Dynamic Bus dispatch Policies

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***Abstract*—The time gap between two successive buses is called headway in transport systems. In moderate/high frequency routes, with moderate/small headways, the random perturbations (traffic conditions, passenger arrivals, etc.), can alter the headway along the route significantly which possibly leads to bunching of buses. Two or more (successive) buses may start travelling together. Bus bunching results in inefficient and unreliable bus service and is one of the critical problems faced by bus agencies. Thus it is imperative to reduce the bunching possibilities (probability). Another important aspect is the expected time that a typical passenger has to wait before the arrival of its bus. If one increases the headway, the bunching chances might reduce, however, may significantly increase the passenger waiting times. We precisely study this inherent trade-off and derive a bus schedule optimal for a joint cost related to all the trips, which is a weighted combination of the two performance measures.**

We consider a system with Markovian travel times, fluid passenger arrivals and derive dynamic headways which control the bus frequency based on the observed system state. The observation is a delayed information of the time gaps between successive bus arrivals at various stops, corresponding to two earlier (previous to previous) trips. We solve the relevant dynamic programming equations to obtain near-optimal policies, and the approximation improves as the load factor reduces. The near-optimal policy turns out to be linear in previous headway and the (earlier) bus-inter-arrival times. Using Monte Carlo based simulations, we demonstrate that the proposed dynamic policies significantly improve (both) the performance measures, in comparison with the previously proposed partial dynamic policies that only depend upon the headways of the previous trips.

***Index Terms*—Bus bunching; Dynamic programming; Waiting times; Markov decision process; Delayed information.**

I. Introduction

Public transport plays an important role in any system. We consider public transport systems like that of buses, trams, metros, local trains etc, for brevity we refer them as bus transport systems. In fact in the later cases, bunching (buses travelling together) is a major issue. Bus agencies desire to provide the best service to the passengers due to heavy competition from other transport services and would strive hard to reduce/eliminate the bunching possibilities.

We consider a bus-transport system, where the buses travel repeatedly along a fixed route consisting of a fixed number of stops. Each bus starts at the depot, traverses all the stops and returns to the depot, while facilitating the transfer of the (encountered) passengers from their origin to their destination. Typically successive buses are designed to depart at depot, according to a pre-designed time-table. The time period between two successive bus-departs is referred to as headway (at depot).

The randomness of travel times, load conditions, etc., leads to random headways at bus stops. These random delays can lead to bunching of two or more buses; the leading bus can get delayed excessively due to a large number of passengers (possibly a random spurt of arrivals) and or due to heavy random traffic en-route, and the trailing bus may have relatively lesser load which can eventually lead to both of them coming close to each other somewhere along the path.

The larger headways at depot reduce the bus bunching, but leads to an increase in the passenger waiting times. Thus one needs to design the headways optimally, to ensure proper trade-off between these two important performance aspects. Hence, one needs to design T-successive depot headways (with $(T+1)$-trips), and hence a need for a finite horizon headway policy. We considered such optimal headway policies in [1], [7], and those policies depend at maximum on the headways of the previous trips.

One can do much better if one has access to a better system state that is influenced by the random fluctuations governing the system. For example , if one can observe the number of passengers waiting in various stops (at bus arrival instances) or an equivalent information in the previous trips, a better headway policy can be designed using this knowledge. In this paper we consider that the bus-inter-arrival times between various stops (of previous trips) are observable, based on which the headway times of the future trips are decided.

The natural tool to design such policies is the theory of Markov decision processes ([8]). However in this system, one will have access only to delayed information: the headway decision for the current bus has to be made immediately after the previous bus departs the depot, the information related to the previous bus trajectory (no delay) is obviously not available for this decision epoch. Further one may not even have the information about some of the trips, previous to the trip that just started. For notational simplicity we assume that 1-delay information is available and derive the optimal policies. One can easily extend this analysis to any arbitrary delay, we provide some initial suggestions regarding the same, while, the exact analysis with arbitrary delay would be considered for future work.

We obtained closed form expressions for an ε-optimal policy, that is ε-optimal under small load factors. Interestingly the policy is linear in the previous trip headways and the bus-inter-arrival times at various stops of the previous trip (information about which is available). We showed numerically that this dynamic policy has significant improvement in comparison with respect to the optimal policy of [1] that dynamically adapts the headways only based on previous trip headways. This improvement is significant even for considerable load factors (upto 0.5). We used Monte-Carlo based simulations to estimate the two performance measures. Thus one can do much better, if there is a possibility to observe more intimate details related to the previous trips. The observation process might be complicated, but the complexity of the proposed policy is negligible.

Related literature

Bus bunching is a critical issue faced by bus agencies and this problem has been thoroughly investigated over past few decades. dHowever to the best of our knowledge none of the literature studies the important trade-off between the bunching chances and the passenger waiting times (see [7] for details on this observation). Other than [1], [7], none of the papers study/consider the probability of bunching. As already mentioned, work in [1], [7] does not consider the fully dynamic policies.

There is a vast literature that studies other topics related to bus bunching and we discuss a few of them here. Existing control strategies are based on ideas like skipping some bus stops (e.g., [3], [6], [10]), limited boarding (e.g., [4], [5]) or forcibly holding the buses at some stops (e.g., [3], [11], [4]) etc. Skipping some stops/ holding control at intermediate bus-stops are not passenger friendly policies. In this paper, we applied holding control only for the depot.

In [4], authors consider minimizing the total sojourn time (travel time between the boarding stop and the destination stop) of all the passengers. In papers like [4], [5], [11], [9], authors control the error variance between ideal and proposed schedules when the number of buses and stops increases to infinity. They assume no bunching. Stochastic models that optimize the holding times to minimize passenger waiting times (defined via sum of squared headways), using the real-time information, are discuused in [2]. They avoid bus bunching, while maintaining the frequency of the buses as high as possible for the next few trips. In many scenarios with high randomness, it is not possible to completely avoid bunching nor is it possible to adhere to the ideal schedules. In such scenarios, it is rather important to reduce the probability of bunching, and we precisely consider this probability. Further we consider a more realistic definition of the passenger waiting times: as the time difference between their arrival and the arrival of their bus.

II. System Model and Problem description

We consider a bus-transport system in which the buses travel in a loop, traversing the given path of M stops and repeating this for $(T+1)$ number of trips. We begin with the description of the details of the system considered and the required assumptions.

A. Bus travel and (stop) inter-arrival times

Let S_k^i be the time taken by the k-th bus to travel between the stops $(i-1)$ and i. We consider Markovian (correlated) travel times, between any two stops. To be precise we assume that

$$S_k^i = S_{k-1}^i + W_k^i, \text{ for any, } k \geq 0, \text{ and } S_0^i = s^i,$$

where W_k^i is the random difference between two successive travel times and $\{s^i\}_i$ are the sojourn times of the first trip. We assume $\{W_k^i\}_k$ are IID (Independent and Identically distributed) *Gaussian random variables with mean* 0 *and variance* ϵ^2 and this is true for all stops i. Further these are independent across the stops.

The boarding time of (all) the passengers at any stop majorly constitutes the dwell time of the bus. We assume the following:

A.1: Gated service: *Only the passengers that arrived to a stop before the arrival of the bus can board.*

A.2 Parallel boarding and de-boarding: *We neglect the time taken to de-board while computing the dwell times.*

A.3 Fluid arrivals and boarding: *The number of passengers arrived to a stop, during a period* t *equals* λt*, where* $\lambda > 0$ *is the arrival rate.* More details about this modelling is provided in the next section. *The time taken to board* X *number of passengers equals* bX*, where* $b > 0$ *is the boarding rate.*

These assumptions are not very restrictive, and are satisfied by most of the commonly used practices in bus transport systems. The fluid arrivals can be justified owing to Elementary Renewal theorem, and because typical (bus) inter-arrival times at any stop would be significant. Further, usually negligible number of passengers arrive during the boarding process.

We require the following additional assumptions to obtain a tractable solution:

A.4 Surplus number of buses: *For any trip, there exists a bus (at depot) to start after the prescribed headway (without having to wait for the return of the previous buses).*

A.5: Order of buses *is maintained throughout the journey, i.e., even if the buses are bunched the next bus will board and depart the stop after the previous bus.* Thus overtaking of buses does not happen.

