit times, then the users visit their TLs at intervals that are exponentially distributed as in a CTBP. Further every visit can potentially change the NU-TLs with post-**P**. If post-**P** is read by the user upon visit, it may share it to a random number of friends. If the post is not read by the user it is not shared. Thus any visit results in 0 shares (let $\zeta = 0$ then) when the post is not read and a random number of shares ($\zeta \geq 0$) otherwise. If this sharing process is independent and identical across various users, one can view the overall number of these shares as IID offsprings. And immediately after the first visit by one of the $X(0)$ TLs, the NU-TLs with post-**P** are exactly like that in the CTBP.

[2]We say a user visits its TL, when it views the contents of its TL.
[3]Users not yet visited their TL, hence unaware of the post on their TL.

- The population dynamics of the CTBP continues with new population in exactly the same way as before. The exponential random variable is memoryless and hence the CTBP changes again after exponentially distributed time, but now with parameter $(X(0)-1+\zeta)\lambda$. And again the dying particle adds a random number of offsprings to the existing population which is independent and identically distributed as ζ. And this continues. Similarly, the NU-TLs with post-**P** can change if one of the existing TLs with post-**P** visits its TL.

However the CTBP just described above does not capture some aspects related to the modelling of the post-propagation process. There is a possibility that the post-**P** disappears from some of the TLs, before the visit by the corresponding users. For example, the post-**P** would disappear from a TL with $N-l+1$ or more shares, if initially post-**P** were at level l. In all the propagation of content in a social network is influenced by two factors: a) the evolution of individual TLs (with post–**P**) when some other posts are shared to them (content on the TL shifts down); and b) the sharing dynamics (of post–**P**) between different TLs.

The CTBP described before considers population of single type, in that, all the particles have same death rate and offspring distribution. However the disappearance of post-**P** from a TL depends upon the level at which the post is available. Further we will see that many more aspects of the dynamics depend upon the level at which the post-**P** resides. Thus clearly the single type CTBP is not sufficient and we will actually require a multi-type continuous time branching process (MTBP). An MTBP describes the population dynamics in the scenarios with finite number of population-types. All the particles belonging to one type have same death rate and offspring distribution, however these parameters could be different across different types. To model the rich behaviour of the propagation dynamics we will require (details in later sections) one type to produce offsprings of other types. This modelling feature is readily available with MTBPs. We will show that the propagation dynamics can be well modelled by a MTBP, where for any $l \leq N$, all the TLs with post-**P** in level l form one type of population.

Content providers and their goals

The content providers (CPs) use social network (e.g., Facebook, Twitter etc) to propagate their posts. The purpose could be advertisement of a particular product/service, or could be spread of a particular post that could improve their reputation etc. The CPs would gain if their posts get viral (i.e. their number increase fast with time) by inherent sharing of the post by the users of the social network. We will precisely discuss the event of virality, and obtain the expression for the probability of virality. In the scenarios with positive probability of virality, which we would refer to as viral scenarios, the CPs might be interested in the rate of virality (the rate at which the number of copies of the post increases with time). In the extinction scenarios, i.e., when the post completely disappears from the social network after some time, the CPs might be interested in the expected number of shares before extinction. One might also be interested in a performance measure that captures the number of shares of a post till time t in viral scenarios. We consider all these aspects in this paper. We shall provide explicit expressions for some relevant performance measures as a function of controllable parameters, while some more are represented as the solution of appropriate FP (fixed point) equations. We use these results for system and user optimization.

Assumptions and applications

We study the true indigenous branching part, for example the number of shares generated by the users themselves sharing the post, while influenced by the quality of post. We assume a TL with posts of the CPs under study is not written[4] with the post of the same CP or the competing CP again. We consider a huge social network and the probability that a particular user with one such post is again written with the same post (or a competing post) is very small. Further, one can find applications that satisfy such assumptions. As an example, consider few organizations that plan to advertise their products using coupon system. Also, consider that these coupons can be shared with friends. But a user with one or two such coupons can't be shared with another coupon at a later point of time. In order to avoid multiple shares to the same user, there is a control mechanism. Any user sharing the coupons to its friends, needs to declare the recipients in a list which disables the share of coupons to the same recipients.

The other assumptions are mentioned as and when required. We conclude with another assumption about the model, which is actually not restrictive. We track only the most important information, that of the (top-most) level that first contains the post of the CP(s).

We consider two important scenarios in this paper. We first consider a single CP and propagation of its post. As already explained we will model this with an N-type MTBP and obtain the performance analysis in the next two sections. From section VI onwards, we study the posts of two competing CPs, and the game theoretic aspects. We will use an MTBP with number of types more than $2N$ to model this two CP scenario.

III. Single Content Provider, Modelling details

We consider a single CP and refer its post briefly as the CP-post. The TLs also contain the other posts, and the movement of these posts can also affect the propagation of the CP-post. However our focus would be on CP-post. We say a user is of type l, if its TL contains the CP-post on level l and if the top $l-1$ levels do not contain the CP-post. Let $X_l(t)$ represent the number of unread users of type l at time t. We will show below that the N-vector valued process $\mathbf{X}(t) := \{X_1(t), X_2(t), \cdots, X_N(t)\}$ is an MTBP under suitable conditions.

Modelling aspects and State transitions

To model this process by an appropriate branching process, one needs to specify the 'death' of an existing parent (a TL with 'unread' CP-post in our case) and the distribution of its offsprings. A user of type l is said to 'die' either when its TL is written by another user or when the user itself wakes-up (visits its TL) and shares the post with some of its friends. In the former event exactly one user of type $(l+1)$ (if $l<N$) is 'born' while the latter event gives birth to a random number of offsprings of types 1 or 2 or $\cdots N$. If $i-1$ *(with $i \le N$) posts are shared with the same user, after the CP-post, then the CP-post is available on the i-th level and we will have an i-type offspring.* We assume that a user produces offsprings of type i with probability ρ_i and that $\rho_1 > 0$. Note that $\sum_i \rho_i = 1$. Some times, users have lethargy to view/read all the posts. We represent this via a level based view probability, r_l, which represents the probability that a typical user views the post on level l. It is reasonable to assume $r_1 \ge r_2 \cdots \ge r_N$. We thus have two types of transitions that modify the MTBP, which we refer to as *shift and share transitions respectively.*

In a CTBP, any one of the existing particles 'dies' after exponentially distributed time while in a discrete time version all the particles of a generation 'die' together. The continuous version mimics a social network scenario better and hence a CTBP can model more accurately than the discrete time version. In fact when the number of copies of CP-post grows fast (i.e., when the post is viral), the time period between two subsequent changes decreases rapidly as time progresses. This is also well captured by CTBP as will be seen below.

Let $\mathcal{G}_1$ represent the subset of users with CP-post at some level, while $\mathcal{G}_2$ contains the other users. We assume the social network (and hence $\mathcal{G}_2$) has infinitely many users and note $\mathcal{G}_1$ at time t has,

$$X(t) := \sum_{l \le N} X_l(t), \tag{1}$$

number of users. Group $\mathcal{G}_2$ has infinite number of agents and this remains the same irrespective of the size of $\mathcal{G}_1$, which is finite at any finite time. Thus the transitions between $\mathcal{G}_2$ and $\mathcal{G}_1$ are more significant and one can neglect the transitions within $\mathcal{G}_1$. It is obvious that we are not interested in transitions within $\mathcal{G}_2$ (users without CP-post). We thus model the action of these groups in the following consolidated manner:

- any user from $\mathcal{G}_1$ wakes up after $exp(\nu)$ (exponentially distributed with parameter ν) time to visit its TL and writes to a random (IID) number of users (predominately) of $\mathcal{G}_2$;
- the TL of any user of $\mathcal{G}_1$ is written by one of the users of $\mathcal{G}_2$, and the time intervals between two successive writes are exponentially distributed with parameter λ.

The state of the network, $\mathbf{X}(t)$, changes when the first of the above mentioned events occurs. At time t we have $X(t)$ (see equation (1)) number of users in $\mathcal{G}_1$ and thus (first) one of them wakes up according to exponential distribution with parameter $X(t)\nu$. Similarly, the first TL/user of group $\mathcal{G}_1$ is written with a post after exponential time with parameter $X(t)\lambda$. Thus the state $\mathbf{X}(t)$, changes after exponential time with parameter $X(t)\lambda + X(t)\nu$. Thus with viral posts, $X(t)$ grows rapidly and hence the rate of transitions increase as time progresses. Considering all the modelling aspects mentioned so far, the IID offsprings generated by one l-type user are summarized as below (w.p. implies with probability):

$$\xi_l = \begin{cases} \mathbf{e}_{l+1}\mathbb{1}_{l<N} & \text{w.p. } \theta := \frac{\lambda}{\lambda+\nu} \text{ and} \\ \zeta \mathbf{e}_i & \text{w.p. } (1-\theta) r_l \rho_i \quad \forall i \le N \\ 0 & \text{w.p. } (1-\theta)(1-r_l). \end{cases} \tag{2}$$

where $\mathbf{e}_l$ represents standard unit vector of size N with one in the l-th position, $\mathbb{1}_A$ represents the indicator, ζ is

[4] We say a TL is written when a friend of it shares a post which changes its content.

the random number of friends to whom the post is shared and r_l is the probability the user views a post on level l. Recall that users (offsprings) of type i are produced with probability ρ_i during the share transitions. From (2) the offspring distribution is independent of time t (i.e., independent of the time the users wake-up), ζ can be assumed independent across users, and hence ξ_l are IID offsprings from any type l user. Further all the transitions occur after memoryless exponential times, and hence $\mathbf{X}(t)$ is an MTBP with N- types (e.g. [13]).

PGFs and post quality factor: Let $f_F(s,\beta)$ be the probability generating function (PGF) of the number of friends, $\mathbb{F}$, of a typical user, parametrized by β. For example, $f_F(s,\beta) = \exp(\beta(s-1))$ stands for Poisson distributed $\mathbb{F}$, $f_F(s,\beta) = (1-\beta)/(1-\beta s)$ stands for geometric $\mathbb{F}$. Let $m = f_F'(1,\beta)$ represent the corresponding mean. When a user reads a post it shares the same to some or all of its friends (ζ of equation (2)), based on its quality. The better the quality, the more the number of shares. Let η represent the quality of the CP-post. *We assume that the mean of the number of shares is proportional to this quality factor.* In other words, $m(\eta) = m\eta$ represents the post quality dependent mean of the random shares. Let $f(s,\eta,\beta)$ represent the PGF of ζ. For example, for Poisson friends the PGF and the expected value of ζ are given respectively by:

$$f(s,\eta,\beta) = f_F(s,\eta\beta) = \exp(\beta\eta(s-1)) \text{ for any } s \text{ and } m(\eta) = \eta\beta.$$

For Geometric friends, one may assume the post quality dependent parameter $\beta_\eta = (1-\beta)/(1-\beta+\beta\eta)$ which ensures $m(\eta) = \eta\beta$. And then the PGF of ζ is given by $f(s,\eta,\beta) = f_F(s,\beta_\eta) = (1-\beta_\eta)/(1-\beta_\eta s)$. One can derive such PGFs for other distributions of $\mathbb{F}$. Further, most of the analysis does not depend upon the distribution of $\mathbb{F}$.

Let $\mathbf{s} := (s_1,\cdots,s_N)$ and $\bar{\mathbf{f}}(\mathbf{s},\eta) := \sum_{i=1}^N f(s_i,\eta,\beta)\rho_i$. The post quality factor dependent PGF, of the offspring distribution of the overall Branching process, is given by (see equation (2)):

$$\boxed{h_l(\mathbf{s}) = \theta\left(s_{l+1}\mathbb{1}_{l<N} + \mathbb{1}_{l=N}\right) + (1-\theta)r_l\bar{\mathbf{f}}(\mathbf{s},\eta) + (1-\theta)(1-r_l)}. \quad (3)$$

A. Generator Matrix

The key ingredient required for analysis of any MTBP is its generator matrix. We begin with the generator for MTBP that represents the evolution of unread TLs with CP-post. We refer to this process briefly as TL-CTBP, the Timeline Continuous Time Branching Process. The generator matrix, A, is given by $A = (a_{lk})_{N\times N}$, where $a_{lk} = a_l\left(\partial h_l(\mathbf{s})/\partial s_k\Big|_{\mathbf{s}=\mathbf{1}} - \mathbb{1}_{\{l=k\}}\right)$ and a_l represents the transition rate of a type-l particle (see [13] for details). For our case, from previous discussions $a_l = \lambda+\nu$ for all l. Further from (3), the matrix A for our single CP case is given by (with $c := (1-\theta)m\eta,\ c_l = c\rho_l$)

$$A = (\lambda+\nu)\begin{bmatrix} c_1r_1-1 & c_2r_1+\theta & \cdots & c_{N-1}r_1 & c_Nr_1 \\ c_1r_2 & c_2r_2-1 & \cdots & c_{N-1}r_2 & c_Nr_2 \\ & & \vdots & & \\ c_1r_{N-1} & c_2r_{N-1} & \cdots & c_{N-1}r_{N-1}-1 & c_Nr_{N-1}+\theta \\ c_1r_N & c_2r_N & \cdots & c_{N-1}r_N & c_Nr_N-1 \end{bmatrix}. (4)$$

The largest eigenvalue and the corresponding eigenvectors of the above generator matrix are instrumental in obtaining analysis of TL-CTBP ([13]) and the following lemma establishes important properties about the same. We also prove that the resulting TL-CTBP is positive regular[5], which is an important property that establishes the simultaneous survival/extinction of all the types of TLs.

Lemma 1. *i) When $0<\theta<1$, the matrix e^{At} for any $t>0$, is positive regular.*
ii)Let α be the maximal real eigenvalue, of the generator matrix A. Then
$\alpha \in \Big(\boldsymbol{r.c}-1,\ \boldsymbol{r.c}-1+\theta\Big)(\lambda+\nu)$ where inner product $\boldsymbol{r.c} := \sum_{i=1}^N r_ic_i$. When the view probabilities have special form $r_l = d_1d_2^l$ (for some $0\le d_1,d_2\le 1$), then

$$\alpha \to (\boldsymbol{r.c}-1+\theta d_2)(\lambda+\nu) \ \ as\ N\to\infty.$$

iii) The left and right eigenvectors $\mathbf{u}$, $\mathbf{v}$ corresponding to α satisfy the following equations (with $\sigma := \alpha/(\lambda+\nu)+1$)

$$\begin{aligned} c_1\boldsymbol{r.u} &= \sigma u_1,\ c_1\boldsymbol{r.v} = \sigma v_N; \quad \text{and} \quad (5)\\ u_l &= \sum_{i=0}^{l-1}\frac{\rho_{l-i}}{\rho_1}\left(\frac{\theta}{\sigma}\right)^i u_1,\ v_l = \sum_{i=0}^{N-l}\left(\frac{\theta}{\sigma}\right)^i\frac{r_{l+i}}{r_N}v_N \quad \forall l\ge 2. \end{aligned}$$

Proof: The proof is provided in Appendix A. ■

IV. Performance analysis for single CP content

If the CP invests sufficiently in advertisement, and ensures a good quality post, the post can get viral. The CP would be interested in many related performance measures, as a function of post-quality factor: a) the exact probability of extinction; b) the rate of explosion or the rate of virality; and c) a measure of total number of shares. We begin with the probability of extinction.

A. Extinction Probabilities-Virality chances

The CP-post is said to be extinct, when it disappears completely from the social network. None of the N-length TLs contain the CP-post eventually (as time progresses towards infinity).

With positive regularity given by Lemma 1.(i), either all the types belonging to TL-CTBP survive or all die together (e.g., [14]). Let q_l be the probability with which the process gets extinct, when TL-CTBP starts with one TL of type l,

$$q_l := P\Big(\mathbf{X}(t) = \mathbf{0} \text{ for some } t>0\Big|\mathbf{X}(0) = \mathbf{e}_l\Big).$$

Let $\mathbf{q} := \{q_1,q_2,\cdots,q_N\}$ represent the vector of extinction probabilities.

Under positive regularity conditions of Lemma 1.(i) when a BP is not extinct, the population grows exponentially fast to infinity ([14, 13] etc). This fact is established for our TL-CTBP in Theorem 1, provided in the later subsections. *Thus we have dichotomy: the post gets viral with exponential rate, when it is not extinct.* And hence the extinction probability actually equals one minus the probability of virality.

Depending upon the context of the problem, for instance an awareness campaign, the CP may be interested in knowing the chances of dissemination of its information to a large population, i.e. virality of its post. Thus, it is important to know the extinction probability in the context of social networks, whose properties are provided below.

Lemma 2. *Assume $0<\theta<1$ and $E[\mathbb{F}log\mathbb{F}]<\infty$ with $\mathbb{F}\log(\mathbb{F}) := 0$ when $\mathbb{F}=0$. Then clearly $E[\zeta log\zeta]<\infty$ for any*

[5] A matrix B is called positive regular (irreducible) if there exists an n such that the matrix B^n has all strict positive entries. A BP is positive regular when its mean matrix is positive regular. With A as generator, the positive regularity is guaranteed if e^A is positive regular (e.g. [13]).

post quality factor η. Hence we have the following:
(i) If $\alpha \le 0$, extinction occurs w.p.1, i.e., $\boldsymbol{q} = \mathbf{1} = (1, \cdots, 1)$;
(ii) If $\alpha > 0$, then[6] $\boldsymbol{q} < \mathbf{1}$. In this case, the extinction probability vector $\mathbf{q}$ is the unique solution of the equation, $h(\boldsymbol{s}) = \boldsymbol{s}$, in the interior of $[0,1]^N$.