A.6: *There is no constraint on capacity of the bus.*

The system may require one or two additional buses to satisfy **A.4**, which is a normal practice to cater for any eventuality. The systems usually operate with small bunching probabilities, as such the event in assumption **A.5** is a rare event. Further this is a common practice in Tram, metro, local train etc., systems. Assumption **A.6** can be restrictive, but is a commonly made assumption in literature and one can consider relaxation of this for the future work.

Because of the above assumptions, the passengers boarding a bus in any trip k and at any stop i equals the ones that arrived during the bus-inter arrival time $I_k^i := A_k^i - A_{k-1}^i$, where A_k^i is the arrival instance of k-bus at stop i. Thus the total number of passengers X_k^i waiting at stop i, at bus arrival instance, equals λI_k^i. Thus the dwell time of $k-$th bus at stop i equals[1]:

$$V_k^i = X_k^i b = b\lambda I_k^i = \rho I_k^i \text{ with } \rho := \lambda b. \tag{1}$$

In the above ρ represents the load factor of the stop[2].

[1]Since bunching is a rare event we neglect the affects of **A**.5 in this part of the modelling.

[2]One can easily consider the case with different load factors at different stops, for notational simplicity we consider the same load factor at all stops.

Let h_k be the headway between $(k-1)$-th and k-th bus at depot. Then the inter-arrival times are given by:

$$I_k^1 = \left(h_k + S_k^1\right) - S_{k-1}^1 = h_k + W_k^1 \text{ for first stop, similarly}$$

$$I_k^i = h_k + \sum_{1\le j\le i} S_k^j + \sum_{1\le j<i} V_k^j - \left(\sum_{j\le i} S_{k-1}^j + \sum_{j<i} V_{k-1}^j\right)$$

$$= h_k + \sum_{1\le j\le i} W_k^j + \rho \sum_{1\le j<i}\left(I_k^j - I_{k-1}^j\right) \text{ for any stop } i. \quad (2)$$

The last equality follows by fluid arrival and gated service assumptions as in (1).

III. Markov Decision Process (MDP)

A. Decision epochs, State and Action spaces

When the $(k-1)$-th bus leaves the depot, the system needs to determine the headway for the k-th bus. A decision at this decision epoch, can depend upon the available system state. As already mentioned, we only have delayed information about the previous bus trajectories, to be precise that of $(k-2)$-th bus. At $(k-1)$-th bus departure (decision epoch), we have access to the following state (see equation (2)):

$$Y_k = (h_{k-1}, \{I_{k-2}^j\}_{1\le j\le M}), \quad (3)$$

the depot-headway of the previous bus and the inter-bus-arrival times at various stops. It is easy to observe that the random vector sequence $\{Y_k\}_k$ forms a Markov Chain, whose evolution depends upon the headway (of the current trip that needs to be decided) and the previous state and hence is a controlled chain.

1) Initial trips (t_0, h_0): Initial trips usually have light load conditions (passenger arrival rates) and are subjected to small variations in traffic, load conditions. We assume that the buses operate during these initial trips (say t_0 of them) at some fixed headway h_0. We consider controlling the depot-headway starting[3] from trip t_0+2, and to keep the notations simple, we refer (t_0+2+k)-th trip by index k. Alternatively one can consider controlling the buses starting from the first trip as in our previous paper ([1]).

B. Performance measures

1) Passenger waiting times: The waiting time of a typical passenger is the time gap between its arrival instance at the stop and the arrival instance of its bus (to the stop). Let $W_{n,k}^i$ be the waiting time of the n-th passenger that boards the k-th bus at stop i. The customer average of the waiting times corresponding to trip k and stop i equal (e.g., [1], [7]):

$$\bar{w}_k^i \triangleq \frac{\bar{W}_k^i}{X_k^i} \text{ with } \bar{W}_k^i := \sum_{n=1}^{X_k^i} W_{n,k}^i.$$

[3]The expressions derived in the next section for 1-delay information are valid only if the load factor (ρ) remains the same for the previous trip also (Lemma 2-3). Hence we make this simplifying assumption. Similarly for d-delay information one requires that load factors remain the same for previous d trips. If required one can easily extend the analysis to varying load factors, and that may be considered in future.

Fluid approximation/arrivals: The passengers are assumed to arrive at regular intervals (of length $1/\lambda$), with λ large. The waiting time of the first passenger during bus inter-arrival period (I_k^i) is approximately[4] $W_{1,k}^i \approx I_k^i$, that of the second passenger is approximately $W_{2,k}^i \approx I_k^i - 1/\lambda$ and so on. Thus as $\lambda \to \infty$, the following (observe it is a Riemann sum) converges:

$$\frac{\bar{W}_k^i}{\lambda} = \frac{1}{\lambda}\sum_{n=0}^{\lambda I_k^i}\left(I_k^i - \frac{n}{\lambda}\right) \to \int_0^{I_k^i}(I_k^i - x)dx = \frac{(I_k^i)^2}{2}. \quad (4)$$

Thus for large λ,

$$\bar{w}_k^i \approx \frac{I_k^i}{2} \text{ because } \quad \bar{W}_k^i \approx \lambda\frac{(I_k^i)^2}{2} \quad \text{and } X_k^i \approx \lambda I_k^i. \quad (5)$$

We *refer this approximation throughout, as the fluid approximation.*

The trip average of the waiting times is given by:

$$\frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M} E[\bar{w}_k^i] = \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{M}\frac{E[I_k^i]}{2},$$

which is one of the components to be optimized. By Lemma 2 (Appendix), the conditional expectation given the state Y_k and the depot-headway decision h_k equals (see (3)):

$$E\left[I_k^i \middle| Y_k, h_k\right] = h_k(1+\rho)^{i-1} - h_{k-1}(i-1)\rho(1+\rho)^{i-2}$$
$$+ \sum_{j=1}^{i-2} I_{k-2}^j (i-j-1)\rho^2(1+\rho)^{i-j-2} \text{ for any } k \ge 1. \quad (6)$$

2) Bunching probability: Starting from the depot, the buses travel on a single route with some headway (time gap between successive arrivals to the same location) between successive buses. If these headways were maintained constant thought their journey, the successive buses would not meet each other. However, because of variability in load/traffic conditions, the above is not always true. A bus can get delayed (to some stop) significantly because of the random fluctuations. The delayed bus has larger number of passengers to board and hence is further delayed for the next stop. The trailing bus has lesser number of passengers and hence departs early from the stop. This continues in the subsequent stops, and there is a possibility of the headway between the two buses becoming zero. This is called *bus bunching.*

Bus bunching increases the waiting times of passengers, further and more importantly wastes the capacity of the trailing buses. Thus the system becomes inefficient. The larger depot headway times decreases the chances of bunching but, however, increases the passenger waiting times. Thus one needs an optimal trade-off.