Proof follows from [13, Theorems 1-2] and by Lemma 1. ■

It is easy to verify that the hypothesis of this lemma are easily satisfied by many distributions. For example, Poisson, Geometric etc satisfy $E[\mathbb{F} log \mathbb{F}] < \infty$.

Virality Threshold: By Lemma 2.(ii) the CP-post *gets viral, i.e., the TL-CTBP survives with non-zero probability*, when $\alpha > 0$. We hence call α as virality threshold. When N is sufficiently large, by Lemma 1.(ii) and Lemma 2.(ii),

$$\alpha \approx ((m\eta(1-\theta)\boldsymbol{\rho}.\boldsymbol{r} - 1) + \theta d_2)(\lambda + \nu) = (m\eta\boldsymbol{\rho}.\boldsymbol{r} - 1)\nu - (1 - d_2)\lambda. \quad (6)$$

It is well known that the BPs survive with positive probability, if the expected number of offsprings is greater than 1. We have an (almost) equivalent of the same, i.e., process can survive when $m\eta(1-\theta)\boldsymbol{\rho}.\boldsymbol{r} > 1$ (see term 1 of (6)), for a BP pitted against the shifting process, the TL-CTBP. Hence the virality chances are influenced by post quality η, shift factor (1-θ), by the types of posts produced as given by $\boldsymbol{\rho}$ and the view probabilities $\mathbf{r}$. In effect the virality chances are influenced by factor, $(1-\theta)\eta\boldsymbol{\rho}.\boldsymbol{r}$.

No-TL case: Majority of works (e.g., [2, 1]) consider study of content propagation without considering TL structure and as mentioned this is an incomplete study. We would like to compare our conclusions with the case when the effects of TLs were neglected. We say it is No-TL case when $N \to \infty$ (i.e., post is never lost) and more importantly when $r_l = 1$ for all l (i.e., post at any level on TL is viewed with same interest). For this case ($d_2 = 1$ in (6)),

$$\alpha_{No-TL} \approx (m\eta - 1)\nu \text{ for any } \boldsymbol{\rho},$$

the post gets viral when $m\eta > 1$. Thus neglecting the effects of TL, one might conclude that the virality chances are influenced only by η (the post quality factor), while in reality the influence is summarized by factor $(1-\theta)\eta\boldsymbol{\rho}.\boldsymbol{r}$. In other words $(1-\theta)\boldsymbol{\rho}.\boldsymbol{r}$ captures the influence of TLs while η is due to post-quality.

By Lemma 2.(ii), the extinction probabilities are obtained by solving $h(\mathbf{s}) = \mathbf{s}$, i.e., by solving (see (3) and for any l):

$$q_l = \theta\Big(q_{l+1}\mathbb{1}_{\{l<N\}} + \mathbb{1}_{\{l=N\}}\Big) + (1-\theta)r_l\,\bar{\mathbf{f}}(\mathbf{q},\eta) + (1-\theta)(1-r_l), \quad (7)$$

The above simplifies to:

$$q_{N-l} = (q_N - 1)\sum_{i=0}^{l}\theta^{l-i}\frac{r_{N-i}}{r_N} + 1 \text{ for any } 1 \le l < N. \quad (8)$$

Solving (7) for No-TL case, $q_l = \theta^{N-l+1} + \bar{\mathbf{f}}(\mathbf{q},\eta)(1-\theta^{N-l+1})$ and then with $N \to \infty$, $q_l \approx q_j$ for any l, j. For example, the extinction probability is almost the same either you start with one CP-post on level 1 or on level 9. This is again a wrong interpretation and the solutions of (8)/(7) provide the correct extinction probabilities which considers the influence of TLs.

Influence of mean number of friends $m = E[\mathbb{F}]$**:** When the mean number m increases, network becomes more active as the sharing of different posts becomes more pronounced. The TLs are flooded with different posts rapidly, so do the TLs containing post-**P**, and one might

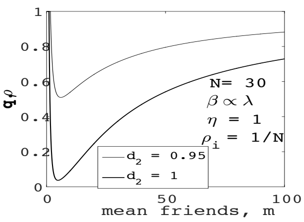


Fig. 1: Extinction vs $m = E[\mathbb{F}]$

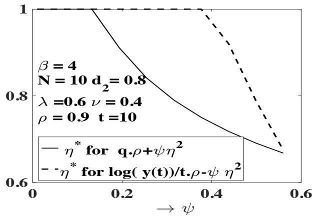


Fig. 2: Optimizers

anticipate an increase in its virality chances. These TLs also receive the other posts rapidly, resulting in rapid shifts to their content. Thus with increase in m, the λ increases, and so does θ. We observe an interesting phenomenon in Figure 1, with respect to the virality chances $q_\rho := \sum_l q_l \rho_l$, when λ is set proportional to mean m. To begin with the virality chances q_ρ improve (decrease) with mean m, as anticipated. However if one increases m further, we notice an increase in q_ρ. Basically increased m implies more shares of post-**P** to new users, but it also implies post-**P** is missed more often. Thus this phenomenon is mainly observed because of time-line structure: if there were infinite TL levels and if any user views all the posts with equal interest, one would not have noticed this. There seems to be *an optimal number of mean friends, which is best suited for post propagation.*

B. Exponential growth and its rate in viral paths

Under the assumptions of Lemma 2, the TL-CTBP satisfies the hypothesis of [13, Theorem 1] and hence we have:

Theorem 1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an appropriate probability space and let $\{\mathscr{F}_t\}$ be the natural filtration for TL-CTBP $\boldsymbol{X}(.)$, i.e., for each t, $\mathscr{F}_t$ is the σ-algebra generated by $\{\boldsymbol{X}(t');\ t' \le t\}$. The process $\{\boldsymbol{u}\cdot\boldsymbol{X}(t)e^{-\alpha t};\ t \ge 0\}$, with $\mathbf{u}, \alpha$ as in Lemma 1, is a non negative martingale (with natural filtration)* and

$$\lim_{t\to\infty} \boldsymbol{X}(t,\omega)e^{-\alpha t} = W(\omega)\boldsymbol{u} \quad \text{for almost all } \omega, \quad (9)$$

where W is a non negative random variable that satisfies[7]: $P_l(W = 0) = q_l$, $E_l[W] = v_l$ for each l, if $\mathbf{u}'.\mathbf{v} = 1$. ■

The CP-post gets extinct for the sample paths with $W = 0$, and it gets viral in the complementary paths (when $W > 0$) ([13]). For viral paths we have: a) **growth rate:** all the components of TL-CTBP grow with time, at the same rate which is given by e^α; and b) **the fraction of unread TLs with post at level** l**:** from (9) eventually (after long time) equals $u_l/(\sum_i u_i)$. We observe here that the growth rate exactly equals the virality threshold, and again (6) more accurately describes the growth rate, and not α_{No-TL}.

C. Viral scenarios: Number of shares till time t

In a social network, it is important to know the spread of post i.e. total number of shares a post gets in a given time-frame (e.g. number of shares in Facebook). Let $Y(t)$ be the accumulated number of shares till time t and let $Y = \lim_{t\to\infty} Y(t)$ (can also be infinity) be the eventual number of shares. It is important to observe here that this number is different from total progeny of the underlying BP, as the former is due to 'share'-offsprings while the latter is due to both 'share' as well as 'shift' offsprings. Also we consider the total shares, irrespective of the level at which the shares are produced. We have the following result (proof in Appendix):

[6]Vector $\mathbf{q} < \mathbf{s}$ if $q_i < s_i$ for all components i.

[7]*We use E_l and P_l to represent the conditional expectation and probability respectively when TL-CTBP starts with one l-type TL.*

Lemma 3. *Let* $\mathbf{y}(t) := [y_1(t)\cdots,y_N(t)]$ *with* $y_l(t) := E_l[Y(t)] = E[Y(t)|\mathbf{X}(0) = \mathbf{e}_l]$, *the expected number of shares till time t when started with one l-type TL, for each l. If* $\alpha > 0$, *we have*

$$\mathbf{y}(t) \;=\; e^{At}\Big(\boldsymbol{1} + (\lambda+\nu)A^{-1}\boldsymbol{k}\Big) - (\lambda+\nu)A^{-1}\boldsymbol{k} \qquad (10)$$

$$\text{where } \boldsymbol{k} = \;[1-\theta, 1-\theta, \cdots, 1-\theta, 1]^T. \qquad \square$$

Proof. Proof is in Appendix A. ■

From [13, Equation(45)] (10) can be approximated (large t):

$$\mathbf{y}(t) \approx e^{\alpha t}\mathbf{v}\mathbf{u}'\Big(\mathbf{1} + (\lambda+\nu)A^{-1}\mathbf{k}\Big) - (\lambda+\nu)A^{-1}\mathbf{k}, \qquad (11)$$

where $\mathbf{v}$, $\mathbf{u}$ are eigen vectors of Lemma 1 such that $\mathbf{v}'.\mathbf{u} = 1$. Thus *the expected number of shares grow exponentially* with time, for viral scenarios. Further the growth rate α is the same as that for the unread posts and equals the virality threshold of (6). From (11), for large t, the expected shares when started with one l-type particle:

$$y_l(t) \approx e_{l,0}e^{\alpha t} \text{ with } e_{l,0} = v_l\sum_{i=1}^N u_i\left(1+\frac{\nu}{\alpha}\right) + v_l\frac{\lambda}{\alpha}u_N. \qquad (12)$$

In Figure 3, we compared the theoretical trajectories with simulation based trajectories, and they match very well.

D. Non-viral scenarios: Expected number of shares

The population gets extinct w.p.1 and then the expected number of total shares is finite. One can directly obtain the expected number of shares, by conditioning on the first transition event:

$$y_l := E_l[Y] = \theta y_{l+1}\mathbb{1}_{\{l<N\}} + (1-\theta)r_l\left(m\eta + m\eta\mathbf{y}.\boldsymbol{\rho}\right), \text{for all } l \le N. \quad (13)$$

On simplifying the above system of equations backward recursively, we obtain the following for any $l \le N$

$$y_l = (1-\theta)m\eta(1+\mathbf{y}.\boldsymbol{\rho})\sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i} \qquad (14)$$

Summing the above over l after multiplying with ρ_l we obtain:

$$\mathbf{y}.\boldsymbol{\rho} = \sum_{l=1}^N \rho_l y_l = (1-\theta)m\eta(1+\mathbf{y}.\boldsymbol{\rho})\sum_l \rho_l \sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i}.$$

Thus the FP equation for $\mathbf{y}.\boldsymbol{\rho}$ is linear and hence we have a unique FP solution for $\mathbf{y}.\boldsymbol{\rho}$ whenever $(1-\theta)m\eta\sum_l\rho_l\sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i} < 1$. If

$$(1-\theta)m\eta\mathbf{r}.\boldsymbol{\rho} - 1 + \theta = \mathbf{r}.\mathbf{c} - 1 + \theta < 0,$$

from Lemma 1.(ii) $\alpha < 0$ and the process would be extinct w.p.1. In this scenario:

$$\begin{aligned}
(1-\theta)m\eta\sum_l \rho_l \sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i} \\
&\le (1-\theta)m\eta\sum_l \rho_l r_l \sum_{i=0}^{N-l}\theta^{N-l-i} \\
&= (1-\theta)m\eta\sum_l \rho_l r_l \frac{1-\theta^{N-l-1}}{1-\theta} \\
&= m\eta\mathbf{r}.\boldsymbol{\rho} < 1
\end{aligned}$$

because $r_1 \ge r_2 \cdots \ge r_N$. We can similarly show, using the limit of the eigenvalue α of Lemma 1, that when the process is extinct w.p.1 the above condition is always satisfied asymptotically. To be more precise the condition is satisfied for all N bigger than a threshold $\tilde{N}$, whenever the process is extinct w.p.1.

We thus have the following unique FP for $\mathbf{y}.\boldsymbol{\rho}$, under the conditions discussed above:

$$\mathbf{y}.\boldsymbol{\rho} = \frac{(1-\theta)m\eta\sum_l\rho_l\sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i}}{1-(1-\theta)m\eta\sum_l\rho_l\sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i}}, \qquad (15)$$

One can substitute the above in equation (14) to obtain y_l for all l and it is easy to verify that the FP is unique by uniqueness of the FP solutions for $\mathbf{y}.\boldsymbol{\rho}$.

Special case: Say $r_i = d_1 d_2^i$, $\rho_i = \tilde{\rho}\rho^i$ (with $\sum_i \rho_i = 1$) for all i, one can easily simplify the above. We have the following

$$\begin{aligned}
\sum_l \rho_l \sum_{i=0}^{N-l}\theta^{N-l-i}r_{N-i} &= d_1\tilde{\rho}\sum_l \rho^l \sum_{i=0}^{N-l}\theta^{N-l-i}d_2^{N-i} \\
&= d_1\tilde{\rho}\sum_l \rho^l d_2^l \sum_{i=0}^{N-l}\theta^{N-l-i}d_2^{N-l-i} \\
&= d_1\tilde{\rho}\sum_l \rho^l d_2^l \sum_{i=0}^{N-l}\theta^i d_2^i \\
&= d_1\tilde{\rho}\sum_l \rho^l d_2^l \frac{(1-(\theta d_2)^{N-l+1})}{1-\theta d_2} \\
&= \frac{d_1\tilde{\rho}}{1-\theta d_2}\left(\rho d_2\sum_{l=0}^{N-1}\rho^l d_2^l - d_2(\theta d_2)^N\rho\sum_{l=0}^{N-1}\rho^l\theta^{-l}\right) \\
&= \frac{d_1\tilde{\rho}}{1-\theta d_2}\left(\rho d_2\frac{1-(\rho d_2)^N}{1-\rho d_2} - (d_2)^{N+1}\rho\frac{\theta^N-\rho^N}{\theta(\theta-\rho)}\right).
\end{aligned}$$

Substituting this into (15) and under the limit $N \to \infty$, we obtain the following compact expression (where $\tilde{\rho} = (1-\rho)/\rho$ is now the limit):

$$\mathbf{y}.\boldsymbol{\rho} \approx \frac{O_{mean}}{1-O_{mean}} \text{ with } O_{mean} := (1-\theta)m\eta\frac{(1-\rho)d_1d_2}{(1-\theta d_2)(1-\rho d_2)}. \quad (16)$$

And then (with $r_l = 1$ for all l) $O_{mean,No-TL} = m\eta$.

V. Optimization

So far we have obtained the probability of extinction $\{q_l\}$, as well as the coefficient of explosion α in the event of virality. The first measure captures the probability of the virality event, while the second one provides the rate of explosion during virality. We also obtained the expected number of shares till time t.

All the performance measures improve with post quality factor η. However, there is a cost for improving the post quality which would be proportional to η. We now discuss the optimal post quality factor by optimizing, $\mathscr{C}(\eta) = pm \pm \psi\eta^2$, where pm represents one of the performance measures, ψ is the trade-off factor for cost of post quality.

A CP may be interested in optimizing the virality chances of its post. In such scenarios the CP wants to optimize the survival chance. So, we minimize $\mathbf{q}.\boldsymbol{\rho} + \psi\eta^2$ or $q_1 + \psi\eta^2$. A CP might have a goal of reaching out to at least B users, it might be interested in optimizing the probability of the total number of shares raising above B, $P(Y > B)$. It is easy to verify that $P(Y > B) \ge (1-q)$, the probability of virality, and that $P(Y > B) \to (1-q)$, as $B \to \infty$. Thus optimizing the *probability of reaching a goal* is approximately the same as optimizing the survival chances.

The CP might be interested in optimizing, the growth rate of the expected shares, by maximizing $\log(y_1(t))/t - \psi\eta^2$. In Figure 2, we compare these optimizers with optimizers of extinction probability. *From the figure, when importance is for growth rate, rather than the virality chances, one needs better quality posts.* When $m < 1$, one can also maximize $y_1 - \psi\eta^2$.

VI. Branching with Competing Content

We now consider the scenario with two competing content providers. The competing CPs are operating in similar kind of business. So they have similar kind of posts, for example posts related to advertising products by e-commerce organisations when CPs operate in e-commerce business. We track the posts of both the CPs and as before, the process is influenced by dynamics of other posts. In general different *CPs have different reputations, one CP can be more influential than the other.* The users can respond more actively to the posts of a more influential CP. Let w_j (≥ 1) be the influence factor of CP-j with $j = 1$ or 2, and we assume that the post quality factor of CP-j is given by $w_j\eta_j$. Thus the CP with high influence factor can obtain good results with lower post quality. To *simplify the notation we use η_j to represent $w_j\eta_j$* and with the new notation, $\eta_j \in [0, 1/w_j]$. As before number of friends, $\mathbb{F}$, is parametrized by β with $E[\mathbb{F}] = m$. The expected number of random shares ζ^j of CP-j equals $m\eta_j$ and with best acceleration, $E[\zeta^j] = m/w_j$.