The bunching probability *is the probability that a bus arrives to a stop before the departure of the previous bus.* It is easy to verify that this is the probability that the dwell time

[4]The residual passenger inter-arrival times at bus-arrival epochs get negligible as $\lambda \to \infty$.

$$\theta^j = \frac{\rho^2}{2}\sum_{i=j+2}^{M}(i-j-1)(1+\rho)^{i-j-2}, \qquad \theta := \frac{1}{2}\sum_{i=1}^{M}(1+\rho)^{i-1} = \frac{(1+\rho)^M-1}{2\rho}, \qquad \bar\theta := \frac{\rho}{2}\sum_{i=1}^{M}(i-1)(1+\rho)^{i-2}$$

$$\psi^j = \rho^2\left((1+\rho)^{M-2-j}(M-j+\rho)\mathbb{1}_{j<M-1} + \mathbb{1}_{j=M-1}\right) \qquad \psi := (1+\rho)^{M-1}, \qquad \bar\psi := \rho(M+\rho)(1+\rho)^{M-2}$$

$$\omega^2 = \epsilon^2\left(\sum_{j=1}^{M}(1+\rho)^{2(M-j)} + \sum_{j=1}^{M-1}\rho^2(1+\rho)^{2(M-1-j)}[M-j+1+\rho]^2+\rho^2\right)$$

$$\eta_{T-k} = \frac{(\theta+\eta_{T-k+1})\bar\psi}{\psi} - \bar\theta - \sum_{j=1}^{M}(1+\rho)^{j-1}\gamma^j_{T-k+1}, \qquad a_{T-k} = \omega\sqrt{-2\log\left(\frac{(\theta+\eta_{T-k+1})\sqrt{2\pi}\omega}{\psi\alpha}\right)},$$

$$\gamma^j_{T-k} = \frac{(\theta+\eta_{T-k+1})}{\psi}\psi^j - \left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma^i_{T-k+1}\right)\mathbb{1}_{j<M-1},$$

$$\delta_{T-k} = \frac{(\theta+\eta_{T-k+1})a_{T-k}}{\psi} + \alpha\left[1-\Phi\left(a_{T-k}\right)\right] + \delta_{T-k+1}.$$

TABLE I: Notations and constants

of $(k-1)$-th bus, V^i_{k-1} given by (1) is greater than the inter arrival time between $(k-1)$ and k-th buses, I^i_k given by (2):

$$P^i_{B_k} = P(V^i_{k-1} > I^i_k) = P(I^i_k - \rho I^i_{k-1} < 0). \tag{7}$$

We consider optimizing the bunching probability of the last stop, as this stop experiences maximum variations. Conditioned on Y_k, h_k the value $I^M_k - \rho I^M_{k-1}$ is Gaussian distributed (see (2)) and from Lemma 3 of Appendix, we have for any $k \ge 1$ (constants are given in Table I):

$$P^\phi\left(I^M_k - \rho I^M_{k-1} < 0 \middle| Y_k, h_k\right) = 1 - \Phi\left(\psi h_k - \bar\psi h_{k-1} + \sum_{j=1}^{M-1}\psi^j I^M_{k-2}\right),$$

$$\Phi(x) := \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\omega^2}}\exp\left(\frac{-t^2}{2\omega^2}\right)dt, \tag{8}$$

where Φ is the cdf of a normal random variable with mean 0 and variance ω^2:

C. The MDP problem

Let $\phi = (d_1 \cdots, d_T)$ be any Markov policy, in that $d_k(y)$ represents the depot headway for the k-th bus if the system observes the state y. *We choose a headway in the range* $[0, \bar h]$ *for some* $\bar h < \infty$. The expected values of the above two cost components depend upon the policy ϕ and the initial trajectories specified by (t_0, h_0) (see sub-section III-A1). To be more specific, given (t_0, h_0) one has probabilistic description of the system state Y_1. Let $E^\phi_{t_0,h_0}$ represent the expectation given the policy and the initial conditions, at times we omit the subscript and superscript to keep notations simple. We have multi-objective (two) optimization and a natural way is to optimize the following weighted combination of the two costs (6), (8):

$$\begin{aligned} J(\phi; h_0, t_0) &= \sum_{k=1}^{T} E^\phi\left[\bar w^i_k\right] + \alpha P^\phi\left(I^M_k - \rho I^M_{k-1} < 0 \middle| Y_k, h_k\right) \\ &= \sum_{k=1}^{T} E^\phi_{t_0,h_0}\left[r(Y_k,h_k)\right] \text{ with} \\ r(Y_k,h_k) &= \sum_{i=1}^{M} E[I^i_k|Y_k,h_k] + \alpha P^\phi\left(I^M_k - \rho I^M_{k-1} < 0 \middle| Y_k, h_k\right) \\ &= h_k\theta - h_{k-1}\bar\theta + \sum_{j=1}^{M-2}\theta^j I^j_{k-2} \\ &\quad + \alpha\left[1 - \Phi\left(\psi_k h_k - \bar\psi_k h_{k-1} + \sum_{j=1}^{M-1}\psi^j_k I^j_{k-2}\right)\right], \end{aligned}$$

where $\alpha > 0$ is the trade-off factor and the constants are in Table I. Our objective is to obtain a policy that optimizes the following for any given (t_0, h_0):

$$v(t_0,h_0) := \inf_\phi J(\phi; t_0, h_0).$$

It is easy to verify that the above value function equals:

$$v(t_0,h_0) = E_{t_0,h_0}\left[v(Y_1)\right],$$

and this can be solved by solving the MDP problem for any given initial condition $y_1 = (h_0, \{I^j_{-1}\}_j)$, i.e., by deriving the value function $v(y_1)$ for any y_1 (e.g., [8]).

IV. Optimal Policies

The optimal policy is obtained by solving dynamic programming (DP) equations using backward induction. The DP equations, for any $k < T$ are given by ([8]):

$$\begin{aligned} v_k(Y_k) &= \inf_{h_k\in[0,\bar h]}\left\{r_k(Y_k,h_k) + E\left[v_{k+1}(Y_{k+1})|Y_k,h_k\right]\right\}, \\ v_{T+1}(Y_{T+1}) &= 0. \end{aligned}$$

From the trip wise running costs (6)-(8), these equations are rewritten as (constants are given in Table I):

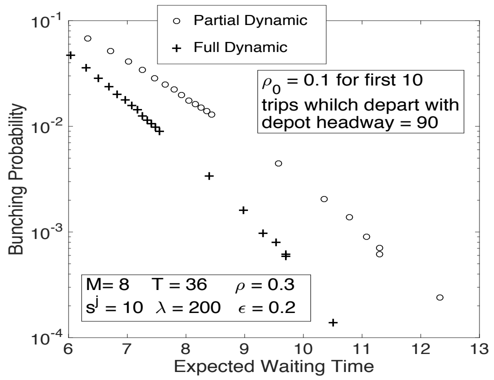

Fig. 1: Comparison between Partial dynamic and Dynamic policies.

Configuration	Bunching probability		Waiting times	
	Dynamic	Partial	Dynamic	Partial
$\epsilon = 0.3,\ \rho = 0.3,\ \alpha$ big	1.95e-02	2.1e-02	18.42	25.29
$\epsilon = 0.3,\ \rho = 0.3,\ \alpha$ small	1.52e-01	1.51e-01	11.33	14.01
$\epsilon = 0.4,\ \rho = 0.3,\ \alpha$ big	1.78e-02	1.77e-02	24.41	34.15
$\epsilon = 0.4,\ \rho = 0.3,\ \alpha$ small	1.36e-01	1.37e-01	14.85	18.95
$\epsilon = 0.2,\ \rho = 0.5,\ \alpha$ big	2.64e-01	2.66e-01	144.23	185.81
$\epsilon = 0.2,\ \rho = 0.5,\ \alpha$ small	6.54e-02	6.51e-02	242.68	380.01

TABLE II: Performance for various configurations with Initial trip details: $\rho_0 = 0.2$, $h_0 = 100$, $t_0 = 12$ and the controlled trip details: $M = 10$ $T = 36$ $s^j = 10$ $\lambda = 200$.

$$
\begin{aligned}
v_k(Y_k) \quad &= \inf_{h_k \in [0,\bar{h}]} \Bigg\{ h_k\theta - h_{k-1}\bar{\theta} + \sum_{j=1}^{M-2} \theta^j I^j_{k-2} \\
&+\alpha \left[1 - \Phi\left(\psi h_k - \bar{\psi} h_{k-1} + \sum_{j=1}^{M-1} \psi^j I^j_{k-2} \right) \right] \Bigg\}. \qquad (9)
\end{aligned}
$$

One can derive optimal policies by solving these DP equations and there are many known numerical techniques to do the same (e.g., [8]). In the following we derive the structure of near optimal policies (closed form expressions) for the case with small load factors:

Theorem 1. *Assume $T > M + 1$. We define the coefficients $\{\eta_k\}_k$, $\{\gamma^j_k\}_{k,j}$ and $\{a_k\}_k$ backward recursively: first set $\eta_{T+1} = 0, \delta_{T+1} = 0$, $\gamma^j_{T+1} = 0$ for all $1 \le j \le M$ and then set the rest of them as in Table I. There exists a $\bar{\rho} > 0$, such[5] that for all $\rho \le \bar{\rho}$:*

$$
\alpha > \frac{(\theta + \eta_{T-k})\sqrt{2\pi}\omega}{\psi} \text{ and } \theta + \eta_{T-k} > 0, \text{ for all } k \ge 0. \qquad (10)
$$

Further for all such ρ, the following is an ε-optimal policy[6] with[7] $\varepsilon = O(\rho)$:

$$
\begin{aligned}
h^*_{T-k}(Y_{T-k}) \quad &= \quad \max\left\{0,\ \min\left\{\bar{h},\ h^{uc}_{T-k}(Y_{T-k})\right\}\right\}, \qquad (11) \\
h^{uc}_{T-k}(Y_{T-k}) \quad &:= \quad \frac{1}{\psi}\left[\bar{\psi} h_{T-k-1} - \sum_{j=1}^{M-1} I^j_{T-k-2}\psi^j + a_{T-k} \right].
\end{aligned}
$$

The expected value function (for any k, Y_{T-k-1}, h_{T-k-1}) equals:

$$
\begin{aligned}
E[v_{T-k}(Y_{T-k})|Y_{T-k-1}, h_{T-k-1}] \quad &= \quad E\Bigg[\eta_{T-k} h_{T-k-1} \\
&- \sum_{j=1}^{M-1} I^j_{T-k-2}\gamma^j_{T-k} + \delta_{T-k} \Bigg| Y_{T-k-1}, h_{T-k-1} \Bigg] + O(\rho).
\end{aligned}
$$

[5]There was a small error in the version submitted to CDC, the second condition was forgotten in equation (10).

[6]The cost under this policy is within ε radius of the optimal cost.

[7]Big O notation: $f = O(\rho)$, as $\rho \to 0$, implies $f(\rho) \le C\rho$ for some constant $C > 0$, for all ρ sufficiently small.

Proof: The details are in Appendix. ■

Thus the ε-optimal policy is affine linear in the previous trip headway and the bus-inter-arrival times. By the above theorem, the policy well approximates the optimal one, as ρ the load factor reduces. We will notice that the policy works well even for nominal load factors (in some examples upto $\rho = 0.5$) in the next section.

V. Numerical analysis

A. Partial Dyanmic policies ([1])

As already mentioned, in our previous work in [1], we derive policies that depend only on previous headways. We refer them as 'partial-dynamic' policies, as they do not consider the random component of the system state Y_k. It is obvious that one can improve with the 'fully' dynamic policies of Theorem 1. In this section, we study the extent of improvement provided by the extra information. Towards this we reproduce the optimal policies of [1], for the purpose of completion. For all load factor $\rho \le \bar{\rho}$ (for some $\bar{\rho} > 0$, the optimal policy is given by ($\mathbf{h}_{T-k} := [h_{T-k-1}, h_{T-k-2} \cdots h_{T-k-M}]$):

$$
\begin{aligned}
h^*_{T-k}(\mathbf{h}_{T-k}) \quad &= \quad \left[-\sum_{l=1}^{M} h_{T-k-l}\psi^p_l + a^p_* \right], \text{ with} \qquad (12) \\
a^p_* \quad &= \quad \frac{\sigma^M_M}{(1+\rho)^{M-1}} \sqrt{-2\log\left(\frac{M\sqrt{2\pi}\sigma^M_M}{2(1-\rho)\alpha} \right)}. \qquad (13) \\
\psi^p_l \quad &= \quad \frac{1}{(1+\rho)^{M-1}} \Bigg((-1)^l \binom{M-1}{l} \rho^l (1+\rho)^{M-1-l} \\
&\qquad - (-1)^{l-1} \binom{M-1}{l-1} \rho^l (1+\rho)^{M-l} \mathbb{1}_{l>0} \Bigg).
\end{aligned}
$$

The constant σ^M_M is available in [1].

B. Experiments

We conduct many Monte Carlo based simulations to compare the proposed dynamic policies with the partial dynamic policies of [1]. We basically generate several sample paths of transport system trajectories, where each sample path is generated using a sample of the random walking times between the stops and the random passenger arrivals at various stops for all the T-trips. We dispatch the buses according to one of the two policies for different values of trade-off factors α and obtain the estimates of the bunching probability and the average passenger waiting times using the sample means.

In Figure 1, we plot the estimates of average passenger waiting times versus the estimates of the bunching probability for different values of α and for both the policies. The details of the experiment are mentioned in the figure itself. We notice a significant improvement with fully dynamic policies. The curve of bunching probability versus expected waiting time obtained with fully dynamic policy is placed well below that with partial dynamic policies. This implies that one can simultaneously improve both the performance measures, when one has access to the more information about the system state. We conducted many more experiments and the observations are similar.

In Table II we consider various system configurations, which is described in the first column. We choose different values of α for the two policies such that the bunching probabilities are almost equal (under both the policies) and these values are reported in next two columns. We then tabulate the corresponding average passenger waiting times in the last two columns. These are the estimates averaged across all the T trips. The different configurations span across different levels of traffic variability (ϵ), different load factors during controllable trips (ρ) and or different level of α/trade-off factors. In all the configurations, we notice a good improvement with fully dynamic policies. Since α were chosen such that the bunching probabilities of both the policies are almost equal, one can study the improvement via the improvement in average passenger waiting times. We observe that improvements are in the range of 21% to 44%.

Extension to arbitrary delays

We think one can easily extend this analysis to arbitrarily delayed information, i.e., for the case when the observation is d-delayed we have, $Y_k = (h_{k-1}, h_{k-2} \cdots, h_{k-d}, \{I^j_{k-d}\}_j)$ with $1 \leq d < M$. Basically the Lemmas 2-3 can easily be extended and we think the rest of the proof can be completed. We would consider this in the immediate future. We conjecture the following would be an ε-optimal policy for some appropriate coefficients $\{\bar{\psi}_r\}$, $\{\psi^j\}$ and $\{a_{T-k}\}$:

$$h^*_{T-k}(y_{T-k}) = \min\Bigg\{\bar{h}, \max\Bigg\{0, \sum_{r=1}^{d} \bar{\psi}_r h_{T-k-r} - \sum_{j=1}^{M-d} I^j_{T-k-d-1}\psi^j + a_{T-k}\Bigg\}\Bigg\}.$$

Note that the above matches with the partial dynamic policy of [1] reproduced in (12) as well as the fully dynamic policy (11) proposed in this paper. It would be further interesting to consider the case where one has partial information (only for some stops) related to some (delayed) trips.

VI. Conclusions

Unlike the popular models considered in literature, we directly studied the inherent trade-off between the two most important aspects of any bus transport system, the bunching possibilities and the passenger waiting times. Further, we formulated a Markov decision processes based problem to derive optimal (depot) dispatch (i.e., headway) policies that depend upon the random state observed at various bus stops of the previous trips. The observation is that of the time gaps between arrivals of the successive buses at the same stop.

We consider systems with Markovian travel times, fluid passenger arrivals and with delayed (one delay) information. The objective function optimized is the sum of a weighted combination of the two performance measures, corresponding to all the trips of the given session. We obtained a near-optimal dynamic policy for small load factors by solving the corresponding finite horizon dynamic programming equations, using backward induction. This policy is linear in previous trip headway and the bus-inter-arrival times corresponding to the earlier trips. We conducted Monte-Carlo based simulations to plot the estimates of the average passenger waiting times and the bunching probability for various trade off factors. We also observed that the proposed dynamic policy performs significantly better than the previously proposed partial dynamic policies of [1]. These partial dynamic policies depend only upon the headways of the previous trips.

References

[1] Stationary and finiate horizon bus schedules for optimal bunching and waiting. Manuscript under preparation http://www.ieor.iitb.ac.in/files/faculty/kavitha/BunchJourn.pdf.

[2] Simon Jonas Youna Berrebi. A real-time bus dispatching policy to minimize headway variance. PhD thesis, 2014.

[3] Cristián E Cortés, Doris Sáez, Freddy Milla, Alfredo Núñez, and Marcela Riquelme. Hybrid predictive control for real-time optimization of public transport systems operations based on evolutionary multi-objective optimization. Transportation Research Part C: Emerging Technologies, 18(5):757–769, 2010.