A. Modelling Details

This can again be modelled by an MTBP, most of the modelling details are same as before and we discuss the differences. We have additional types of TLs and the group $\mathcal{G}_1$ is further divided into three sub groups, as below:

Mixed types are the TLs having the posts of both the CPs. Let $X_{l,k}(t)$ denote the number of users with CP1-post on the l-th level and CP2-post on the k-th level of their TLs at time t. Such TLs are referred as (l,k) type TLs. In this paper we consider the analysis with initial TLs having the posts of both the CPs on the top levels i.e. we begin with either (1,2) or (2,1) type TLs. It is not difficult to start with other types of TLs, but the expressions become complicated and we would like to explain the results in a simpler manner. With a shift transition, a (1,2) type (a (2,1) type) gets converted to a (2,3) type ((3,2) type resp.), which further gets converted to (3,4) ((4,3) resp.) type with another shift and so on. We thus have $2(N-1)$ mixed types, which at time t are given by, $\mathbf{X}_{mx}(t) = (\mathbf{X}_{mx1}(t), \mathbf{X}_{mx2}(t))$ with

$$\mathbf{X}_{mx1}(t) \;:=\; \{X_{1,2}(t), X_{2,3}(t), \cdots, X_{N-1,N}(t)\},$$
$$\mathbf{X}_{mx2}(t) \;:=\; \{X_{2,1}(t), X_{3,2}(t), \cdots, X_{N,N-1}(t)\}.$$

Exclusive CP types comprises all the users with post of only one CP. Let $\mathbf{X}^1_{ex}(t) := \{X_{1,0}(t), \cdots, X_{N,0}(t)\}$ and $\mathbf{X}^2_{ex}(t) := \{X_{0,1}(t), \cdots, X_{0,N}(t)\}$ represent the numbers of exclusive CP1 and CP2 type TLs at time t. The group $\mathcal{G}_2$, as before, has all the other TLs without the post of either CP.

Transitions Recall that a 'shift' transition occurs when a user of $\mathcal{G}_2$ writes to the top level of a user/TL of $\mathcal{G}_1$. Exclusive CP TLs are changed in same way as in single CP case. While for mixed types, the position of each post slides down by one level. For example an (l,k) type TL (with either $l = k+1$ or $k-1$) gets converted to $(l+1, k+1)$ type when $l, k < N$, while $(N-1, N)$ and $(N, N-1)$ type TLs get converted to exclusive CP types $(N, 0)$ and $(0, N)$ respectively.

Share transition protocol of exclusive CP types is same as in single CP case. While a mixed type TL, say $(l, l+1)$, undergoes the following changes, when subjected to 'share' transition

i) The user first views the CP1-post w.p. r_l and shares the same to some of its friends, as before;

ii) The CP2-post is below CP1-post and recall the posts are of similar nature. *The motivation of the users to read the second post of similar nature would be lesser.* We assume that the user views the second post w.p. δ;

iii) When user views/reads both the posts, it can share CP1-post alone to some of its friends, CP2-post alone to some others and both the posts to some more. Other wise, only CP1-post is shared. TLs of exclusive types are produced when it shares only CP1-post or CP2-post. While mixed type TLs are produced when it shares both the posts; and

iv) When only one CP-post is shared, e.g., CP1-post, it can produce type $(i, 0)$ w.p. $\bar{\rho}_i$ where $1 \leq i < N$, with $\sum_{i=1}^{N-1} \bar{\rho}_i = 1$. It can't produce $(N, 0)$ type as the user has already discarded one post, that of CP2. Recall a TL of type i is produced when $(i-1)$ more posts are shared to it after the CP-post. When both the posts are shared to the same friend, mixed types $(i+1, i)$, $(i, i+1)$ (with $i < N$) are produced respectively w.p. $p\bar{\rho}_i$ and $(1-p)\bar{\rho}_i$. With high probability CP1-post is shared first followed by sharing of CP2-post, as we start with $(l, l+1)$ parent. Hence the order of the posts in the recipient TLs would be reversed with high probability and hence p would in general be larger than $(1-p)$.

We have similar transitions with $(l+1, l)$ type TLs.

B. PGF and the generator Matrix

The PGF for the two-CP case can be obtained using the above modelling details. The random number of friends, *to whom both the posts are shared, is parametrized by $\eta_1\eta_2$, while the number of shares of exclusive CP1-post or CP2-post is parametrized by $\eta_1(1-\eta_2)$ or $\eta_2(1-\eta_1)$ respectively.* For example if the number of friends $\mathbb{F}$ is Poisson with parameter β, then random (sampled) number with whom both the posts are shared would be Poisson with parameter $\beta\eta_1\eta_2$. With $\mathbf{s} := \{\mathbf{s}^1_{ex}, \mathbf{s}^2_{ex}, \mathbf{s}_{mx1}, \mathbf{s}_{mx2}\}$, $\mathbf{s}_{mx1} = \{s_{l,l+1}\}$, $\mathbf{s}_{mx2} = \{s_{l+1,l}\}$ and $\bar{\mathbf{g}}(\mathbf{s}', \eta) := \sum_{i=1}^{N-1} f\left(s'_i, \eta, \beta\right)\bar{\rho}_i$, the PGF for $(l, l+1)$ and $(l+1, l)$ type equals (see (3)),

$$\begin{aligned}
h_{l,l+1}(\mathbf{s}) \;=\; & \theta\left(s_{l+1,l+2}\mathbb{1}_{l<N-1} + s_{N,0}\mathbb{1}_{l=N-1}\right) + (1-\theta)(1-r_l) \\
& +(1-\theta)r_l\Big[(1-\delta)\bar{\mathbf{g}}\left(\mathbf{s}^1_{ex}, \eta_1\right) + \delta\Big(\left((1-p)\bar{\mathbf{g}}\left(\mathbf{s}_{mx1}, \eta_1\eta_2\right) + \; p\bar{\mathbf{g}}\left(\mathbf{s}_{mx2}, \eta_1\eta_2\right)\right) \\
& \qquad \bar{\mathbf{g}}\left(\mathbf{s}^1_{ex}, \eta_1(1-\eta_2)\right)\bar{\mathbf{g}}\left(\mathbf{s}^2_{ex}, \eta_2(1-\eta_1)\right)\Big)\Big], \\
h_{l+1,l}(\mathbf{s}) \;=\; & \theta\left(s_{l+2,l+1}\mathbb{1}_{l<N-1} + s_{0,N}\mathbb{1}_{l=N-1}\right) + (1-\theta)(1-r_l) \\
& +(1-\theta)r_l\Big[(1-\delta)\bar{\mathbf{g}}\left(\mathbf{s}^2_{ex}, \eta_2\right) + \delta\Big(\left(p\bar{\mathbf{g}}\left(\mathbf{s}_{mx1}, \eta_1\eta_2\right) + \; (1-p)\bar{\mathbf{g}}\left(\mathbf{s}_{mx2}, \eta_1\eta_2\right)\right) \\
& \qquad \bar{\mathbf{g}}\left(\mathbf{s}^1_{ex}, \eta_1(1-\eta_2)\right)\bar{\mathbf{g}}\left(\mathbf{s}^2_{ex}, \eta_2(1-\eta_1)\right)\Big)\Big].
\end{aligned}$$

The pgf for exclusive types is as in single CP case, e.g., $h_{l,0}(\mathbf{s}) = h_l(\mathbf{s}^1_{ex})$ of (3). The generator matrix $\mathbb{A}$ has the following block structure,

$$\mathbb{A} = \begin{bmatrix} A_{mx} & A^1_{mx,ex} & A^2_{mx,ex} \\ \mathbf{0} & A^1_{ex} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^2_{ex} \end{bmatrix}, \tag{17}$$

where: a) matrices A_{ex}^j for $j = 1,2$ are same as in (25), with constant c replaced by $c_j := (1-\theta)m\eta_j$ and represent the transitions within exclusive CP types; b) matrix A_{mx} corresponds to transitions within the mixed types and is given by the following when types are arranged in the order $(1,2),(2,1),(2,3),(3,2),\cdots,(N-1,N),(N,N-1)$,

$$A_{mx} = \begin{bmatrix} z'_1 r_1 - 1 & z_1 r_1 & \theta + z'_2 r_1 & \cdots & z'_{N-1} r_1 & z_{N-1} r_1 \\ z_1 r_1 & z'_1 r_1 - 1 & z_2 r_1 & \cdots & z_{N-1} r_1 & z'_{N-1} r_1 \\ z'_1 r_2 & z_1 r_2 & z'_2 r_2 - 1 & \cdots & z'_{N-1} r_2 & z_{N-1} r_2 \\ z_1 r_2 & z'_1 r_2 & z_2 r_2 & \cdots & z_{N-1} r_2 & z'_{N-1} r_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z'_1 r_{N-2} & z_1 r_{N-2} & z'_2 r_{N-2} & \cdots & \theta + z'_{N-1} r_{N-2} & z_{N-1} r_{N-2} \\ z_1 r_{N-2} & z'_1 r_{N-2} & z_2 r_{N-2} & \cdots & z_{N-1} r_{N-2} & \theta + z'_{N-1} r_{N-2} \\ z'_1 r_{N-1} & z_1 r_{N-1} & z'_1 & \cdots & z'_{N-1} r_{N-1} - 1 & z_{N-1} r_{N-1} \\ z_1 r_{N-1} & z'_1 r_{N-1} & z_2 r_{N-1} & \cdots & z_{N-1} r_{N-1} & z'_{N-1} r_{N-1} - 1 \end{bmatrix},$$

where with $c_{mx} := \delta(1-\theta)\eta_1\eta_2 m$, $z'_i := (1-p)c_{mx}\bar{\rho}_i$ and $z_i := pc_{mx}\bar{\rho}_i$ for all i; and c) matrices $A_{mx,ex}^j$ for $j = 1,2$ represent the transitions from mixed to exclusive CP types and are defined using the following constants

$$c_{mx}^j := (1-\theta)\left[(1-\delta)m\eta_j + \delta m(1-p)\eta_j(1-\eta_{-j})\right], \text{ where}$$
$$-j := 1\mathbb{1}_{\{j=2\}} + 2\mathbb{1}_{\{j=1\}} \text{ is the usual game theoretic notation.}$$

Because of **0** sub-matrices of (17), $\mathbb{A}$ is not positive regular, TL-CTBP is decomposable (e.g., [10, 9]) and the analysis is drastically different. We carry out the analysis by first identifying, analysing the independent population types of our TL-CTBP.

C. Analysis of Mixed population

From the block structure of generator matrix $\mathbb{A}$ (17) (see also the PGFs), it is clear that the sub group of types corresponding to mixed populations, $\{(l,k): l \geq 1, k \geq 1 \text{ and } l = k+1 \text{ or } k = l+1\}$, survive on their own. A mixed type can be produced only by another mixed type, the mixed types can produce exclusive CP types, but not the other way round (see (17)). Thus the extinction/virality analysis of the mixed population can be obtained independently. To begin with we have the following result (proof in Appendix B):

Theorem 2. *i) If $0 < \theta, p < 1$, matrix $e^{A_{mx}t}$ for any $t > 0$, is positive regular.*
ii) Let α_{mx} be the largest eigenvalue of the generator matrix A_{mx}. Then $\alpha_{mx} \in \big((c_{mx}\mathbf{r}.\bar{\rho}-1),\ (c_{mx}\mathbf{r}.\bar{\rho}-1+\theta)\big)(\lambda+\nu)$, where $\mathbf{r}$ is redefined as $(r_1 \cdots r_{N-1})$. When $r_l = d_1 d_2^l$ for all l:

$$\boxed{\alpha_{mx} \to (c_{mx}\mathbf{r}.\bar{\rho} - 1 + \theta d_2)(\lambda+\nu),\ c_{mx} := \delta(1-\theta)\eta_1\eta_2 m \text{ as } N \to \infty.}$$

iii) Further the left eigenvector $\mathbf{u}_{mx} = (u_{mx,1},\cdots,u_{mx,2N-2})$ corresponding to α_{mx} satisfies for any $l \leq N-1$:

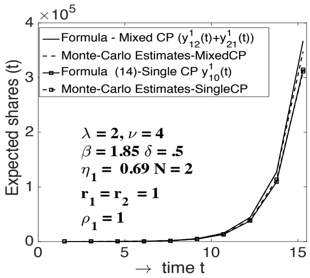
$$u_{mx,2l-1} = \sum_{i=0}^{l-1} \frac{\bar{\rho}_{l-i}}{\bar{\rho}_1}\left(\frac{\theta}{\sigma}\right)^i u_{mx,1};\ u_{mx,2l} = \sum_{i=0}^{l-1} \frac{\bar{\rho}_{l-i}}{\bar{\rho}_1}\left(\frac{\theta}{\sigma}\right)^i u_{mx,2}.$$

iv) The process $\{\boldsymbol{u}_{mx} \cdot \boldsymbol{X}_{mx}(t)e^{-\alpha_{mx}t}\}$ is a non-negative martingale, $\lim_{t\to\infty} \boldsymbol{X}_{mx}(t,\omega)e^{-\alpha_{mx}t} = W_{mx}(\omega)\boldsymbol{u}_{mx}$ for almost all ω. ■

From (ii) the mixed TLs get viral when $c_{mx}\mathbf{r}.\bar{\rho} > 1$, and the rate of explosion on viral paths equals α_{mx}. Important point to note here is that mixed population can get extinct, i.e.,

$$P\Big(\mathbf{X}_{mx}(t) = 0 \text{ for some } t > 0 \Big| \mathbf{X}_{mx}(0) = \mathbf{e}_{l,k}\Big) = 1 \text{ for all } (l,k)$$

(when $\alpha_{mx} < 0$), but can leave out few exclusive CP types which can get viral (if $\alpha_j > 0$). This is always possible because α_{mx} is mostly less than each α_j. For example, consider the case with $\rho_l = \bar{\rho}_l = \mathbb{1}_{\{l=1\}}$ and as $N \to \infty$ $\alpha_j \to (1-\theta)m\eta_j - 1 + \theta d_2$ while $\alpha_{mx} \to c_{mx} - 1 + \theta d_2 = (1-\theta)\delta m\eta_1\eta_2 - 1 + \theta d_2$.



$$e_{1,2}^1 = \frac{\theta e_{2,0}^1 + (1-\theta)\big(\delta - (1-\delta)(1-\eta_2)\big)e_{1,0}^1 m\eta_1\alpha_\lambda}{1-(1-\theta)^2(1-\delta)^2 m\eta_1^2\eta_2^2\alpha_\lambda^2} + \frac{\big((1-\theta)^2(1-\delta)^2 m\eta_1\eta_2(1-\eta_2)\big)e_{1,0}^1 m\eta_1\alpha_\lambda}{1-(1-\theta)^2(1-\delta)^2 m\eta_1^2\eta_2^2\alpha_\lambda^2},$$

$$e_{2,1}^1 = (1-\theta)(1-\delta)m\eta_1\alpha_\lambda\big(e_{1,0}^1(1-\eta_2) + e_{1,2}^1\eta_2\big),$$

$$\alpha_\lambda = \frac{\lambda+\nu}{\alpha_1+\lambda+\nu} \quad \boxed{N = 2, r_1 = r_2 = \rho_1 = 1\rho_2 = 0.}$$

Fig. 3: Time evolution of expected shares, $y_{l,k}^1(t) \approx e_{l,k}^1 e^{\alpha_1 t}$.

VII. CP WISE PERFORMANCE MEASURES UNDER COMPETITION

Mixed types keep producing exclusive type TLs as well as mixed types till their extinction i.e. when none of the TLs contain both the CP posts. Once mixed types get extinct, the left over exclusive CP-types don't influence each other (see (17)). Nevertheless, their survival/growth depends upon the effects created by mixed-types before death. When mixed gets viral, total CP population is clearly influenced by mixed-types. In all, the evolution of population corresponding to a particular CP depends upon the competition, whether or not the source of the competition dies (eventually). Recall TL-CTBP is not positive regular, population corresponding to a particular CP can get extinct w.p.1 while the other can get viral with positive probability. Further, they can have different growth rates, when viral. Thus we require individual CP-wise performance measures.

A. CP-wise extinction probabilities

By Lemma 1 sub-matrix A_{ex}^j of (17) is irreducible, thus all exclusive types of one CP survive/die together. By Theorem 2, matrix A_{mx} is irreducible. We say CP-j is extinct when all the mixed, exclusive CP-j types gets extinct and define its probability as (for any $l, k = l+1$ or $l-1$ and where $\mathbf{e}_{l,k}$ is unit vector with one only at l,k position):

$$q_{l,k}^j = P\Big(\mathbf{X}_{ex}^j(t) = \mathbf{0},\ \mathbf{X}_{mx}(t) = \mathbf{0} \text{ for some } t > 0 \Big| \mathbf{X}(0) = \mathbf{e}_{l,k}\Big).$$

Let $\mathbf{q}^j := \{\mathbf{q}_{ex}^j, \mathbf{q}_{mx1}^j, \mathbf{q}_{mx2}^j\}$ with $\mathbf{q}_{ex}^j := \{q_{l,0}^j\}_l$, $\mathbf{q}_{mx1}^j := \{q_{l,l+1}^j\}_l$ and $\mathbf{q}_{mx2}^j := \{q_{l+1,l}^j\}_l$. By again conditioning on the events of first transition we obtain $\mathbf{q}_{mx1}^1$ via FP equations:

$$\begin{aligned} q_{l,l+1}^1 &= \theta\Big(q_{l+1,l+2}^1 \mathbb{1}_{\{l<N-1\}} + \mathbb{1}_{\{l<N-1\}} q_{N,0}^1\Big) + (1-\theta)(1-r_l) \\ &+ (1-\theta)r_l\Big[\delta\ \Big(p\bar{\mathbf{g}}\big(\mathbf{q}_{mx2}^1,\eta_{12}\big) + (1-p)\bar{\mathbf{g}}\big(\mathbf{q}_{mx1}^1,\eta_{12}\big)\Big)\bar{\mathbf{g}}\big(\mathbf{q}_{ex}^1,\eta_1(1-\eta_2)\big) \\ &+ (1-\delta)\bar{\mathbf{g}}\big(\mathbf{q}_{ex}^1,\eta_1\big)\Big], \text{ with } \bar{\mathbf{g}}(\mathbf{s}',\eta) := \sum_{i=1}^{N-1} f(s'_i,\eta,\beta)\bar{\rho}_i,\ \eta_{12} := \eta_1\eta_2. \end{aligned}$$

One can write FP equations for $\mathbf{q}_{mx2}^1$, $\mathbf{q}_{mx1}^2$ and $\mathbf{q}_{mx2}^2$ in a similar way and extinction probabilities starting from exclusive CP types $\mathbf{q}_{ex}^j$ are same as that of the single CP. The FP equations have unique solution (proof in Appendix B):

Lemma 4. *When $\mathbf{q}_{ex}^j < \mathbf{1} = (1,\cdots,1)$, we have unique solution in the interior of $[0,1]^{2N-2}$, i.e., $\mathbf{q}_{mx1}^j < \mathbf{1}$, $\mathbf{q}_{mx2}^j < \mathbf{1}$. When $\mathbf{q}_{ex}^j = \mathbf{1}$, $(\mathbf{q}_{mx1}^j, \mathbf{q}_{mx2}^j) = \mathbf{1}$ is the unique solution, under extra assumption that $\rho_N = 0$ and $\bar{\rho}_i = \rho_i$ for all $i < N$.* ■

B. CP-wise expected number of shares

Let $Y^j_{l,k}(t)$ be the total number of shares of CP-j post till time t and let $y^j_{l,k}(t)$ represent its expected value, when started with one (l,k) where $k = l+1$ or $l-1$ type TL. These include mixed shares as well as exclusive CP shares belonging to CP-j. The time evolution of expected shares depends upon many factors in this competitive scenario, we have some initial results which we discuss here.