[4] Felipe Delgado, Juan Muñoz, Ricardo Giesen, and Aldo Cipriano. Real-time control of buses in a transit corridor based on vehicle holding and boarding limits. Transportation Research Record: Journal of the Transportation Research Board, (2090):59–67, 2009.

[5] Felipe Delgado, Juan Carlos Munoz, and Ricardo Giesen. How much can holding and/or limiting boarding improve transit performance? Transportation Research Part B: Methodological, 46(9):1202–1217, 2012.

[6] Liping Fu, Qing Liu, and Paul Calamai. Real-time optimization model for dynamic scheduling of transit operations. Transportation Research Record: Journal of the Transportation Research Board, (1857):48–55, 2003.

[7] M Venkateswararao Koppisetti, Veeraruna Kavitha, and Urtzi Ayesta. Bus schedule for optimal bus bunching and waiting times. In 2018 10th International Conference on Communication Systems & Networks (COMSNETS), pages 607–612. IEEE, 2018.

[8] Martin L Puterman. Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons, 2014.

[9] GE Sánchez-Martínez, HN Koutsopoulos, and NHM Wilson. Real-time holding control for high-frequency transit with dynamics. Transportation Research Part B: Methodological, 83:1–19, 2016.

[10] Aichong Sun and Mark Hickman. The real–time stop–skipping problem. Journal of Intelligent Transportation Systems, 9(2):91–109, 2005.

[11] Yiguang Xuan, Juan Argote, and Carlos F Daganzo. Dynamic bus holding strategies for schedule reliability: Optimal linear control and performance analysis. Transportation Research Part B: Methodological, 45(10):1831–1845, 2011.

VII. Appendix

Lemma 2. *The conditional expectation of inter arrival times given the state Y_k (from (3)) and h_k equals:*

$$E\left[I_k^i \middle| Y_k, h_k\right] = (1+\rho)^{i-1}h_k - (i-1)\rho(1+\rho)^{i-2}h_{k-1} + \sum_{j=1}^{i-2}(i-j-1)\rho^2(1+\rho)^{i-j-2}I_{k-2}^j. \quad \blacksquare \tag{14}$$

Proof: From equation (2), the inter arrival times are Gaussian distributed. The proof is based on mathematical induction. We begin with $i=1$, From equation (2),

$$I_k^1 = h_k + W_k^1, \text{ and hence} E\left[I_k^1 \middle| Y_k, h_k\right] = h_k.$$

Assume the result (equation (14)) is true for $i = n$, and consider $i = n+1$. Again from (2)

$$\begin{aligned} I_k^{n+1} &= h_k + \sum_{1\le j\le n+1} W_k^j + \rho \sum_{1\le j<n+1}\left(I_k^j - I_{k-1}^j\right), \\ &= W_k^{n+1} + (1+\rho)I_k^n - \rho I_{k-1}^n \text{ and then} \end{aligned}$$

$$\begin{aligned} &E\left[I_k^{n+1}\middle| Y_k, h_k\right] \\ &= (1+\rho)E\left[I_k^n \middle| Y_k, h_k\right] - \rho E\left[I_{k-1}^n \middle| Y_k, h_k\right] \\ &= (1+\rho)\Big\{(1+\rho)^{n-1}h_k - (n-1)\rho(1+\rho)^{n-2}h_{k-2} \\ &\quad + \sum_{j=1}^{n-2}(n-j-1)\rho^2(1+\rho)^{n-j-2}I_{k-2}^j\Big\} \\ &\quad -\rho\Big\{(1+\rho)^{n-1}h_{k-1} - \rho\sum_{j=1}^{n-1}(1+\rho)^{n-1-j}I_{k-2}^j\Big\} \\ &= (1+\rho)^n h_k - n\rho(1+\rho)^{n-1}h_{k-1} \\ &\quad + \sum_{j=1}^{n-1}(n-j)\rho^2(1+\rho)^{n-j-1}I_{k-2}^j. \end{aligned}$$

Hence lemma is verified for $i = n+1$. ■

Lemma 3. *For any $2 \le i \le M$ we have:*

$$\begin{aligned} &I_k^i - \rho I_{k-1}^i \\ &= (1+\rho)^{i-1}h_k - (i+\rho)\rho(1+\rho)^{i-2}h_{k-1} + \textstyle\sum_{j=1}^i (1+\rho)^{i-j}W_k^j \\ &\quad - \textstyle\sum_{j=1}^i \left((i-j+1+\rho)\rho(1+\rho)^{i-j-1}\mathbb{1}_{j<i} + \rho\mathbb{1}_{j=i}\right)W_{k-1}^j \\ &\quad + \textstyle\sum_{j=1}^{i-1}\left(\rho^2(1+\rho)^{i-2-j}(i-j+\rho)\mathbb{1}_{j<i-1} + \rho^2\mathbb{1}_{j=i-1}\right)I_{k-2}^j. \end{aligned} \tag{15}$$

The bunching probability of $(k-1)$ and k-th bus at stop i given the state Y_k (from (3))and h_k equals:

$$P\left(I_k^i - \rho I_{k-1}^i < 0 \middle| Y_k, h_k\right) = 1 - \Phi\left(\psi h_k - \bar{\psi}h_{k-1} + \sum_{j=1}^{i-1}\psi^j I_{k-1}^j\right),$$

where the constants are given in Table I. ■

Proof: From the equation (2), the inter arrival times are Gaussian distributed. The proof is based on mathematical induction. We begin with $i=2$, and from equation (2),

$$\begin{aligned} I_k^2 - \rho I_{k-1}^2 &= h_k + W_k^1 + W_k^2 + \rho(I_k^1 - I_{k-1}^1) \\ &\quad -\rho\left(h_{k-1} + W_{k-1}^1 + W_{k-1}^2 + \rho(I_{k-1}^1 - I_{k-2}^1)\right), \\ &= (1+\rho)h_k + (1+\rho)W_k^1 + W_k^2 \\ &\quad -(2+\rho)\rho h_{k-1} - (2+\rho)\rho.W_{k-1}^1 - \rho W_{k-1}^2 + \rho^2 I_{k-2}^1. \end{aligned}$$

Hence the lemma is verified for $i=1$. Assuming the result is true for $i=n$, i.e., the equation (15) is true for all $i \le n$, we prove the result for $i = n+1$. One can write easily from (2),

$$I_k^{n+1} = W_k^{n+1} + (1+\rho)I_k^n - \rho I_{k-1}^n.$$

Hence,

$$\begin{aligned} &I_k^{n+1} - \rho I_{k-1}^{n+1} \\ &= W_k^{n+1} + (1+\rho)I_k^n - \rho I_{k-1}^n - \rho\left(W_{k-1}^{n+1} + (1+\rho)I_{k-1}^n - \rho I_{k-2}^n\right) \\ &= (1+\rho)I_k^n + W_k^{n+1} - (2+\rho)\rho I_{k-1}^n - \rho W_{k-1}^{n+1} + \rho^2 I_{k-2}^n \\ &= (1+\rho)(I_k^n - \rho I_{k-1}^n) + W_k^{n+1} - \rho^2 I_{k-1}^n - \rho W_{k-1}^{n+1} + \rho^2 I_{k-2}^n, \\ I_{k-1}^n &= (1+\rho)^{n-1}h_{k-1} \\ &\quad + \sum_{j=1}^n (1+\rho)^{n-j}W_{k-1}^j - \rho\sum_{j=1}^{n-1}(1+\rho)^{n-1-j}I_{k-2}^j. \end{aligned}$$

From the above equations, one can prove the hypothesis easily for $i = n+1$. ■

Proof of Theorem 1: Condition (10) can be proved in exactly similar lines as in [1, Lemma 5]. Note here that the coefficients are defined in a deterministic manner (do not depend upon any random variables) and hence this proof is easy, when one follows similar logic as in [1, Lemma 5].