Exclusive types can get viral ($\alpha_j > 0$), mixed TLs get extinct w.p. 1 ($\alpha_{mx} < 0$): In this case our conjecture is that the CP-wise expected shares of CP-j grows at rate α_j. We initially assume that each $y^j_{l,k}(t)$ has the form $\sum_i^{N'} e^j_{l,k,i} e^{\bar{\alpha}_i t}$ for any finite positive sequence $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{N'}\}$ and appropriate sequence of constants $\{e^j_{l,k,i}\}$ and then write down the FP equations in Appendix C. When we solve the FP equations for the case with $\alpha_{mx} < 0$, we notice that the co-efficient $e^j_{l,k,i}$ is non-zero (in fact positive) only when the corresponding $\bar{\alpha}_i = \alpha_j$ or 0 else. In Appendix C we simplified the above mentioned FP and the required coefficients are solution of the linear equations of Appendix C.

Let $Y^j(t)$ be the total number of shares of CP-j post till time t and let $y^j(t)$ represent its expected value, when started with one (1,2) and one (2,1) type TLs and hence

$$y^j(t) = y^j_{1,2}(t) + y^j_{2,1}(t) \text{ when } N = 2.$$

We obtain closed form expressions for $\{e^1_{l,k}\}$ for the special case of $N = 2$, $r_1 = r_2 = \rho_1 = 1$, $\rho_2 = 0$ in Appendix C and the same are provided in the right hand side of Figure 3. These results require theoretical justification, we currently provide a simulation based justification in Figure 3. We obtain the estimates of expected shares at ten different time points, of one CP operating in competitive (curves without markers) as well as single CP (curves with markers) environment, using Monte-Carlo simulations. We basically generate many sample paths of TL-CTBP (7500 samples) and compute the expected shares using sample averages. We also plot the theoretical estimates, using formula (12) for single CP case and the formula provided in figure itself for competitive environment, at the same points. We observe a very good match in both the sets of curves. *An important observation: from the expressions provided in Figure 3, the co-efficient $e^j_{1,0} > e^j_{1,2}$ but the growth rate remains the same in competitive as well as single CP environment.*

When exclusive as well as mixed can get viral: we conjecture that (see Appendix C)

$$y^j_{l,k}(t) \approx e_{l,k} e^{\alpha_j t} + g_{l,k} e^{\alpha^*_j t}, \ \frac{\alpha^*_j}{\lambda+\nu} = c_{mx} \sum_l \bar{\rho}_l \sum_{i=0}^{N-l-1} \theta^i r_{l+i} - 1 \leq \frac{\alpha_{mx}}{\lambda+\nu}.$$

Thus the expected shares *grow like a sum of two exponential curves, when mixed can also survive.*

C. Expected Number of Shares for non-viral scenarios

With $m < 1$, any post gets extinct w.p. 1. Basically we have a moderately active network. We obtain the total expected shares (before extinction) when started with one TL of various types, $\{y^j_{l,k}\}$ with $y^j_{l,k} := E[\lim_{t\to\infty} Y^j(t) | \mathbf{X}(0) = \mathbf{e}_{l,k}]$. These can again be obtained by solving appropriate FP equations (see Appendix D). Let $\mathbf{y}^j_{mx1} := \{y^j_{l,l+1}\}$, $\mathbf{y}^j_{mx2} := \{y^j_{l+1,l}\}$ and $\mathbf{y}^j_{mx} := \mathbf{y}^j_{mx1} + \mathbf{y}^j_{mx2}$. For the special case with $\bar{\rho}_l = \bar{\rho}^l / \sum_{i=1}^{N-1} \bar{\rho}^i$, $r_l = d_1 d_2^l$ and as $N \to \infty$, the FP equations are solved to obtain (Appendix D):

$$\mathbf{y}^j_{mx} \cdot \bar{\boldsymbol{\rho}} \to \frac{\left(2c_j\delta\left[(1+\mathbf{y}^j_{exj}\cdot\bar{\boldsymbol{\rho}})(1-\eta_{-j})+\eta_{-j}\right] + c_j(1-\delta)(1+\mathbf{y}^j_{exj}\cdot\bar{\boldsymbol{\rho}})\right) O_{mx}}{1 - c_{mx} O_{mx}}$$

with

$$O_{m_x} \to \frac{d_1 d_2 (1-\bar{\rho})}{(1-d_2\bar{\rho})(1-\theta d_2)} \text{ and where } -j := 2\mathbb{1}_{\{j=1\}} + 1\mathbb{1}_{\{j=2\}}.$$

Here $\mathbf{y}^j_{exj}$ is given by (13)-(15), and $\{y^j_{l,k}\}$ with $k = l+1$ or $l-1$ can be computed uniquely using $\mathbf{y}^j_{mx}$.

VIII. Game Theoretic Aspects

Competition between CPs can be significant and we study its effects using game theoretic approach. Each CP has profit in form of various performance measures, while will have to pay for post quality. CPs like to optimize their own cost,

$$\mathscr{C}_j(\eta_j, \eta_{-j}) = pm_j(\eta_j, \eta_{-j}) \pm \psi w_j^2 \eta_j^2 \text{ with } \eta_j \in \left[0, 1/w_j\right] \quad \forall j, \quad (18)$$

which is influenced by the post of the other and hence we have a non-cooperative game. The performance $pm_j(.,.)$ could be the expected number of shares or the probability of virality. Well known Nash Equilibrium (NE) is a solution

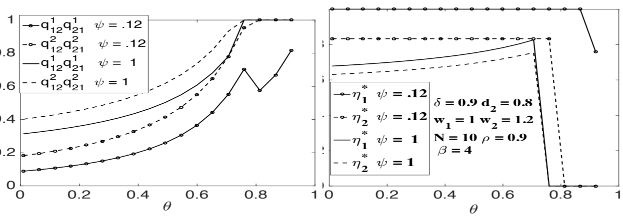


Fig. 4: NE for Extinction probabilities

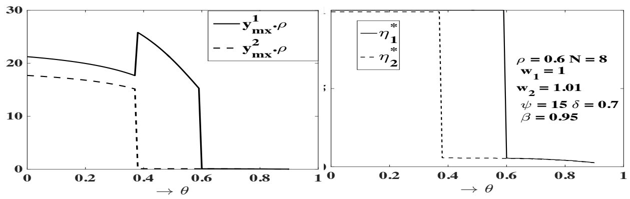


Fig. 5: NE for Expected Shares

concept. NE is an equilibrium strategy, to be precise a pair of post quality factors (η^*_1, η^*_2) one for each CP, deviating unilaterally from which neither of the CPs would benefit. Existence and uniqueness would be a topic of future research. We compute NE using gradient and best-response dynamics based numerical algorithm for Poisson friends, $\mathbb{F}$.

We begin with $pm_j = q^j_{1,2} q^j_{2,1}$ in (18), the extinction probability when started with one (1,2) and one (2,1) TL. We plot the NE and extinction probability at NE in Figure 4. The system parameters are indicated in the figures. We consider two CPs with respective influence parameters $w_1 = 1$, $w_2 = 1.2$. We study NE as function of the shift parameter θ. For small values of θ, the losses due to shifting (post lost before the user visits the TL) are less and hence we notice that NE (η^*_1, η^*_2) is close to $(1, 1/w_2)$. In fact, when $\psi = .12$ is small (curves with markers) the NE equals $(1, 1/w_2)$ for all θ less than .75. As θ increases η^*_2 corresponding to less influential CP, decreases below $1/w_2$, in fact it is close to 0. While we notice that the more influential CP has $\eta^*_1 = 1$ till $\eta^*_1 = 0.9$. Interestingly there is a jump in extinction probability of CP1 at the transition point of η_2^* (see Fig. 4). This is because the competition has reduced suddenly and CP1 has much better benefit.

In Fig. 5 we plot NE corresponding to expected shares. The behaviour is similar. Here again we notice a big jump in CP1 performance (solid line in right sub-figure) at $\psi = 0.4$. Thus in presence of competition the returns are limited even for more influential CP1 (beyond $\psi = 0.6$, expected shares $\approx$ zero). If CP1 is willing to invest more (ψ small) it can suppress the competitor (CP2 here) and enjoy absolute monarchy. There exists a range of trade-off factors (e.g., $\psi \in [0.4, 0.6]$ in figure) for which this is possible.

No-TL case: We again set $N \to \infty$ and $r_l \equiv 1$. With this $\alpha_{mx} \approx (m\eta_1\eta_2\delta - 1)\nu$, which again indicates an inflated growth rate as well as the virality chances of mixed population. Even the influence of competition, inherent only due to time-line structures, can not be considered appropriately as this majorly depends upon the response[8] of a typical user based on the levels at which the competing posts reside. Currently we consider only $(l, l+1)$ or $(l+1, l)$ type TLs, but one can easily generalize our study to consider all the combinations.

Conclusions

We say a post gets viral in a social network, if the number of users shared with the post grow fast with time. The users shared with the post, can share the same to some of their friends and if this continues rigorously the post gets viral. The propagation of a post is influenced by the activity of the network: the post on a TL (time-line) gets shifted down by one level with every new share and can disappear before the owner of the TL visits the network. Thus the post can get extinct (eventually zero users with post) either due to excessive shifts or because of lack of interest. The competition can further reduce survival chances. For example consider two agents of similar business/agenda advertising their information using the same social network. The users of the network may not be interested in viewing and subsequent sharing of two posts of similar type and hence may neglect one of them.

We modelled the propagation of content over a huge social network using multi-type branching processes. One important observation is that, either the post gets extinct or it gets viral (in fact users with post grow exponentially fast with time). We obtain the probability of virality, based on post-quality and network local parameters like the view probabilities at different TL levels, distribution of friends of a typical user etc. We also showed that the expected number of shares, which are different from expected total progeny, due to presence of TLs, grow at the same rate as the number of unread posts.

The multi-type branching processes resulting from competitive scenario are decomposable. As a result the virality probability, the growth rate of expected shares etc., are different for different types of TLs. However the performance remains the same for all the types belonging to one content provider. We derived closed form expressions or fixed point equations to obtain the these common performance measures. We formulated non-cooperative game problem using the CP-wise performance measures and study the influence of system parameters on NE.

So far in literature, we did not find a work that considers the influence of structure of time-lines on post-propagation. We compared our results with the results that one would obtain without TLs. The two sets of conclusions are drastically different: a) study without TLs shows that even less interesting posts can get viral; b) it also indicates very large growth rates in comparison with the results obtained by considering the structure of TLs; c) More interestingly we observed that viral chances can get reduced when the network gets more active; d) study without TLs cannot capture accurately the competition prevalent in the propagation of posts of competing content providers, which occurs due to the inherent structure of time-lines.

This just opened up many more open questions, both related to branching processes and the application (social network) and our future work would be towards answering these questions. One needs to estimate the parameters (e.g., view probabilities) from real data to derive more useful conclusions from this study. This would be a topic of future research. However, the theoretical study has already provided good insights, which were missing when one neglects the influence of the structure of TLs.

[8] For example, parameters like δ can capture these effects.

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Proofs are in the next Page.

Appendix A: Single CP related proofs

Proof of Lemma 1: i) The matrix e^{At} for any $t>0$ is positive regular iff e^{A} is ([13]), because $A+I$ has only non-negative entries. Thus it is sufficient to prove e^{A} is positive regular. Without loss of generality we can drop the multiplier $\lambda+\nu$. Then the matrix A can be written in the following way $A=A_1+A_2$, where

$$A_1=\begin{bmatrix} c_1r_1 & c_2r_1+\theta & \cdot & c_{N-1}r_1 & c_Nr_1 \\ c_1r_2 & c_2r_2 & \cdot & c_{N-1}r_2 & c_Nr_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1r_{N-1} & c_2r_{N-1} & \cdot & c_{N-1}r_{N-1} & c_Nr_{N-1}+\theta \\ c_1r_N & c_2r_N & \cdot & c_{N-1}r_N & c_Nr_N \end{bmatrix}.$$

and the matrix $A_2=Diag(-1)$. Thus $e^{A}=e^{A_1}e^{A_2}=e^{-1}e^{A_1}$ since matrices commute. For any i, one can express

$$e^{A_i} \quad = \quad I+A_i+\frac{A_i^2}{2!}+\frac{A_i^3}{3!}+\cdots, \tag{19}$$

where I is the identity matrix. Also $e^{A_2}=e^{-1}I$ commutes with e^{A_1}. A matrix is positive regular if there exists an n such that A^n has all positive entries. If $c_l>0$ and $r_l>0$ for all l, then A_1 is trivially positive regular and hence e^{A} is also positive regular.

Consider a general case, where some of the constants can be zero, in particular consider the case with $c_l=0\ \forall l>1$ and $c_1>0$. For this case:

$$A_1=\begin{bmatrix} c_1r_1 & \theta & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ c_1r_2 & 0 & \theta & 0 & \cdot & 0 & 0 & 0 & 0 \\ c_1r_3 & 0 & 0 & \theta & \cdot & 0 & 0 & 0 & 0 \\ c_1r_4 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1r_{N-3} & 0 & 0 & 0 & \cdot & 0 & \theta & 0 & 0 \\ c_1r_{N-2} & 0 & 0 & 0 & \cdot & 0 & 0 & \theta & 0 \\ c_1r_{N-1} & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \theta \\ c_1r_N & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then it is clear that

$$A_1^2=\begin{bmatrix} c_1^2r_1^2+\theta c_1r_2 & \theta c_1r_1 & \theta^2 & \cdot & 0 & 0 \\ c_1^2r_1r_2+\theta c_1r_3 & \theta c_1r_2 & 0 & \cdot & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ c_1^2r_1r_{N-2}+\theta c_1r_{N-1} & \theta c_1r_{N-2} & 0 & \cdot & 0 & \theta^2 \\ c_1^2r_1r_{N-1}+\theta c_1r_N & \theta c_1r_{N-1} & 0 & \cdot & 0 & 0 \\ c_1^2r_1r_N & \theta c_1r_N & 0 & \cdot & 0 & 0 \end{bmatrix} \tag{20}$$

The third power $A_1^3=A_1^2A_1$ will have first three columns positive because the first two columns in A_1^2 is have strict positive terms and the first 2×3 sub matrix of A_1

$$\begin{bmatrix} c_1r_1 & \theta & 0 \\ c_1r_2 & 0 & \theta \end{bmatrix}$$

has at least one positive entry in every column. Continuing this way one can verify that A_1^N has all positive entries by induction. Basically once A_1^n has first n columns with only positive entries, because the first $n\times(n+1)$ sub-matrix of A_1 has atleast one positive entry in every column, the matrix $A_1^{n+1}=(A_1^n)\times A_1$ will have its first $n+1$ columns with only positive entries. Further A^n has only non negative entries for any $n\in\mathbb{N}$. From (19) it is direct that e^{A_1} is positive regular and so is $e^{-1}e^{A_1}$.

For the general case, when only some of $\{c_l\}$ are non-zero, since terms are non-negative the positive regularity follows from above case and expansion (19). The result is true as long as $c_1>0$.

Proof of parts (ii)-(iii): We proved that e^{A} is positive regular. By Frobenius-Perron theory of positive regular matrices: a)there exists an eigenvalue, call it e^{α}, of matrix e^{A} whose algebraic and geometric multiplicities are one and which dominates all the other eigenvalues in absolute sense. In fact α would be real eigenvalue of matrix A and it dominates the real components of all other eigenvalues of the matrix A; b) there exists a left eigenvector $\mathbf{u}$ and a right eigenvector $\mathbf{v}$, both with all positive components, corresponding to α. Fix one such set of left and right eigenvectors $\mathbf{u}$, $\mathbf{v}$.

Note that the eigenvectors of matrices A and e^{A} are the same. Any left eigenvector of α, in particular $\mathbf{u}$, satisfies $\mathbf{u}A=\alpha\mathbf{u}$ and hence we get the following system of equations relating $\mathbf{u}$ and α

$$(\lambda+\nu)c_1\mathbf{r.u}-(\lambda+\nu)u_1=\alpha u_1 \text{ or in other words } c_1\mathbf{r.u} \quad = \quad \frac{\alpha+\lambda+\nu}{\lambda+\nu}u_1, \text{ and similarly}$$

$$c_l\mathbf{r.u}+\theta u_{l-1} \quad = \quad \frac{\alpha+\lambda+\nu}{\lambda+\nu}u_l, \quad l\geq 2. \tag{21}$$

Simplifying the above we obtain the following relation among various components of left eigenvector $\mathbf{u}$: for any $l \le N$

$$u_l = \sum_{i=0}^{l-1} \frac{\rho_{l-i}}{\rho_1} \left(\frac{\theta}{\sigma}\right)^i u_1; \quad \sum_{i=1}^{N} u_i = \sum_{l=1}^{N} \frac{\rho_l}{\rho_1} \sum_{i=0}^{N-l} \left(\frac{\theta}{\sigma}\right)^i u_1 \text{ where } \sigma := \frac{\alpha+\lambda+\nu}{\lambda+\nu}. \tag{22}$$

Following exactly similarly procedure we obtain the relation among various components of right eigenvector $\mathbf{v}$ as given by (5). This completes the proof of part (iii).

Fix $\mathbf{u}$, $\mathbf{v}$ as before, and consider the following linear function of σ':

$$P(\sigma') := (\mathbf{r.c})\,\mathbf{r.u} + \theta \sum_{i=1}^{N-1} r_{i+1} u_i - \sigma' \mathbf{r.u} \tag{23}$$

where $\mathbf{r.c} := \sum_{i=1}^{N} r_i c_i$ etc. Multiplying either side of the equation (21) with r_l and then summing over l we notice that σ is a zero of $P(.)$. In other words, eigenvalue $\alpha = (\sigma^* - 1)(\lambda+\nu)$, where σ^* is a zero of $P(.)$ Because $u_i > 0$ for all l, $\mathbf{r.u} > 0$ and similarly $\mathbf{r.c} > 0$. Thus σ is the only zero of $P(.)$. It is clear that

$$P(\mathbf{r.c}) = \theta \sum_{i=1}^{N-1} r_{i+1} u_i > 0.$$

Since r_is are monotonic, i.e., because $r_1 \ge r_2 \ge \cdots \ge r_N$,

$$P(\mathbf{r.c} + \theta) = \theta \sum_{i=1}^{N-1} r_{i+1} u_i - \theta \mathbf{r.u} < 0.$$

Thus the only zero of $P(.)$ lies in the open interval interval $\left(\mathbf{r.c}, \quad \mathbf{r.c} + \theta\right)$. Thus $\alpha \in \left(\mathbf{r.c} - 1, \quad \mathbf{r.c} + \theta - 1\right)(\lambda+\nu)$.

Consider the special case with $r_l = d_1 d_2^l$, where d_1 and $d_2 \le 1$ are constants, then clearly the only root of equation (23) σ equals

$$\sigma = \mathbf{r.c} + \theta d_2 \frac{\sum_{i=1}^{N-1} r_i u_i}{\mathbf{r.u}} = \mathbf{r.c} + \theta d_2 \left(1 - \frac{r_N u_N}{\mathbf{r.u}}\right).$$

Now we study the convergence of σ as $N \to \infty$. It is obvious that the eigenvectors/eigenvalues corresponding to different N would be different. We would normalize them by choosing the eigenvector $\mathbf{u}$ with $u_1 = 1$ for any N. With such a choice, it is clear from (22) that u_N remains bounded even when we let $N \to \infty$. Thus as $N \to \infty$

$$\sigma = \mathbf{r.c} + \theta d_2 \left(1 - \frac{r_N u_N}{\mathbf{r.u}}\right) \to \mathbf{r.c} + \theta d_2 \quad \text{as} \quad N \to \infty \quad \because (r_N \to 0).$$

Thus, as the number of TL levels increase the largest eigenvalue, α of matrix A converges to $(\mathbf{r.c} + \theta d_2 - 1)(\lambda+\nu)$. ■

Proof of Lemma 3: Let $\mathbf{j_x} = \{\mathbf{j_{x_1}}, \mathbf{j_{x_2}}, \cdots, \mathbf{j_{x_N}}\}$ be the number of TLs of type $1, 2, \cdots, N$ respectively and y be the total number of shares. It is easy to observe that $y \geq \sum_i j_{x_i}$. We write it in short form as $y \geq \mathbf{j_x}$. Define $\mathbf{s_x^{j_x}} := \Pi_i s_{x_i}^{j_{x_i}}$ then the PGF of TL-CTBP can be written as

$$F_1(\mathbf{s}, t) = \sum_{\mathbf{j_x}=0}^{\infty} \sum_{y \geq \mathbf{j_x}}^{\infty} P_{(\mathbf{e}_1,1)\to(\mathbf{j_x},y)}\big(t\big)\mathbf{s_x^{j_x}} s_y^y \quad \text{and,}$$

$$\frac{\delta F_1(\mathbf{s}, t)}{\delta t} = \sum_{\mathbf{j_x}=0}^{\infty} \sum_{y \geq \mathbf{j_x}}^{\infty} P'_{(\mathbf{e}_1,1)\to(\mathbf{j_x},y)}\big(t\big)\mathbf{s_x^{j_x}} s_y^y.$$

This is obtained by conditioning on the events of the first transition, noting that the populations generated by two parents evolve independently of each other and the procedure is similar to the standard procedure used in these kind of computations (e.g., [12]). Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_N)$ represent the offsprings produced by one parent and let $\bar{\xi} := \sum_i \xi_i$. Via backward equation $P'_{1k}(t) = \sum_j q_{1j} P_{jk}(t)$, in our case it is

$$\frac{\delta F_1(\mathbf{s}, t)}{\delta t} = (\lambda+\nu)\Big((1-\theta)r_1 \sum_{\boldsymbol{\xi}} \sum_{\mathbf{j_x}=0}^{\infty} \sum_{y \geq \mathbf{j_x}}^{\infty} P_1(\boldsymbol{\xi}) P_{\left(\xi,\bar{\xi}+1\right)\to(\mathbf{j_x},y)}\Big(t\Big)\mathbf{s_x^{j_x}} s_y^y + \theta F_2(\mathbf{s}, t) - \sum_{\mathbf{j_x}=0}^{\infty} \sum_{y \geq \mathbf{j_x}}^{\infty} P_{(\mathbf{e}_1,1)\to(\mathbf{j_x},y)}\big(t\big)\mathbf{s_x^{j_x}} s_y^y + (1-\theta)(1-r_1)s_y\Big)$$

$$\frac{\delta F_1(\mathbf{s}, t)}{\delta t} = (\lambda+\nu)\Big((1-\theta)r_1 \sum_{\boldsymbol{\xi}} P_1(\boldsymbol{\xi}) \Pi_{i=1}^N \Big(\sum_{\mathbf{j_x}=0}^{\infty} \sum_{y \geq \mathbf{j_x}}^{\infty} P_{(\mathbf{e}_i,1)\to(\mathbf{j_x},y)}\big(t\big)\mathbf{s_x^{j_x}} s_y^y\Big)^{\xi_i} s_y + \theta F_2(\mathbf{s}, t) - F_1(\mathbf{s}, t) + (1-\theta)(1-r_1)s_y\Big)$$

$$\frac{\delta F_1(\mathbf{s}, t)}{\delta t} = (\lambda+\nu)\Big((1-\theta)r_1 s_y f_1\Big(\mathbf{F}(\mathbf{s}, t)\Big) + \theta F_2(\mathbf{s}, t) - F_1(\mathbf{s}, t) + (1-\theta)(1-r_1)s_y\Big)$$

where $\mathbf{F}(\mathbf{s}, t) := \{F_1(\mathbf{s}, t), F_2(\mathbf{s}, t), \cdots, F_N(\mathbf{s}, t)\}$. Similarly we can write for any l

$$\frac{\delta F_l(\mathbf{s}, t)}{\delta t} = (\lambda+\nu)\Big((1-\theta)r_l s_y f_l\Big(\mathbf{F}(\mathbf{s}, t)\Big) + \theta\Big(\mathbb{1}_{l<N} F_{l+1}(\mathbf{s}, t) + s_y \mathbb{1}_{l=N}\Big) - F_l(\mathbf{s}, t) + (1-\theta)(1-r_l)s_y\Big).$$

Let $\dot{y}_l(t) = \frac{\delta^2 F_l(\mathbf{s},t)}{\delta t \delta s_y}|_{\mathbf{s}=1}$ $\forall l = \{1, 2, \cdots, N\}$ represent the time derivative of number shares when started with a type l progenitor, till time t. Then we have the following expression for $\dot{y}_1(t)$

$$\begin{aligned}
\dot{y}_1(t) &= \Big(\lambda+\nu\Big)\Big((1-\theta)r_1 f_1(1) + (1-\theta)r_1 \sum_{i=1}^N \frac{\delta f_1(\mathbf{F}(\mathbf{s}, t)}{\delta F_i(\mathbf{s}, t)} \frac{\delta F_i(\mathbf{s}, t)}{\delta s_y}\Big|_{\mathbf{s}=1} + (1-\theta)(1-r_1)1 \\
&+ \theta \frac{\delta F_2(\mathbf{s}, t)}{\delta s_y}\Big|_{\mathbf{s}=1} - \frac{\delta F_1(\mathbf{s}, t)}{s_y}\Big|_{\mathbf{s}=1}\Big) \\
&= \Big(\lambda+\nu\Big)\Big((1-\theta)r_1 + (1-\theta)(1-r_1) + (1-\theta)r_1 m\eta \sum_{i=1}^N \rho_i y_i(t) + \theta y_2(t) - y_1(t)\Big) \\
&= \Big(\lambda+\nu\Big)\Big(1-\theta + r_1 \sum_{i=1}^N c_i y_i(t) + \theta y_2(t) - y_1(t)\Big).
\end{aligned}$$

Similarly we can write the above for any l

$$\dot{y}_l(t) = \big(\lambda+\nu\big)\Big(1-\theta + r_l \sum_{i=1}^N c_i y_i(t) + \theta y_{l+1}(t)\mathbb{1}_{l<N} - y_l(t) + \theta \mathbb{1}_{l=N}\Big). \tag{24}$$

In matrix form the above can be written as

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_{N-1}(t) \\ \dot{y}_N(t) \end{bmatrix} = (\lambda+\nu) \begin{bmatrix} c_1 r_1 - 1 & c_2 r_1 + \theta & \cdots & c_{N-1} r_1 & c_N r_1 \\ c_1 r_2 & c_2 r_2 - 1 & \cdots & c_{N-1} r_2 & c_N r_2 \\ & \vdots & & & \\ c_1 r_{N-1} & c_2 r_{N-1} & \cdots & c_{N-1} r_{N-1} - 1 & c_N r_{N-1} + \theta \\ c_1 r_N & c_2 r_N & \cdots & c_{N-1} r_N & c_N r_N - 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N-1}(t) \\ y_N(t) \end{bmatrix} + (\lambda+\nu) \begin{bmatrix} 1-\theta \\ 1-\theta \\ \vdots \\ 1-\theta \\ 1 \end{bmatrix}.$$

Solving the above set of equations we obtain:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N-1}(t) \\ y_N(t) \end{bmatrix} = e^{At} \begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_{N-1}(0) \\ y_N(0) \end{bmatrix} + e^{At} \int_0^t e^{-As}(\lambda+\nu) \begin{bmatrix} 1-\theta \\ 1-\theta \\ \vdots \\ 1-\theta \\ 1 \end{bmatrix} ds \tag{25}$$

$$= e^{At} \begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_{N-1}(0) \\ y_N(0) \end{bmatrix} + e^{At} A^{-1}\big(I - e^{-At}\big)(\lambda+\nu) \begin{bmatrix} 1-\theta \\ 1-\theta \\ \vdots \\ 1-\theta \\ 1 \end{bmatrix} \tag{26}$$

With $\mathbf{y}(t) := \{y_1(t), y_2(t), \cdots, y_N(t)\}$, we can represent the above as:

$$\mathbf{y}(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N-1}(t) \\ y_N(t) \end{bmatrix} = e^{At}\Big(\mathbf{1} + (\lambda+\nu)A^{-1}\mathbf{k}\Big) - (\lambda+\nu)A^{-1}\mathbf{k} \text{ with } \mathbf{k} := [1-\theta, 1-\theta, \cdots, 1-\theta, 1]^T$$

$$\begin{aligned}
\mathbf{y}(t) &= e^{At}\Big(\mathbf{1} + (\lambda+\nu)A^{-1}\mathbf{k}\Big) - (\lambda+\nu)A^{-1}\mathbf{k} \\
&\approx \mathbf{v}\mathbf{u}' e^{\alpha t}\Big(\mathbf{1} + (\lambda+\nu)A^{-1}\mathbf{k}\Big) - (\lambda+\nu)A^{-1}\mathbf{k} \\
&\approx e^{\alpha t}\Big(\mathbf{v}\sum_{i=1}^{N} u_i + \frac{\lambda+\nu}{\alpha}\mathbf{v}\mathbf{u}'\mathbf{k}\Big) - (\lambda+\nu)A^{-1}\mathbf{k} \\
&\approx \mathbf{v}e^{\alpha t}\left(\sum_i u_i\Big(1 + \frac{\lambda+\nu}{\alpha}(1-\theta)\Big) + \frac{\lambda+\nu}{\alpha}u_N\right) - (\lambda+\nu)A^{-1}\mathbf{k} \\
&\approx \mathbf{v}e^{\alpha t}\sum_i u_i\left(1 + \frac{1-\theta}{\mathbf{r}.\mathbf{c} - 1 + \theta d_2}\right) - (\lambda+\nu)A^{-1}\mathbf{k}.
\end{aligned}$$

□

APPENDIX B: MIXED POPULATION PROOFS

Proof of Theorem 2: Here the generator matrix A_{mx} is

$$\begin{bmatrix}
z_1' r_1 - 1 & z_1 r_1 & \theta + z_2' r_1 & \dots & z_{N-1}' r_1 & z_{N-1} r_1 \\
z_1 r_1 & z_1' r_1 - 1 & z_2 r_1 & \dots & z_{N-1} r_1 & z_{N-1}' r_1 \\
z_1' r_2 & z_1 r_2 & z_2' r_2 - 1 & \dots & z_{N-1}' r_2 & z_{N-1} r_2 \\
z_1 r_2 & z_1' r_2 & z_2 r_2 & \dots & z_{N-1} r_2 & z_{N-1}' r_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
z_1' r_{N-2} & z_1 r_{N-2} & z_2' r_{N-2} & \dots & \theta + z_{N-1}' r_{N-2} & z_{N-1} r_{N-2} \\
z_1 r_{N-2} & z_1' r_{N-2} & z_2 r_{N-2} & \dots & z_{N-1} r_{N-2} & \theta + z_{N-1}' r_{N-2} \\
z_1' r_{N-1} & z_1 r_{N-1} & z_1' & \cdots & z_{N-1}' r_{N-1} - 1 & z_{N-1} r_{N-1} \\
z_1 r_{N-1} & z_1' r_{N-1} & z_2 r_{N-1} & \dots & z_{N-1} r_{N-1} & z_{N-1}' r_{N-1} - 1
\end{bmatrix}. \tag{27}$$

First we prove that $e^{A_{mx}}$ is positive regular for any $0 < \theta, p < 1$. As in the case of Lemma 1, we prove the result for special case with $z_l = z_l' = 0 \ \forall l > 1$ and $z_1 > 0$, $z' > 0$. The result again follows for general case because all the terms involved are non-negative. For this special case the matrix $A_{mx} + I$ has the following form with all $z_l r_k$ or $z_l' r_k$ terms being strictly positive because $0 < p < 1$:

$$A_{mx} + I = \begin{bmatrix}
z_1' r_1 & z_1 r_1 & \theta & 0 & \dots & 0 & 0 & 0 & 0 \\
z_1 r_1 & z_1' r_1 & 0 & \theta & \dots & 0 & 0 & 0 & 0 \\
z_1' r_2 & z_1 r_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
z_1 r_2 & z_1' r_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
z_1' r_{N-2} & z_1 r_{N-2} & 0 & 0 & \dots & 0 & 0 & \theta & 0 \\
z_1 r_{N-2} & z_1' r_{N-2} & 0 & 0 & \dots & 0 & 0 & 0 & \theta \\
z_1' r_{N-1} & z_1 r_{N-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
z_1 r_{N-1} & z_1' r_{N-1} & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{bmatrix};$$

Positive regularity of a matrix is determined by existence of all positive terms in some power of the given matrix. Since the matrix $A_{mx} + I$ has only non-negative entries it is sufficient to check zero, non-zero structure (the location of zero and non zero terms in the given matrix and not the exact values) of the resulting powers of the matrices $(A_{mx} + I)^n$. The matrix $(A_{mx} + I)$ is exactly similar in zero non-zero structure as the second power A_1^2 given by (20) of the single CP matrix. Thus positive regularity follows in exactly similar lines.

Proof of parts (ii)-(iii): We follow exactly the same procedure as in the proof of parts (ii)-(iii) of Lemma 1. Hence we mention only the differences with respect to that proof.