The rest of proof is based on backward mathematical induction. We begin with $k=0$. Then the corresponding DP equations are (see Table I, and equation (9)),

$$\begin{aligned} v_T(Y_T) &= \min_{h_T\in[0,\bar{h}]} r_T(h_T, Y_T), \\ &= \min_{h_T\in[0,\bar{h}]}\Bigg\{h_T\theta - h_{T-1}\bar{\theta} + \sum_{j=1}^{M-2}\theta^j I_{T-2}^j \\ &\quad +\alpha\left[1 - \Phi\left(\psi h_T - \bar{\psi}h_{T-1} + \sum_{j=1}^{M-1}\psi^j I_{T-2}^j\right)\right]\Bigg\}. \end{aligned}$$

The above optimization of r_T is like the objective function of Theorem 6. By this theorem, the optimal policy and value function are respectively[8]:

$$\begin{aligned} h_T^*(Y_T) &= \begin{cases} \bar{h} & \text{on } \bar{\mathcal{A}}_T \\ 0 & \text{on } \underline{\mathcal{A}}_T \\ \frac{1}{\psi}\left[\bar{\psi}h_{T-1} - \sum_{j=1}^{M-1} I_{T-2}^j\psi^j + a_T\right], & \text{on } \mathcal{A}_T^c \end{cases} \\ \underline{\mathcal{A}}_T &:= \left\{-\sum_{j=1}^{M-1}\psi^j I_{T-2}^j < -a_T - \bar{\psi}h_{T-1}\right\}, \\ \bar{\mathcal{A}}_T &= \left\{-\sum_{j=1}^{M-1}\psi^j I_{T-2}^j > \bar{h}\psi - a_T - \bar{\psi}h_{T-1}\right\} \text{ and} \end{aligned}$$

with constants, e.g., a_T, η_T, γ_T^j, as in Table I and

[8] In the CDC paper, the negative sign in left hand terms of the sets $\bar{\mathcal{A}}_T$ and $\underline{\mathcal{A}}_T$ is missing.

$$
v_T(Y_T) = \begin{cases} \bar{h}\theta - h_{T-1}\bar{\theta} + \sum_{j=1}^{M-2} \theta^j I_{T-2}^j \\ +\alpha \left[1 - \Phi\left(\psi\bar{h} - \bar{\psi}h_{T-1} + \sum_{j=1}^{M-1} \psi^j I_{T-2}^j\right)\right] \\ \hfill \text{on } \bar{\mathcal{A}}_T \\ -h_{T-1}\bar{\theta} + \sum_{j=1}^{M-2} \theta^j I_{T-2}^j \\ +\alpha \left[1 - \Phi\left(-\bar{\psi}h_{T-1} + \sum_{j=1}^{M-1} \psi^j I_{T-2}^j\right)\right] \\ \hfill \text{on } \underline{\mathcal{A}}_T \\ \eta_T h_{T-1} - \sum_{j=1}^{M-1} I_{T-2}^j \gamma_T^j + \delta_T, \qquad \text{on, } \mathcal{A}_T^c, \end{cases} \tag{16}
$$

with $\mathcal{A}_T := \underline{\mathcal{A}}_T \cup \bar{\mathcal{A}}_T$. Further for any Y_{T-1}, and h_{T-1}:

$$
\begin{aligned}
E_{Y_{T-1}, h_{T-1}} :=& \quad E\left[v_T(Y_T);\ \mathcal{A}_T \middle| Y_{T-1}, h_{T-1}\right] \\
= E_{Y_{T-1}, h_{T-1}} & \left[\eta_T h_{T-1} - \sum_{j=1}^{M-1} I_{T-2}^j \gamma_T^j + \delta_T\ ;\ \mathcal{A}_T\right] + \Gamma_{T-1},
\end{aligned}
$$

$$
\begin{aligned}
\text{with, } \Gamma_{T-1} := E_{Y_{T-1}, h_{T-1}} & \left[\bar{h}\theta - h_{T-1}\bar{\theta} + \sum_{j=1}^{M-2} \theta^j I_{T-2}^j \right. \\
& \left. +\alpha\left(1 - \Phi\left(\psi\bar{h} - \bar{\psi}h_{T-1} + \sum_{j=1}^{M-1} \psi^j I_{T-2}^j\right)\right)\ ; \bar{\mathcal{A}}_T\right] \\
& +E_{Y_{T-1}, h_{T-1}}\left[- h_{T-1}\bar{\theta} + \sum_{j=1}^{M-2} \theta^j I_{T-2}^j \right. \\
& \left. +\alpha\left(1 - \Phi\left(-\bar{\psi}h_{T-1} + \sum_{j=1}^{M-1} \psi^j I_{T-2}^j\right)\right)\ ; \underline{\mathcal{A}}_T\right] \\
& -E_{Y_{T-1}, h_{T-1}}\left[\eta_T h_{T-1} - \sum_{j=1}^{M-1} I_{T-2}^j \gamma_T^j + \delta_T;\ \mathcal{A}_T\right],
\end{aligned} \tag{17}
$$

where by Lemma 5 (from Table I, ρ factors out from some coefficients ($\{\psi^j, \gamma_T^j\}_j$), others are bounded uniformly in ρ):

$$
\begin{aligned}
|\Gamma_{T-1}| &\leq \beta_{T-1}(Y_{T-1}, h_{T-1}) \text{ with} \\
\beta_{T-1}(Y_{T-1}, h_{T-1}) &:= \rho\left|\sum_j c_T^j I_{T-3}^j\right| + \rho\left|\sum_j u_T^j I_{T-3}^j\right| \\
& \quad +\rho c_T^h h_{T-1} + c_T\rho.
\end{aligned} \tag{18}
$$

The coefficients in the definition of β_{T-1} are appropriately defined, converge to finite constants as $\rho \to 0$ and do not depend upon Y_{T-1} or h_{T-1} (see Table I, equation (17)). Thus in all, for any Y_{T-1}, and h_{T-1} (from (16)) we have:

$$
\begin{aligned}
E[v_T(Y_T)|Y_{T-1}, h_{T-1}] &= \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \\
+E_{Y_{T-1}, h_{T-1}} & \left[\eta_T h_{T-1} - \sum_{j=1}^{M-1} I_{T-2}^j \gamma_T^j + \delta_T\ \right].
\end{aligned} \tag{19}
$$

Clearly $\beta_{T-1} = O(\rho)$, policy (11) is optimal for all Y_T and so the result is true for $k = 0$. Now we will prove the result for $k = 1$. Using DP equations

$$
\begin{aligned}
v_{T-1}(Y_{T-1}) &= \inf_{h_{T-1} \in [0, \bar{h}]} J(h_{T-1}; Y_{T-1}) \text{ with} \\
J(h_{T-1}; Y_{T-1}) &:= r_{T-1}(h_{T-1}, Y_{T-1}) + E[v_T(Y_T)|Y_{T-1}, h_{T-1}].
\end{aligned} \tag{20}
$$

Fix Y_{T-1}. By Lemma 4, and equation (19) the objective function $J(\cdot)$ can be split and rewritten as:

$$
\begin{aligned}
J(h_{T-1}) &= \tilde{J}(h_{T-1}) + \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \text{ with} \\
\tilde{J}(h_{T-1}) &:= h_{T-1}(\theta + \eta_T) - h_{T-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma_T^j\right) \\
& \quad + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma_T^i\right) I_{T-3}^j \\
& -\alpha\left(1 - \Phi\left(\psi h_{T-1} - \bar{\psi}h_{T-2} + \sum_{j=1}^{M-1}\psi^j I_{T-3}^j\right)\right) + \delta_T.
\end{aligned} \tag{21}
$$