Let $\mathbf{u}_{mx} = \{u_{mx,1}, u_{mx,2}, \cdots, u_{mx,2N-3}, u_{mx,2N-2}\}$ be the left eigenvector of A_{mx}, corresponding to largest eigenvalue α_{mx}, both of which exist because of positive regularity given by part (i).

On solving $\mathbf{u}_{mx} A_{mx} = \alpha \mathbf{u}_{mx}$ we have the following system of equations:

$$\begin{aligned}
z_1 \mathbf{r}.\mathbf{u}_{mx,e} + z_1' \mathbf{r}.\mathbf{u}_{mx,o} &= \sigma_{mx} u_{mx,1}, \ \ z_l \mathbf{r}.\mathbf{u}_{mx,e} + z_l' \mathbf{r}.\mathbf{u}_{mx,o} + \theta u_{mx,2l-3} = \sigma_{mx} u_{mx,2l-1}; \ \forall l \geq 2 \\
z_1' \mathbf{r}.\mathbf{u}_{mx,e} + z_1 \mathbf{r}.\mathbf{u}_{mx,o} &= \sigma_{mx} u_{mx,2}, \ \ z_l' \mathbf{r}.\mathbf{u}_{mx,e} + z_l \mathbf{r}.\mathbf{u}_{mx,o} + \theta u_{mx,2l-2} = \sigma_{mx} u_{mx,2l}; \ \forall l \geq 2.
\end{aligned} \tag{28}$$

where $\mathbf{r}.\mathbf{u}_{mx,o} := \sum_{i=1}^{N-1} r_i u_{mx,2i-1}$, $\mathbf{r}.\mathbf{u}_{mx,e} := \sum_{i=1}^{N-1} r_i u_{mx,2i}$ and $u_{mx,-1}(u_{mx,-2}) := 0$. Recall $c_{mx} = \delta(1-\theta) m \eta_1 \eta_2$, the above equations can be rewritten as

$$c_{mx}\bar{\rho}_1\left(p\mathbf{r}.\mathbf{u}_{mx,e}+(1-p)\mathbf{r}.\mathbf{u}_{mx,o}\right) = \sigma_{mx}u_{mx,1}, \quad c_{mx}\bar{\rho}_l\left(p\mathbf{r}.\mathbf{u}_{mx,e}+(1-p)\mathbf{r}.\mathbf{u}_{mx,o}\right)+\theta u_{mx,2l-3} = \sigma_{mx}u_{mx,2l-1};\ \forall l\geq 2$$
$$c_{mx}\bar{\rho}_1\left((1-p)\mathbf{r}.\mathbf{u}_{mx,e}+p\mathbf{r}.\mathbf{u}_{mx,o}\right) = \sigma_{mx}u_{mx,2}, \quad c_{mx}\bar{\rho}_l\left((1-p)\mathbf{r}.\mathbf{u}_{mx,e}+p\mathbf{r}.\mathbf{u}_{mx,o}\right)+\theta u_{mx,2l-2} = \sigma_{mx}u_{mx,2l-2};\ \forall l\geq 2$$

Now on multiplying with r_i and adding all even and odd term equations separately. We have the following

$$c_{mx}\mathbf{r}.\bar{\rho}\left(p\mathbf{r}.\mathbf{u}_{mx,e}+(1-p)\mathbf{r}.\mathbf{u}_{mx,o}\right)+\theta\sum_{i=1}^{N-1} r_i u_{mx,2i-3} = \sigma_{mx}\mathbf{r}.\mathbf{u}_{mx,o} \quad \text{; for odd terms,}$$
$$c_{mx}\mathbf{r}.\bar{\rho}\left((1-p)\mathbf{r}.\mathbf{u}_{mx,e}+p\mathbf{r}.\mathbf{u}_{mx,o}\right)+\theta\sum_{i=1}^{N-1} r_i u_{mx,2i-2} = \sigma_{mx}\mathbf{r}.\mathbf{u}_{mx,e} \quad \text{; for even terms.}$$

Further on adding above equations we have the following linear equation

$$P(\sigma_{mx}) = c_{mx}\mathbf{r}.\bar{\rho}\left(\mathbf{r}.\mathbf{u}_{mx,o}+\mathbf{r}.\mathbf{u}_{mx,e}\right)+\theta\sum_{i=1}^{N-1} r_i\left(u_{mx,2i-3}+u_{mx,2i-2}\right)-\sigma_{mx}\left(\mathbf{r}.\mathbf{u}_{mx,o}+\mathbf{r}.\mathbf{u}_{mx,e}\right), \tag{29}$$

and $\sigma_{mx}=(\alpha_m x+\lambda+\nu)/(\lambda+\nu)$ would be the only zero of it. Now $P\left(c_{mx}\mathbf{r}.\bar{\rho}\right)>0$ and $P\left(c_{mx}\mathbf{r}.\bar{\rho}+\theta\right)<0$ (again using monotonicity of the reading probabilities). And due to similar reasons as in in singe CP case the largest eigenvalue lies in the interval $\alpha_{mx}\in\left(c_{mx}\mathbf{r}.\bar{\rho}-1,\ \ c_{mx}\mathbf{r}.\bar{\rho}+\theta-1\right)(\lambda+\nu)$.
Let us assume $r_l=d_1d_2^l$, once $u_{mx,2N-2}+u_{mx,2N-3}$ are both bounded for any N. Thus the root of equation (29) for this special case

$$\sigma_{mx}=c_{mx}\mathbf{r}.\bar{\rho}+\theta d_2\frac{\sum_{i=1}^{N-2} r_i\left(u_{mx,2i-1}+u_{mx,2i}\right)}{\left(\mathbf{r}.\mathbf{u}_{mx,o}+\mathbf{r}.\mathbf{u}_{mx,e}\right)} c_{mx}\mathbf{r}.\bar{\rho}+\theta d_2\left(1-\frac{r_{N-1}\left(u_{mx,2N-2}+u_{mx,2N-3}\right)}{\mathbf{r}.\mathbf{u}_{mx,o}+\mathbf{r}.\mathbf{u}_e}\right),$$

converges to the following because $r_N=d_1d_2^N\to 0$,

$$\sigma_{mx}\to c_{mx}\mathbf{r}.\bar{\rho}+\theta d_2 \quad \text{as} \quad N\to\infty.$$

So, as the number of TL levels increases the largest eigenvalue, α_{mx} of matrix A_{mx} converges to $\left(c_{mx}\mathbf{r}.\bar{\rho}+\theta d_2-1\right)(\lambda+\nu)$.
Now we write the expression of $u_{mx,2l-1}$ and $u_{mx,2l}$ in terms of $u_{mx,1}$ and $u_{mx,2}$ receptively as done before in single CP and is given below for any $l\leq N-1$, after simplifying equations (28)

$$u_{mx,2l-1}=\sum_{i=0}^{l-1}\frac{\bar{\rho}_{l-i}}{\bar{\rho}_1}\left(\frac{\theta}{\sigma_{mx}}\right)^i u_{mx,1}; u_{mx,2l}=\sum_{i=0}^{l-1}\frac{\bar{\rho}_{l-i}}{\bar{\rho}_1}\left(\frac{\theta}{\sigma_{mx}}\right)^i u_{mx,2}.$$

■

Proof of Lemma 4: *Existence:* We consider $\mathbf{q}_{ex}^1$ a constant vector, as explained below. We have a continuous mapping from $[0,1]^{2N-2}$ into $[0,1]^{2N-2}$ i.e. over compact set. By *Brouwer's fixed point theorem* there exists a solution to the given system of equations.

Uniqueness: The exclusive CP1 types evolve on their own and by Lemma 2 we have unique solution to the relevant fixed point equations in unit cube $[0,1]^N$ which provide the extinction probabilities of CP1 population when started with one of its exclusive types. That is, we have unique $\mathbf{q}_{ex}^1=\{q_{l,0}^l\}_l$ which represents the extinction probabilities for any given set of system parameters. We treat them as constants while studying the fixed point equations of the other equations that provide the extinction probabilities when started with a mixed population, $(\mathbf{q}_{mx1}\mathbf{q}_{mx2})$. One can rewrite the fixed point equations corresponding to this set of the extinction probabilities as below for any $l<N$, after suitable simplification:

$$\begin{aligned} q_{l,l+1}^1 &= K_{1l}(pg_{mx1}+(1-p)g_{mx2})\bar{\mathbf{g}}\left(\mathbf{q}_{ex}^1,\eta_1(1-\eta_2)\right)+K_{2l}(1-\delta)\bar{\mathbf{g}}\left(\mathbf{q}_{ex}^1,\eta_1\right)+K_{3l}+\theta^{N-l}q_{N,0}^1 \\ q_{l+1,l}^1 &= K_{1l}\left((1-p)g_{mx1}+pg_{mx2}\right)\bar{\mathbf{g}}\left(\mathbf{q}_{ex}^1,\eta_1(1-\eta_2)\right)+K_{2l}(1-\delta)+K_{3l}+\theta^{N-l} \end{aligned} \tag{30}$$

where

$$\begin{aligned} K_{2l} &= (1-\theta)\sum_{i=0}^{N-l-1}\theta^i r_{l+i}; \quad K_{3l}=(1-\theta)\sum_{i=0}^{N-l-1}\theta^i(1-r_{l+i}) \\ K_{1l} &= K_{2l}\delta \\ g_{mx1} &= \bar{\mathbf{g}}\left(\mathbf{q}_{mx1},\eta_1\eta_2\right); \quad g_{mx2}=\bar{\mathbf{g}}\left(\mathbf{q}_{mx2},\eta_1\eta_2\right). \end{aligned}$$

Consider the following weighted sum over l, of terms $f(q_{l,l+1}^1,\ \eta_1\eta_2,\ \beta)$ and $f(q_{l+1,l}^1,\ \ \eta_1\eta_2,\ \beta)$

$$\sum_{l=1}^{N-1}\bar{\rho}_l f(q_{l,l+1}^1) \quad \text{and} \quad \sum_{l=1}^{N-1}\bar{\rho}_l f(q_{l+1,l}^1),$$

and note that these precisely equal g_{mx1} and g_{mx2} respectively. Thus using the right hand side (RHS) of equation (30), we have the following two dimensional equation, $\Psi = (\Psi_1, \Psi_2)$, whose fixed point provides (g_{mx1}, g_{mx2}):

$$\Psi_1(g_1, g_2) = \sum_{i=1}^{N-1} f\left(\theta^{N-i} q^1_{N,0} + K_{1i}\left(pg_1 + (1-p)g_2\right)\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) + K_{2i}(1-\delta)\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1\right) + K_{3i}, \quad \eta_1\eta_2, \quad \beta\right)\bar{\rho}_i$$

$$\Psi_2(g_1, g_2) = \sum_{i=1}^{N-1} f\left(\theta^{N-i} + K_{1i}\left((1-p)g_1 + pg_2\right)\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) + K_{2i}(1-\delta) + K_{3i}, \quad \eta_1\eta_2, \quad \beta\right)\bar{\rho}_i.$$

It is easy verify for any l that

$$K_{1l} + K_{2l}(1-\delta) + K_{3l} = K_{2l} + K_{3l} = (1-\theta)\sum_{i=0}^{N-l-1}\theta^i = \left(1-\theta^{N-l}\right);$$

and hence that

$$\theta^{N-l} + K_{1l} + K_{2l}(1-\delta) + K_{3l} = 1.$$

Thus for any $\mathbf{q}^1_{ex} \le \mathbf{1}$ we[9] have:

$$\begin{aligned} \theta^{N-l} + \quad & K_{1l}\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) + K_{2l}(1-\delta) + K_{3l} \le 1 \\ \theta^{N-l} q^1_{N,0} + \quad & K_{1l}\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) + K_{2l}(1-\delta)\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1\right) + K_{3l} \le 1. \end{aligned} \tag{31}$$

Case 1 When $\mathbf{q}^1_{ex} < \mathbf{1}$: When $\mathbf{q}^1_{ex} < 1$, $\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1\right) < 1$ as well as $\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) < 1$ and so we have strict inequality in (31) and thus $\Psi_j(1,1) < 1$ for each j. Consider $j = 1$ without loss of generality. Thus $\Psi_1(1, g_2) < 1$ for any $g_2 \le 1$. Consider the one-variable function $g \to \Psi_1(g, g_2)$, represented by

$$\Psi_1^{g_2}(g) := \Psi_1(g, g_2),$$

for any fixed g_2, which is clearly a continuous and monotone function. Let $id(g) := g$ represent the identity function. From the definition of Ψ clearly $\Psi_1^{g_2}(0) > 0$ for any g_2. Hence $\Psi_1^{g_2}(0) - id(0) > 0$ while $\Psi_1^{g_2}(1) - id(1) < 0$. Thus by intermediate value theorem as applied to the (continuous) function $\Psi_1^{g_2}(.) - id(.)$, there exists at least one point at which it crosses the 45-degree line, the straight line through origin (0,0) and (1,1). Note that the intersection points of this 45-degree line and a function are precisely the fixed points of that function.

It is easy to verify that the derivative of the function $\Psi_1^{g_2}$ (partial derivative of Ψ_1 with respect to the second variable) is positive. Thus $\Psi_1^{g_2}(.)$ for any fixed g_2 is continuous increasing strict convex function. If $\Psi_1^{g_2}(.)$ function were to cross 45-degree line more than once before reaching $\Psi_1^{g_2}(1) < 1$ at 1, then it would have to cross the 45-degree line three times (recall $\Psi_1^{g_2}(0) > 0$). However this is not possible because any strict convex real function crosses any straight line at maximum twice. Thus there exists exactly one point in interval $[0,1]$ at which $\Psi_1^{g_2}(.)$ crosses 45-degree line, which would be its unique fixed point.

Thus for any g_2 there exists a unique fixed point of the mapping $\Psi_1^{g_2}(.)$ in the interval $[0,1]$ and call the unique fixed point as $\mathbf{g}^*(g_2)$. It is easy to verify that this fixed point is minimizer of the following objective function parametrized by g_2:

$$\min_{g\in[0,1]} \Phi(g, g_2) \text{ with } \Phi(g, g_2) := \left(\Psi_1(g, g_2) - g\right)^2.$$

The function Φ is jointly continuous, convex in (g, g_2) and the domain of optimization is same for all g_2. Further for each g_2 by previous arguments there exists unique optimizer in $[0,1]$. Thus by [15, Maximum Theorem for Convex Functions], the fixed point function $\mathbf{g}^*(.)$ is continuous, and convex function.

We now obtain the overall (two dimensional) fixed point via the solution of the following one-dimensional fixed equation

$$\boldsymbol{\Gamma}(g) := \Psi_2(\mathbf{g}^*(g), g).$$

$$f\left(\theta^{N-i} + K_{1l}\left((1-p)g_1 + pg_2\right)\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right) + K_{2l}(1-\delta) + K_{3l}, \quad \eta_1\eta_2, \quad \beta\right)$$

Let $K_{4l} := \theta^{N-l} + K_{2l}(1-\delta) + K_{3l}$ and $K_{5l} := K_{1l}\bar{\mathbf{g}}\left(\mathbf{q}^1_{ex}, \eta_1(1-\eta_2)\right)$. With these definitions:

$$\boldsymbol{\Gamma}(g) = \sum_{l=1}^{N-1} f\left(K_{4l} + K_{5l}\left((1-p)\mathbf{g}^*(g) + pg\right), \ \eta_1\eta_2, \beta\right)\bar{\rho}_l$$

[9]Here $\le$ represents the usual partial order between two Euclidean vectors, i.e., $\mathbf{a} < \mathbf{b}$ if and only if $a_i < b_i$ for all i and $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for all i .

Consider any $0 \le \gamma, g, g' \le 1$ and by convexity of $\mathbf{g}^*$ and monotonicity of Ψ_2 we have

$$
\begin{aligned}
\boldsymbol{\Gamma}(\gamma g + (1-\gamma) g') &= \sum_{l=1}^{N-1} f\Big(K_{4l} + K_{5l}\Big((1-p)\mathbf{g}^*(\gamma g + (1-\gamma)g') + p\Big[\gamma g + (1-\gamma) g'\Big]\Big), \eta_1\eta_2, \beta\Big)\bar{\rho}_l \\
&\le \sum_{l=1}^{N-1} f\Big(K_{4l} + K_{5l}\Big((1-p)\Big[\gamma \mathbf{g}^*(g) + (1-\gamma)\mathbf{g}^*(g')\Big] + p\Big[\gamma g + (1-\gamma) g'\Big]\Big), \eta_1\eta_2, \beta\Big)\bar{\rho}_l \\
&= \sum_{l=1}^{N-1} f\Big(K_{4l} + K_{5l}\Big(\gamma\Big[(1-p)\mathbf{g}^*(g) + pg\Big] + (1-\gamma)\Big[(1-p)\mathbf{g}^*(g') + pg'\Big]\Big), \eta_1\eta_2, \beta\Big)\bar{\rho}_l \\
&\overset{\text{By convexity of } f}{\le} \sum_{l=1}^{N-1} \Big(\gamma f\Big(K_{4l} + K_{5l}\Big[(1-p)\mathbf{g}^*(g) + pg\Big],\ \eta_1\eta_2, \beta\Big) \\
&\qquad + (1-\gamma) f\Big(K_{4l} + K_{5l}\Big[(1-p)\mathbf{g}^*(g') + pg'\Big],\ \eta_1\eta_2, \beta\Big)\Big)\bar{\rho}_l \\
&= \gamma \boldsymbol{\Gamma}(g) + (1-\gamma)\boldsymbol{\Gamma}(g').
\end{aligned}
$$

This shows that $\boldsymbol{\Gamma}$ is convex, further we have $\boldsymbol{\Gamma}(1) < 1$ and $\boldsymbol{\Gamma}(0) > 0$. Note here that $\mathbf{g}^*(0) > 0$ because $\Psi_1(0,0) > 0$. Thus using similar arguments as before we establish the existence of unique fixed point g_2^* for function $\boldsymbol{\Gamma}$. Therefore $(\mathbf{g}^*(g_2^*), g_2^*)$ represents the unique fixed point, in unit cube $[0,1]^2$, of the two dimensional function Ψ. This establishes the existence and uniqueness of extinction probabilities (g_{12}, g_{21}).

The uniqueness of other extinction probabilities is now direct from equation (30).

Case 2 When $\mathbf{q}_{ex}^1 = 1$: Consider that we begin with one of the following three TLs: one exclusive type $(l,0)$, one mixed type $(l,l+1)$ or one mixed type $(l+1,l)$. Consider the scenario in which the CP-1 population gets extinct at the first transition epoch itself, when started with one $(l,0)$ type. This can happen (see (7)) if one of the following two events occur: a) the TL does not view CP1-post (w.p. r_l); or b) the TL shares CP1-post to none (0) of its friends. In either of the two events the CP-1 population gets extinct even when started with mixed TLs $(l,l+1)$ or $(l+1,l)$. Thus the event of extinction at first transition epoch starting with one $(l,0)$ TL implies extinction at first transition epoch when started with either one $(l,l+1)$ TL or one $(l+1,l)$ TL. Say the number of shares at first transition epoch were non-zero and say they equal x_i of type $(i,0)$ for each i when started with one $(l,0)$ type TL. This proof is given under extra assumption that $\rho_N = 0$ and that $\bar{\rho}_i = \rho_i$. *We assume the following is the scenario under assumption.* When we start with mixed type $(l,l+1)$ (or $(l+1,1)$ type respectively), CP1-post is shared with $\sum_i x_i$ number of Friends as when started with exclusive $(l,0)$ type. Out of these some are now converted to mixed TLs because the parent TL also shares CP-2 post. And a converted type $(i,0)$ offspring becomes $(i,i+1)$ offspring w.p. p (w.p. $(1-p)$ respectively) and $(i+1,i)$ w.p. $(1-p)$ (w.p. $(1-p)$ respectively). When started with mixed type $(l+1,l)$ it is possible that some out of $\sum x_i$ shares of CP1 post are discarded (w.p. δ) because the TL would have viewed the CP2-post first and would be discouraged to view CP1-post. Thus in either case, with or without extinction at first transition epoch, the resulting events are inclined towards survival with bigger probability when started with one exclusive $(l,0)$ type than when started with either of the mixed type TLs. Basically the aforementioned arguments can be applied recursively to arrive to this conclusion and hence the probabilities of extinctions satisfy the following inequalities:

$$q_{l,0}^1 \le q_{l,l+1}^1 \ \text{ and } \ q_{l,0}^1 \le q_{l+1,l}^1 \ \text{for any } l < N-1.$$

Further with $\mathbf{q}_{ex}^1 = 1$, it easy to verify that $\Psi_i(1,1) = 1$ for $i = 1$ as well as 2. Thus we have unique extinction probabilities, $\mathbf{q}_{mx1} = \mathbf{1}$ and $\mathbf{q}_{mx2} = \mathbf{1}$. ∎

Appendix C: Time evolution of expected number of shares in competitive Scenario

Let τ represent the time period at which the first transition occurs to the TL-CTBP and note that τ is exponentially distributed with parameter $\lambda + \nu$. Without loss of generality consider CP-1, neglect the superscript 1 in the following computation. We represent the largest eigenvalue of Lemma 1 corresponding to CP-1 by α_1. From (12), when started with exclusive CP types we have following structure: $y_{l,0}(t) = y_l(t) = d_{l,0} + e_{l,0}e^{\alpha_1 t}$. The co-efficients $\{d_{l,0}\}$ $\{e_{l,0}\}$ are (approximately) provided in (12).

To begin with, we assume the following structure for fixed point waveform, $y_{l,k}(t) = d_{l,k} + e_{l,k}e^{\alpha_1 t} + g_{l,k}e^{\bar{\alpha} t}$ for all l,k and we will show that $g_{l,k} = 0$ under certain conditions. We will require a term proportional to $e^{\alpha_1 t}$. Basically without this term, because of exclusive CP terms the waveform can't be a fixed point waveform.

Conditioning on the events of the first transition, as before, and now conditioning also on the time instance of the first transition and then taking the expectation, we obtain the following FP equations:

$$
\begin{aligned}
y_{l,l+1}(t) &= \mathbb{1}_{l<N-1}\theta\int_0^\infty y_{l+1,l+2}(t-\tau)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau + \mathbb{1}_{l=(N-1)}\theta\int_0^\infty y_{N,0}(t-\tau)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau + (1-\theta)(1-\delta)(1-r_l)0 \\
&\quad +(1-\theta)(1-\delta)r_l m\eta_1\left(1+\int_0^\infty \sum_{i<N}\bar\rho_i y^1_{i,0}(t-\tau)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau\right) \\
&\quad +(1-\theta)\delta r_l m\eta_1\eta_2\left(1+\int_0^\infty \sum_{i<N}\bar\rho_i\Big((1-p)y^1_{i,i+1}(t-\tau)+py^1_{i+1,i}(t-\tau)\Big)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau\right) \\
&\quad +(1-\theta)\delta r_l m\eta_1(1-\eta_2)\left(1+\int_0^\infty \sum_{i<N}\bar\rho_i y_{i,0}(t-\tau)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau\right) + (1-\theta)\delta(1-r_l)0.
\end{aligned}
$$

Since, for example

$$
\int_0^\infty y_{l,k}(t-\tau)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau = \int_0^\infty \Big(d_{l,k}+e_{l,k}e^{\alpha_1(t-\tau)}+g_{l,k}e^{\bar\alpha(t-\tau)}\Big)(\lambda+\nu)e^{-(\lambda+\nu)\tau}d\tau = d_{l,k}+e_{l,k}e^{\alpha_1 t}\alpha_{1\lambda}+g_{l,k}e^{\bar\alpha t}\bar\alpha_\lambda \text{ with}
$$

$$
\bar\alpha_\lambda := \frac{\lambda+\nu}{\lambda+\nu+\bar\alpha} \text{ and } \alpha_{1\lambda} := \frac{\lambda+\nu}{\lambda+\nu+\alpha_1}, \tag{32}
$$

the above equation simplifies to:

$$
\begin{aligned}
y_{l,l+1}(t) &= \mathbb{1}_{l<N-1}\theta\Big(d_{l+1,l+2}+e_{l+1,l+2}e^{\alpha_1 t}\alpha_{1\lambda}+g_{l+1,l+2}e^{\bar\alpha t}\bar\alpha_\lambda\Big) + \mathbb{1}_{l=(N-1)}\theta\Big(d_{N,0}+e_{N,0}e^{\alpha_1 t}\alpha_{1\lambda}\Big) \\
&\quad +(1-\theta)(1-\delta)r_l m\eta_1\left(1+\sum_{i<N}\bar\rho_i\Big(d_{i,0}+e_{i,0}e^{\alpha_1 t}\alpha_{1\lambda}\Big)\right) \\
&\quad +(1-\theta)\delta r_l m\eta_1\eta_2\left(1+\sum_{i<N}\bar\rho_i\Big((1-p)\Big(d_{i,i+1}+e_{i,i+1}e^{\alpha_1 t}\alpha_{1\lambda}+g_{i,i+1}e^{\bar\alpha t}\bar\alpha_\lambda\Big)+p\Big(d_{i+1,i}+e_{i+1,i}e^{\alpha_1 t}\alpha_{1\lambda}+g_{i+1,i}e^{\bar\alpha t}\bar\alpha_\lambda\Big)\Big)\right) \\
&\quad +(1-\theta)\delta r_l m\eta_1(1-\eta_2)\left(1+\sum_{i<N}\bar\rho_i\Big(d_{i,0}+e_{i,0}e^{\alpha_1 t}\alpha_{1\lambda}\Big)\right).
\end{aligned}
$$

Recall $c_1 = (1-\theta)m\eta_1$ and define the following:

$$
\begin{aligned}
\bar d_{mx1} &:= c_1\left(1+\sum_{i=1}^{N-1}\Big(\delta(1-\eta_2)d_{i,0}+\delta\eta_2(1-p)d_{i,i+1}+\delta\eta_2 p d_{i+1,i}+(1-\delta)d_{i,0}\Big)\bar\rho_i\right) \\
&= c_1 + c_1\Big(\delta(1-\eta_2)+(1-\delta)\Big)d_{ex} + \ c_{mx}\big((1-p)d_{mx1}+pd_{mx2}\big), \text{ and} \\
\bar e_{mx1} &:= c_1\sum_{i=1}^{N-1}\Big(\delta\eta_2(1-p)e_{i,i+1}+\delta\eta_2 p e_{i+1,i}\Big)\bar\rho_i, \\
&:= c_{mx}\big((1-p)e_{mx1}+pe_{mx2}\big), \\
\bar g_{mx1} &:= c_{mx}\big((1-p)g_{mx1}+pg_{mx2}\big), \\
\bar e_{ex,1} &= c_1\Big(\delta(1-\eta_2)+(1-\delta)\Big)e_{ex} \text{ where} \\
d_{mx1} &:= \sum_{i=1}^{N-1} d_{i,i+1}\bar\rho_i \quad d_{mx2} := \sum_{i=1}^{N-1} d_{i+1,i}\bar\rho_i \ \text{ and } d_{ex} := \sum_{i=1}^{N-1} d_{i,0}\bar\rho_i \\
e_{mx1} &:= \sum_{i=1}^{N-1} e_{i,i+1}\bar\rho_i \quad e_{mx2} := \sum_{i=1}^{N-1} e_{i+1,i}\bar\rho_i \\
g_{mx1} &:= \sum_{i=1}^{N-1} g_{i,i+1}\bar\rho_i \quad g_{mx2} := \sum_{i=1}^{N-1} g_{i+1,i}\bar\rho_i \text{ and } \ e_{ex} := \sum_{i=1}^{N-1} e_{i,0}\bar\rho_i
\end{aligned}
$$

With the help of these definitions one can rewrite these as:

$$
y_{l,l+1}(t) = \mathbb{1}_{l<N-1}\theta\Big(d_{l+1,l+2}+e_{l+1,l+2}e^{\alpha_1 t}\alpha_{1\lambda}+g_{l+1,l+2}e^{\bar\alpha t}\Big) + \mathbb{1}_{l=(N-1)}\theta\Big(d_{N,0}+e_{N,0}e^{\alpha_1 t}\alpha_{1\lambda}\Big) \ + r_l\big(\bar d_{mx1}+\bar e_{mx1}e^{\alpha_1 t}\alpha_{1\lambda}+\bar g_{mx1}e^{\bar\alpha t}\bar\alpha_\lambda+\bar e_{ex,1}e
$$

Simplifying we obtain:

$$
y_{l,l+1}(t) = \big(\bar d_{mx1}+\bar e_{mx1}e^{\alpha_1 t}\alpha_{1\lambda}+\bar g_{mx1}e^{\bar\alpha t}\bar\alpha_\lambda+\bar e_{ex,1}e^{\alpha_1 t}\alpha_{1\lambda}\big)\sum_{i=0}^{N-l-1}\theta^i r_{l+i} + \big(d_{N,0}+\alpha_{1\lambda}e_{N,0}e^{\alpha_1 t}\big)\theta^{N-l} \tag{33}
$$

One can write down similar equations for $(l+1,l)$ types

$$
\begin{aligned}
y_{l+1,l}(t) &= \mathbb{1}_{l<N-1}\theta\Big(d_{l+2,l+1}+e_{l+2,l+1}e^{\alpha_1 t}\alpha_{1\lambda}+g_{l+2,l+1}e^{\bar\alpha t}\bar\alpha_\lambda\Big) \\
&\quad +(1-\theta)\delta r_l m\eta_1\eta_2\left(1+\sum_{i<N}\bar\rho_i\Big((1-p)\Big(d_{i+1,i}+e_{i+1,i}e^{\alpha_1 t}\alpha_{1\lambda}+g_{i+1,i}e^{\bar\alpha t}\bar\alpha_\lambda\Big)+p\Big(d_{i,i+1}+e_{i,i+1}e^{\alpha_1 t}\alpha_{1\lambda}+g_{i,i+1}e^{\bar\alpha t}\bar\alpha_\lambda\Big)\Big)\right) \\
&\quad +(1-\theta)\delta r_l m\eta_1(1-\eta_2)\left(1+\sum_{i<N}\bar\rho_i\Big(d_{i,0}+e_{i,0}e^{\alpha_1 t}\alpha_{1\lambda}\Big)\right).
\end{aligned}
$$

Again with

$$
\begin{aligned}
\bar{d}_{mx2} &:= c_1\delta\left(1+\sum_{i=1}^{N-1}\Big((1-\eta_2)d_{i,0}+\eta_2 p d_{i,i+1}+\eta_2(1-p)d_{i+1,i}\Big)\bar{\rho}_i\right)\\
&= c_1\delta + c_1\delta(1-\eta_2)d_{ex} + c_{mx}\left(pd_{mx1}+(1-p)d_{mx2}\right)\\
\bar{e}_{mx2} &:= c_1\delta\sum_{i=1}^{N-1}\left(\eta_2 p e_{i,i+1}+\eta_2(1-p)e_{i+1,i}\right)\bar{\rho}_i\\
&= c_{mx}\left(pe_{mx1}+(1-p)e_{mx2}\right)\\
\bar{g}_{mx2} &= c_{mx}\left(pg_{mx1}+(1-p)g_{mx2}\right)\\
\bar{e}_{ex,2} &= c_1\delta(1-\eta_2)e_{ex}
\end{aligned}
$$

one can rewrite the above set of equations, after simplifying as below:

$$
y_{l+1,l}(t) = \left(\bar{d}_{mx2}+\bar{e}_{mx2}e^{\alpha_1 t}\alpha_{1\lambda}+\bar{g}_{mx2}\bar{\alpha}_\lambda e^{\bar{\alpha}t}+\bar{e}_{ex,2}e^{\alpha_1 t}\alpha_{1\lambda}\right)\sum_{i=0}^{N-l-1}\theta^i r_{l+i} \tag{34}
$$

Multiply equation (34) with $\bar{\rho}_l$ and summing up we obtain (see (46) for definition of O_{mx}):

$$
\sum_l \bar{\rho}_l y_{l+1,l}(t) = d_{mx2}+e_{mx2}e^{\alpha_1 t}+g_{mx2}e^{\bar{\alpha}t} = \left(\bar{d}_{mx2}+\bar{e}_{mx2}e^{\alpha_1 t}\alpha_{1\lambda}+\bar{g}_{mx2}\bar{\alpha}_\lambda e^{\bar{\alpha}t}+\bar{e}_{ex,2}e^{\alpha_1 t}\alpha_{1\lambda}\right)O_{mx}
$$

And multiply equation (33) with $\bar{\rho}_l$ and summing up we get:

$$
\begin{aligned}
&\sum_l \bar{\rho}_l y_{l,l+1}(t) = d_{mx1}+e_{mx1}e^{\alpha_1 t}+g_{mx1}e^{\bar{\alpha}t}\\
&\quad= \left(\bar{d}_{mx1}+\bar{e}_{mx1}e^{\alpha_1 t}\alpha_{1\lambda}+\bar{g}_{mx1}e^{\bar{\alpha}t}\bar{\alpha}_\lambda+\bar{e}_{ex,1}e^{\alpha_1 t}\alpha_{1\lambda}\right)O_{mx}+\left(d_{N,0}+\alpha_{1\lambda}e_{N,0}e^{\alpha_1 t}\right)\sum_l \bar{\rho}_l\theta^{N-l}.
\end{aligned}
$$

Summing up the above two equations we obtain:

$$
\begin{aligned}
&d_{mx1}+d_{mx2}+(e_{mx1}+e_{mx2})e^{\alpha_1 t}+(g_{mx1}+g_{mx2})e^{\bar{\alpha}t}\\
&= \left(\bar{d}_{mx1}+\bar{d}_{mx2}+(\bar{e}_{mx1}+\bar{e}_{mx2})\alpha_{1\lambda}e^{\alpha_1 t}+(\bar{g}_{mx1}+\bar{g}_{mx2})\bar{\alpha}_\lambda e^{\bar{\alpha}t}+(\bar{e}_{ex,1}+\bar{e}_{ex,2})e^{\alpha_1 t}\alpha_{1\lambda}\right)O_{mx}+\left(d_{N,0}+\alpha_{1\lambda}e_{N,0}e^{\alpha_1 t}\right)\sum_l \bar{\rho}_l\theta^{N-l}.
\end{aligned} \tag{35}
$$

The above FP equation is valid for all t. Thus FP solution can be obtained by equating the co-efficient of $e^{\bar{\alpha}t}$ terms on either side of the above equation, and co-efficient of $e^{\alpha_1 t}$ terms on either side of the above equation separately and then equating the left over terms on either side. On comparing the above mentioned coefficients/terms on both sides and simplifying we obtain the following:

$$
d_{mx1}+d_{mx2} = \Big(c_{mx}\left(d_{mx1}+d_{mx2}\right)+c_1+c_1\delta+c_1\Big(2\delta(1-\eta_2)+(1-\delta)\Big)d_{ex}\Big)O_{mx}+d_{N,0}\sum_l\bar{\rho}_l\theta^{N-l},
$$

which implies

$$
d_{mx1}+d_{mx2} = \frac{\Big(c_1+c_1\delta+c_1\Big(2\delta(1-\eta_2)+(1-\delta)\Big)d_{ex}\Big)O_{mx}+d_{N,0}\sum_l\bar{\rho}_l\theta^{N-l}}{1-c_{mx}O_{mx}};
$$

in a similar way

$$
e_{mx1}+e_{mx2} = \alpha_{1\lambda}\Big(c_{mx}\left(e_{mx1}+e_{mx2}\right)+c_1 e_{ex}\left(2\delta(1-\eta_2)+1-\delta\right)\Big)O_{mx}+\alpha_{1\lambda}e_{N,0}\sum_l\bar{\rho}_l\theta^{N-l}, \tag{36}
$$

implying,

$$
e_{mx1}+e_{mx2} = \alpha_{1\lambda}\frac{c_1 e_{ex}\left(2\delta(1-\eta_2)+1-\delta\right)\Big)O_{mx}+\alpha_{1\lambda}e_{N,0}\sum_l\bar{\rho}_l\theta^{N-l}}{1-\alpha_{1\lambda}c_{mx}O_{mx}};
$$

and finally:

$$
g_{mx1}+g_{mx2} = (\bar{g}_{mx1}+\bar{g}_{mx2})\bar{\alpha}_\lambda O_{mx} = c_{mx}(g_{mx1}+g_{mx2})\bar{\alpha}_\lambda O_{mx}
$$

Note that

$$
g_{mx1}+g_{mx2} = c_{mx}(g_{mx1}+g_{mx2})\bar{\alpha}_\lambda O_{mx} \tag{37}
$$

can have a solution only when $\bar{\alpha}_\lambda O_{mx}c_{mx}=1$. In other words it is possible only when $c_{mx}O_{mx}>1$ (from (32) that $\bar{\alpha}_\lambda<1$). Also it is easy to note that without $e^{\alpha_1 t}$ term (in $y_{l,t}(t)$) we could not have solved the fixed point equation, i.e., $e_{mx2}+e_{mx1}$ can not be zero, as seen from (36).