The function $\tilde{J}(\cdot)$ is like the objective function of Theorem 6. By this theorem, the optimal policy and the optimal objective function,

$$
\tilde{v}_{T-1}(Y_{T-1}) := \inf_{0 \leq h_{T-1} \leq \bar{h}} \tilde{J}(h_{T-1}(Y_{T-1})), \tag{22}
$$

respectively equal:

$$
\tilde{h}_{T-1}^*(Y_{T-1}) = \begin{cases} \bar{h} & \text{on } \bar{\mathcal{A}}_{T-1} \\ 0 & \text{on } \underline{\mathcal{A}}_{T-1} \\ \frac{1}{\psi}\left[\bar{\psi}h_{T-2} - \sum_{j=1}^{M-1} I_{T-3}^j\psi^j + a_{T-1}\right], & \text{on } \mathcal{A}_{T-1}^c \end{cases}
$$

$$
\begin{aligned}
\mathcal{A}_{T-1} &:= \underline{\mathcal{A}}_{T-1} \cup \bar{\mathcal{A}}_{T-1}, \\
\underline{\mathcal{A}}_{T-1} &:= \left\{-\sum_{j=1}^{M-1}\psi^j I_{T-3}^j < -a_{T-1} - \bar{\psi}h_{T-2}\right\} \\
\bar{\mathcal{A}}_{T-1} &= \left\{-\sum_{j=1}^{M-1}\psi^j I_{T-3}^j > \psi\bar{h} - a_{T-1} - \bar{\psi}h_{T-2}\right\},
\end{aligned}
$$

with constants, e.g., a_{T-1}, η_{T-1}, γ_{T-1}^j, as in Table I and

$$
\tilde{v}_{T-1}(Y_{T-1}) = \begin{cases} \bar{h}(\theta + \eta_T) - h_{T-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma_T^j\right) \\ + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma_T^i\right) I_{T-3}^j \\ -\alpha\left[1 - \Phi\left(\psi\bar{h} - \bar{\psi}h_{T-2} + \sum_{j=1}^{M-1}\psi^j I_{T-3}^j\right)\right] + \delta_T \\ \hfill \text{on } \bar{\mathcal{A}}_{T-1} \\ -h_{T-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma_T^j\right) \\ + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma_T^i\right) I_{T-3}^j \\ -\alpha\left[1 - \Phi\left(-\bar{\psi}h_{T-2} + \sum_{j=1}^{M-1}\psi^j I_{T-3}^j\right)\right] + \delta_T \\ \hfill \text{on } \underline{\mathcal{A}}_{T-1} \\ \eta_{T-1}h_{T-2} - \sum_{j=1}^{M-1} I_{T-3}^j\gamma_{T-1}^j + \delta_{T-1}, \qquad \text{on } \bar{\mathcal{A}}_{T-1}^c. \end{cases}
$$

Further from (20)-(22) and (18), for any Y_{T-1}

$$
\begin{aligned}
v_{T-1}(Y_{T-1}) \leq & \inf_{0 \leq h_{T-1} \leq \bar{h}} \tilde{J}(h_{T-1}(Y_{T-1})) \\
& + \sup_{0 \leq h_{T-1} \leq \bar{h}} |\Gamma_{T-1}(h_{T-1})|, \\
\leq & \quad \tilde{v}_{T-1}(Y_{T-1}) + \varepsilon,
\end{aligned} \tag{23}
$$

with

$$
\begin{aligned}
\varepsilon &:= \sup_{0 \leq h_{T-1} \leq \bar{h}} \beta_{T-1}(h_{T-1}, Y_{T-1}). \\
&= \rho\left|\sum_j c_T^j I_{T-3}^j\right| + \rho\left|\sum_j u_T^j I_{T-3}^j\right| + \rho c_T^h\bar{h} + c_T\rho.
\end{aligned}
$$

Again using (19)-(21) we have:

$$
\begin{aligned}
v_{T-1}(Y_{T-1}) \leq & \left(\tilde{J}(\tilde{h}_{T-1}^*(Y_{T-1})) + \Gamma_{T-1}(\tilde{h}_{T-1}^*(Y_{T-1}))\right) \\
& +\varepsilon - \Gamma_{T-1}(\tilde{h}_{T-1}^*(Y_{T-1})) \\
\leq & \quad J(\tilde{h}_{T-1}^*(Y_{T-1})) + 2\varepsilon.
\end{aligned}
$$

Thus $\tilde{h}^*_{T-1}(Y_{T-1})$ is an ε-optimal policy (note $\varepsilon = O(\rho)$) for any Y_{T-1} and thus (11) is true for $k = 1$.

Further as in the previous step (i.e., step with $k = 0$) one can write (e.g., once again $P(\mathcal{A}_{T-1}) = O(\rho)$, and so on)

$$
\begin{aligned}
&E[\tilde{v}_{T-1}(Y_{T-1})|Y_{T-2}, h_{T-2}] \\
&= E_{Y_{T-2},h_{T-2}}\left[\eta_{T-2}h_{T-2} - \textstyle\sum_{j=1}^{M-1} I^j_{T-3}\gamma^j_{T-1} + \delta_{T-1}\ \right] + \tilde{\Gamma}_{T-2},
\end{aligned}
$$

where $\tilde{\Gamma}_{T-2}$ can be upper-bounded like in (18). Define

$$
\begin{aligned}
\Gamma_{T-2} := {} & E_{Y_{T-2},h_{T-2}}\left[v_{T-1}(Y_{T-1}) - \tilde{v}_{T-1}(Y_{T-1})|Y_{T-2}, h_{T-2}\right] \\
& + \tilde{\Gamma}_{T-2}
\end{aligned}
$$

and note that[9] $$|\Gamma_{T-2}| \le \varepsilon + \left|\tilde{\Gamma}_{T-2}\right|.$$

Recall $\tilde{\Gamma}_{T-2}$ can be upper-bounded like in (18) and using Lemma 5, it is easy to verify the following:

$$\begin{aligned}
|\Gamma_{T-2}| &\le \beta_{T-1}(Y_{T-2}, h_{T-2}) \text{ where} \\
\beta_{T-1}(Y_{T-2}, h_{T-2}) &= \rho\left|\sum_j c^j_{T-1} I^j_{T-4}\right| + \rho\left|\sum_j u^j_{T-1} I^j_{T-4}\right| \\
&\quad + \rho c^h_{T-1} h_{T-2} + c_{T-1}\rho,
\end{aligned}$$

for appropriate coefficients that converge to finite constants as $\rho \to 0$ and which do not depend upon Y_{T-2} or h_{T-2}. This completes the proof for $k = 1$.

The rest of the proof is completed using induction, assume the result is true for $k = n$. Now we will prove the result for $k = n + 1$. Using DP equations

$$\begin{aligned}
v_{T-n-1}(Y_{T-n-1}) &= \inf_{h_{T-n-1} \in [0,\bar{h}]} J(h_{T-n-1}; Y_{T-n-1}) \text{ with} \\
J(h_{T-n-1}; Y_{T-n-1}) &:= r_{T-n-1}(h_{T-n-1}, Y_{T-n-1}) + E[v_T(Y_{T-n}) | Y_{T-n-1}, h_{T-n-1}].
\end{aligned} \tag{24}$$

Fix Y_{T-n-1}. By Lemma 4, the objective function $J(\cdot)$ can be split and rewritten as:

$$\begin{aligned}
J(h_{T-n-1}) &= \tilde{J}(h_{T-n-1}) + \Gamma_{T-n-1}(Y_{T-n-1}, h_{T-n-1}) \text{ with} \\
\tilde{J}(h_{T-n-1}) &:= h_{T-n-1}(\theta + \eta_{T-n}) - h_{T-n-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma^j_{T-n}\right) + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma^i_{T-n}\right) I^j_{T-n-3} \\
&\quad -\alpha\left(1 - \Phi\left(\psi h_{T-n-1} - \bar{\psi} h_{T-n-2} + \sum_{j=1}^{M-1}\psi^j I^j_{T-n-3}\right)\right) + \delta_{T-n}.
\end{aligned} \tag{25}$$