Conclusions:

1) When $c_{mx}O_{mx} < 1$ there is no solution to the above one-dimensional FP equation (37). However in this case $\alpha_{1\lambda}c_{mx}O_{mx} < 1$ which uniquely determines $e_{mx1} + e_{mx2}$ and hence all other $\{e_{l,k}\}$ constants as before. In other words, the only possible fixed point wave form equals:

$$\boxed{y_{l,k}(t) = d_{l,k} + e_{l,k}e^{\alpha_1 t}.}$$

2) In the limit $N \to \infty$ and with special structure $r_l = d_1 d_2^l$, $\alpha_{mx}/(\lambda+\nu) \approx c_{mx}\mathbf{r}.\bar{\boldsymbol{\rho}} - 1 + \theta d_2$ Further for this special case we have

$$O_{mx} = d_1 \sum_l \bar{\rho}_l \sum_{i=0}^{N-l-1} \theta^i d_2^{l+i} = \sum_l \bar{\rho}_l r_l \sum_{i=0}^{N-l-1} \theta^i d_2^i \le \sum_l \bar{\rho}_l r_l \frac{1-(\theta d_2)^{N-l}}{1-\theta d_2} \le \mathbf{r}.\bar{\boldsymbol{\rho}} \frac{1}{1-\theta d_2} \text{ for any } N.$$

In fact as $N \to \infty$

$$O_{mx} \to \mathbf{r}.\bar{\boldsymbol{\rho}} \frac{1}{1-\theta d_2}.$$

(i) In this case when $\alpha_{mx} < 0$, i.e., if mixed gets extinct w.p.1 then $\alpha_{mx} \approx c_{mx}\mathbf{r}.\bar{\boldsymbol{\rho}} - 1 + \theta d_2 \le 0$, i.e., $c_{mx}\mathbf{r}.\bar{\boldsymbol{\rho}} \le (1-\theta d_2)$ and thus $c_{mx}O_{mx} \le c_{mx}\mathbf{r}.\bar{\boldsymbol{\rho}}/(1-\theta d_2) < 1$. Thus there is no solution for one-dimensional FP equation (37) and hence

$$y_{l,k}(t) = d_{l,k} + e_{l,k}e^{\alpha_1 t}.$$

(ii) If $\alpha_{mx} > 0$, mixed types can survive. In this case (for large N)

$$\alpha_{mx}/(\lambda+\nu) \approx c_{mx}\mathbf{r}.\bar{\boldsymbol{\rho}} - 1 + \theta d_2 > 0 \text{ which implies } c_{mx}O_{mx} \approx c_{mx}\frac{\mathbf{r}.\bar{\boldsymbol{\rho}}}{1-\theta d_2} > 1.$$

Therefore, with

$$\bar{\alpha} = \alpha^* := \Big(c_{mx}O_{mx} - 1\Big)(\lambda+\nu)$$

we have:

$$\boxed{y_{l,k}(t) = d_{l,k} + e_{l,k}e^{\alpha_1 t} + g_{l,k}e^{\alpha^* t}.}$$

Here $g_{l,k} = 1 - d_{l,k} - e_{l,k}$ to satisfy the initial condition that $y_{l,k}(0) = 1$, while closed form expressions for $d_{l,k}$ and $e_{l,k}$ are already derived.

3) Point (2).(ii) is true for any general case that satisfies $c_{mx}O_{mx} > 1$.

4) We are yet to understand the form of explosion for other cases, e.g., when $c_{mx}O_{mx} > 1$ as well as $c_{mx}O_{mx}\alpha_{1\lambda} > 1$.

Once we have solutions for $d_{mx1} + d_{mx2}$, $e_{mx1} + e_{mx2}$ and $g_{mx1} + g_{mx2}$ one can derive the other co-efficients as before.
■

APPENDIX D

Expected number of shares: With $\underline{m<1}$, any post gets extinct w.p. 1. Basically we have a moderately active network. We obtain the total expected shares (before extinction) when started with one TL of various types, $\{y_{l,k}^j\}$ with $y_{l,k}^j := E[\lim_{t\to\infty} Y^j(t) | \mathbf{X}(0) = \mathbf{e}_{l,k}]$. These can again be obtained by solving appropriate FP equations These FP equations are obtained by conditioning on the events of first transition, as before. Here we have some additional events depending upon the starting TL. Like when a mixed type TL is subjected to the 'share transition', then we can have shares exclusively of post of one of the CPs, and or shares of both the posts. Whereas when an exclusive CP-type TL is subjected to a 'share transition', only exclusive types are engendered, as in single CP. With 'shift' transition we have similar changes as in single CP case.

Below we obtain the expected shares for CP-1 without loss of generality and hence suppress the superscript j for remaining discussions.

Let $Y_{l,k} = \lim_{t\to\infty} Y_{l,k}(t)$ be the total number of shares of CP1-post, before extinction, when started with one TL of (l,k) type (with $k = l+1$ or $l-1$). Let $y_{l,k} := E\left[Y_{l,k}\right]$ be its expected value. The total number of shares of any CP post is finite on extinction paths. Thus by conditioning on the events of first transition epoch, one can write the following recursive equations for any $l < N$:

$$\begin{aligned} y_{l,l+1} \;=\;& \theta\Big(\mathbb{1}_{\{l<N-1\}} y_{l+1,l+2} + \mathbb{1}_{\{l=N-1\}} y_{N,0}\Big) \\ &+(1-\theta) r_l (1-\delta) m\eta_1 (1 + \mathbf{y}_{ex1}.\bar{\boldsymbol{\rho}}) \\ &+ (1-\theta) r_l \delta m \eta_1 \Big[(1-\eta_2)(1+\mathbf{y}_{ex1}.\bar{\boldsymbol{\rho}}) \\ &\qquad + \eta_2 (1 + p\mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} + (1-p)\mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}})\Big] \end{aligned} \tag{38}$$

where $\mathbf{y}_{ex1} = \{y_{1,0}^1, y_{2,0}^1, \cdots, y_{N-1,0}\}$. And again for any $l < N$,

$$
\begin{aligned}
y_{l+1,l} &= \mathbb{1}_{\{l<N-1\}}\theta y_{l+2,l+1} \\
&+ (1-\theta) r_l \delta m \eta_1 \Big[(1-\eta_2)(1+\mathbf{y}_{ex1}.\bar{\boldsymbol{\rho}}) \\
&\qquad + \eta_2\big(1+(1-p)\mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}}+p\mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}}\big)\Big].
\end{aligned} \tag{39}
$$

One can easily solve the above set of linear equations to obtain the fixed point solution, by first obtaining the solutions for

$$
\begin{aligned}
&\mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}}+\mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}} \text{ with} \\
&\quad \mathbf{y}_{mx1} := \{y^1_{1,2}, y^1_{2,3}, \cdots, y^1_{N-1,N}\} \text{ and} \\
&\quad \mathbf{y}_{mx2} := \{y^1_{2,1}, y^1_{3,2}, \cdots, y^1_{N,N-1}\}.
\end{aligned}
$$

Below we carry out the same for the special case with view probabilities given by: $r_i = d_1 d_2^i$. Recall $c_j = (1-\theta)m\eta_j$ and define the following which will be used only in this subsection:

$$
\begin{aligned}
B_{ex1,\delta} &= c_1\delta(1-\eta_2)\left(1+\mathbf{y}_{ex1}.\bar{\boldsymbol{\rho}}\right), \\
B_{ex1,1-\delta} &= c_1(1-\delta)(1+\mathbf{y}_{ex1}.\bar{\boldsymbol{\rho}}), \\
\bar{C}_{mx1} &= c_{mx}\left(1+\left(p\mathbf{y}_{mx1}+(1-p)\mathbf{y}_{mx2}\right).\bar{\boldsymbol{\rho}}\right) \text{ and} \\
\bar{C}_{mx2} &= c_{mx}\left(1+\left((1-p)\mathbf{y}_{mx1}+p\mathbf{y}_{mx2}\right).\bar{\boldsymbol{\rho}}\right)
\end{aligned} \tag{40}
$$

The first three quantities can be computed from exclusive CP expected number of shares given by equations (13)-(15), while the remaining are obtained by solving the above FP equations. Then we can rewrite the equations (38)-(39) in the following manner for the special case[10] with $r_i = d_1 d_2^i$

$$
\begin{aligned}
y_{l,l+1} &= \theta y_{l+1,l+2} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) d_1 d_2^l \\
&\qquad\qquad \text{for } l<N-1 \text{ and} \\
y_{N-1,N} &= \theta y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) d_1 d_2^{N-1}.
\end{aligned}
$$

Solving these equations using backward recursion:

$$
y_{N-2,N-1} = \theta^2 y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right)\left(\theta d_1 d_2^{N-1} + d_1 d_2^{N-2}\right).
$$

and then continuing in a similar way

$$
y_{N-l,N-l+1} = \theta^l y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) d_1 d_2^{N-l}\left[\sum_{i=0}^{l-1}(\theta d_2)^i\right].
$$

One can rewrite it as the following for any $l<N$:

$$
\begin{aligned}
y_{l,l+1} = &\ \theta^{N-l} y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) d_1 d_2^l\left[\textstyle\sum_{i=0}^{N-l-1}(\theta d_2)^i\right] \\
= &\ \theta^{N-l} y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) d_1 d_2^l \tfrac{1-(\theta d_2)^{N-l}}{1-\theta d_2}.
\end{aligned} \tag{41}
$$

In exactly similar lines for any $l<N$:

$$
y_{l+1,l} = \theta y_{l+2,l+1} + \left(\bar{C}_{mx1}+B_{ex1,\delta}\right) d_1 d_2^l.
$$

This simplifies to the following for any $l<N$:

$$
y_{l+1,l} = \theta^{N-l} + \left(\bar{C}_{mx1}+B_{ex1,\delta}\right) d_1 d_2^l \frac{1-(\theta d_2)^{N-l}}{1-\theta d_2}. \tag{42}
$$

Multiplying the left hand sides of the equations (41) and (42) with $\bar{\rho}_l$ and summing it up we obtain $\mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}}$ and $\mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}}$ respectively:

$$
\mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} = \sum_{l<N}\bar{\rho}_l\theta^{N-l}y_{N,0} + \left(B_{ex1,1-\delta}+\bar{C}_{mx2}+B_{ex1,\delta}\right) O_{mx} \tag{43}
$$

$$
\mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}} = \sum_{l<N}\bar{\rho}_l\theta^{N-l} + \left(\bar{C}_{mx1}+B_{ex1,\delta}\right) O_{mx} \tag{44}
$$

$$
O_{mx} := d_1\sum_l \frac{\left(d_2^l\bar{\rho}_l\right)-(\theta d_2)^N\left(\bar{\rho}_l/\theta^l\right)}{(1-\theta d_2)}. \tag{45}
$$

[10]One can easily write down the equations for general case, but are avoid to simplify the notations.

Note that for general r_l which need not be $d_1 d_2^l$ we will have

$$O_{mx} := \sum_{l<N} \bar{\rho}_l \sum_{i=0}^{N-l-1} \theta^i r_{l+i}. \tag{46}$$

On adding equations (43) and (45)

$$\begin{aligned} \mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} + \mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}} &= \sum_{l<N} \bar{\rho}_l \theta^{N-l}(1+y_{N,0}) \\ &\quad + \left(B_{ex1,1-\delta} + \bar{C}_{mx1} + \bar{C}_{mx2} + 2B_{ex1,\delta}\right) O_{mx}. \end{aligned}$$

This implies using (40)

$$\begin{aligned} \mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} + \mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}} &= \sum_{l<N} \bar{\rho}_l \theta^{N-l}(1+y_{N,0}) \\ &\quad + \left(B_{ex1,1-\delta} + c_{mx}\left(2 + \mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} + \mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}}\right) + 2B_{ex1,\delta}\right) O_{mx}. \end{aligned}$$

Thus we have unique fixed point solution (when $c_{mx} O_{mx} < 1$) for $\mathbf{y}^1_{mx}.\bar{\boldsymbol{\rho}} := \mathbf{y}_{mx1}.\bar{\boldsymbol{\rho}} + \mathbf{y}_{mx2}.\bar{\boldsymbol{\rho}}$, which equals

$$\mathbf{y}^1_{mx}.\bar{\boldsymbol{\rho}} = \frac{\sum_{l<N} \bar{\rho}_l \theta^{N-l}(1+y_{N,0}) + \left(2\left(B_{exj,\delta} + c_{mx}\right) + B_{exj,1-\delta}\right) O_{m_x}}{1 - c_{mx} O_{m_x}}.$$

In the above $y_{N,0}$ and $\mathbf{y}_{ex1}$ of equation (40) can be obtained using single CP expressions (15) and (14).

We obtain further simpler expressions for the special case, when $\bar{\rho}_l = \tilde{\bar{\rho}} \bar{\rho}^l$ (with $\bar{\rho} < 1$) with

$$\tilde{\bar{\rho}} = \frac{1}{\sum_{i=1}^{N-1} \bar{\rho}^i} = \frac{(1-\bar{\rho})}{\bar{\rho}(1-\bar{\rho}^{N-1})}$$

and when $N \to \infty$. Observe that

$$O_{mx} = d_1 \tilde{\bar{\rho}} \sum_{i=1}^{N-1} \frac{\left(d_2 \bar{\rho}\right)^i - (d_2\theta)^N \left(\frac{\bar{\rho}}{\bar{\theta}}\right)^i}{1 - \theta d_2} \to \frac{d_1 d_2 (1-\bar{\rho})}{\left(1 - d_2 \bar{\rho}\right)(1 - \theta d_2)} ,$$

as $N \to \infty$ because

$$\tilde{\bar{\rho}} \sum_{i=0}^{N-1} (d_2\theta)^N \left(\frac{\bar{\rho}}{\theta}\right)^i = (d_2\theta)^N \frac{(1-\bar{\rho}^N)}{(1-\bar{\rho})} \frac{\theta^N - \bar{\rho}^N}{\theta - \rho} \theta^{-N+1} \to 0.$$

In a similar way

$$\tilde{\bar{\rho}} \theta^N \sum_{l<N} (\bar{\rho}/\theta)^l = \tilde{\bar{\rho}} \theta \frac{\theta^N - \bar{\rho}^N}{\theta - \bar{\rho}} \to 0.$$

And $y_{N,0}$ can be bounded as $N \to \infty$, because of (14) and (16). Thus as $N \to \infty$ for any $j = 1,2$:

$$\mathbf{y}^j_{mx}.\bar{\boldsymbol{\rho}} \quad \to \quad \frac{\left(2c_j\delta\left[(1+\mathbf{y}^j_{exj}.\bar{\boldsymbol{\rho}})(1-\eta_{-j}) + \eta_{-j}\right] + c_j(1-\delta)(1+\mathbf{y}^j_{exj}.\bar{\boldsymbol{\rho}})\right) O_{mx}}{1 - c_{mx} O_{mx}}$$

$$\text{with } O_{m_x} \quad \to \quad \frac{d_1 d_2 (1-\bar{\rho})}{\left(1 - d_2\bar{\rho}\right)(1 - \theta d_2)} \text{ and where } -j := 2\mathbb{1}_{\{j=1\}} + 1\mathbb{1}_{\{j=2\}}.$$

Here $\mathbf{y}^j_{exj}$ is given by (13)-(15), and $\{y^j_{l,k}\}$ with $k = l+1$ or $l-1$ can be computed uniquely using $\mathbf{y}^j_{mx}$.