The function $\tilde{J}(\cdot)$ is like the objective function of Theorem 6. By this theorem, the optimal policy and the optimal objective function,

$$\tilde{v}_{T-n-1}(Y_{T-n-1}) := \inf_{0 \le h_{T-n-1} \le \bar{h}} \tilde{J}(h_{T-n-1}(Y_{T-n-1})), \tag{26}$$

respectively equal:

$$\begin{aligned}
\tilde{h}^*_{T-n-1}(Y_{T-n-1}) &= \begin{cases} \bar{h} & \text{on } \bar{\mathcal{A}}_{T-n-1} \\ 0 & \text{on } \underline{\mathcal{A}}_{T-n-1} \\ \frac{1}{\psi}\left[\bar{\psi} h_{T-n-2} - \sum_{j=1}^{M-1} I^j_{T-n-3}\psi^j + a_{T-n-1}\right], & \text{on } \mathcal{A}^c_{T-n-1} \end{cases} \\
\mathcal{A}_{T-n-1} &:= \underline{\mathcal{A}}_{T-n-1} \cup \bar{\mathcal{A}}_{T-n-1},\ \underline{\mathcal{A}}_{T-n-1} := \left\{-\sum_{j=1}^{M-1}\psi^j I^j_{T-n-3} < -a_{T-n-1} - \bar{\psi} h_{T-n-2}\right\} \\
\bar{\mathcal{A}}_{T-n-1} &= \left\{-\sum_{j=1}^{M-1}\psi^j I^j_{T-n-3} > \psi\bar{h} - a_{T-n-1} - \bar{\psi} h_{T-n-2}\right\},
\end{aligned}$$

with constants, e.g., a_{T-n-1}, η_{T-n-1}, γ^j_{T-n-1}, as in Table I and

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[9] It is clear from (21) that

$$\begin{aligned}
J(h_{T-1}) &= \tilde{J}(h_{T-1}) + \Gamma_{T-1}(Y_{T-1}, h_{T-1}) \\
&\ge \tilde{J}(h_{T-1}) - |\Gamma_{T-1}(Y_{T-1}, h_{T-1})| \\
&\ge \tilde{J}(h_{T-1}) - \sup_{h_{T-1} \in [0,\bar{h}]} |\Gamma_{T-1}(Y_{T-1}, h_{T-1})|, \\
\text{so, } v_{T-1}(Y_{T-1}) &\ge \tilde{v}_{T-1}(Y_{T-1}) - \sup_{h_{T-1} \in [0,\bar{h}]} |\Gamma_{T-1}(Y_{T-1}, h_{T-1})| \\
&\ge \tilde{v}_{T-1}(Y_{T-1}) - \varepsilon,
\end{aligned}$$

and hence from (23), $|v_{T-1} - \tilde{v}_{T-1}| \le \varepsilon$.

$$\tilde{v}_{T-n-1}(Y_{T-n-1}) = \begin{cases} \bar{h}(\theta + \eta_{T-n}) - h_{T-n-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma_{T-n}^j\right) \\ + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma_{T-n}^i\right) I_{T-n-3}^j \\ -\alpha\left[1 - \Phi\left(\psi\bar{h} - \bar{\psi}h_{T-n-2} + \sum_{j=1}^{M-1}\psi^j I_{T-n-3}^j\right)\right] + \delta_{T-n} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \bar{\mathcal{A}}_{T-n-1} \\ \\ -h_{T-n-2}\left(\bar{\theta} + \sum_{j=1}^{M-1}(1+\rho)^{j-1}\gamma_{T-n}^j\right) \\ + \sum_{j=1}^{M-2}\left(\theta^j + \rho\sum_{i=j+1}^{M-1}(1+\rho)^{i-1-j}\gamma_{T-n}^i\right) I_{T-n-3}^j \\ -\alpha\left[1 - \Phi\left(-\bar{\psi}h_{T-n-2} + \sum_{j=1}^{M-1}\psi^j I_{T-n-3}^j\right)\right] + \delta_{T-n} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \underline{\mathcal{A}}_{T-n-1} \\ \\ \eta_{T-n-1}h_{T-n-2} - \sum_{j=1}^{M-1} I_{T-n-3}^j\gamma_{T-n-1}^j + \delta_{T-n-1}, \quad \text{on } \bar{\mathcal{A}}_{T-n-1}^c. \end{cases}$$

Further from (24)-(26) and (18), for any Y_{T-n-1}

$$\begin{aligned} v_{T-n-1}(Y_{T-n-1}) &\leq \inf_{0 \leq h_{T-n-1} \leq \bar{h}} \tilde{J}(h_{T-n-1}(Y_{T-n-1})) + \sup_{0 \leq h_{T-n-1} \leq \bar{h}} |\Gamma_{T-n-1}(h_{T-n-1})|, \\ &\leq \tilde{v}_{T-n-1}(Y_{T-n-1}) + \varepsilon \text{ with} \qquad (27) \\ \varepsilon &:= \sup_{0 \leq h_{T-n-1} \leq \bar{h}} \beta_{T-n-1}(h_{T-n-1}, Y_{T-n-1}). \\ &= \rho\left|\sum_j c_{T-n}^j I_{T-n-3}^j\right| + \rho\left|\sum_j u_{T-n}^j I_{T-n-3}^j\right| + \rho c_{T-n}^h \bar{h} + c_{T-n}\rho. \end{aligned}$$

$$\begin{aligned} v_{T-n-1}(Y_{T-n-1}) &\leq \left(\tilde{J}(\tilde{h}_{T-n-1}^*(Y_{T-n-1})) + \Gamma_{T-n-1}(\tilde{h}_{T-n-1}^*(Y_{T-n-1}))\right) + \varepsilon - \Gamma_{T-n-1}(\tilde{h}_{T-n-1}^*(Y_{T-n-1})) \\ &\leq J(\tilde{h}_{T-n-1}^*(Y_{T-n-1})) + 2\varepsilon. \end{aligned}$$

Thus $\tilde{h}_{T-n-1}^*(Y_{T-n-1})$ is an ε-optimal policy (note $\varepsilon = O(\rho)$) for any Y_{T-n-1} and thus (11) is true for $k = n+1$.

Further as in the previous step (i.e., step with $k=0$) one can write (e.g., once again $P(\mathcal{A}_{T-n-1}) = O(\rho)$, and so on)

$$E[\tilde{v}_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}] \quad = \quad E_{Y_{T-n-2},h_{T-n-2}}\left[\eta_{T-n-2}h_{T-n-2} - \sum_{j=1}^{M-1} I_{T-n-3}^j\gamma_{T-n-1}^j + \delta_{T-n-1}\right] + \tilde{\Gamma}_{T-n-2},$$

where $\tilde{\Gamma}_{T-n-2}$ can be upper-bounded like in (18). Define

$$\Gamma_{T-n-2} := E_{Y_{T-n-2},h_{T-n-2}}\left[v_{T-n-1}(Y_{T-n-1}) - \tilde{v}_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}\right] + \tilde{\Gamma}_{T-n-2}$$

and note that

$$|\Gamma_{T-n-2}| \leq \varepsilon + \left|\tilde{\Gamma}_{T-n-2}\right|.$$

Using the above definition we have:

$$E[v_{T-n-1}(Y_{T-n-1})|Y_{T-n-2}, h_{T-n-2}] \quad = \quad E_{Y_{T-n-2},h_{T-n-2}}\left[\eta_{T-n-2}h_{T-n-2} - \sum_{j=1}^{M-1} I_{T-n-3}^j\gamma_{T-n-1}^j + \delta_{T-n-1}\right] + \Gamma_{T-n-2}.$$

Recall $\tilde{\Gamma}_{T-n-2}$ can be upper-bounded like in (18) and using Lemma 5, it is easy to verify the following:

$$\begin{aligned} |\Gamma_{T-n-2}| &\leq \beta_{T-n-1}(Y_{T-n-2}, h_{T-n-2}) \text{ where} \\ \beta_{T-n-1}(Y_{T-n-2}, h_{T-n-2}) &= \rho\left|\sum_j c_{T-n-1}^j I_{T-n-4}^j\right| + \rho\left|\sum_j u_{T-n-1}^j I_{T-n-4}^j\right| + \rho c_{T-n-1}^h h_{T-n-2} + c_{T-n-1}\rho, \end{aligned}$$

for appropriate coefficients that converge to finite constants as $\rho \to 0$ and which do not depend upon Y_{T-n-2} or h_{T-n-2}. This completes the proof for $k = n+1$. ■

Lemma 4. *For any $k \geq 1$, Y_{T-k} and h_{T-k}*

$$\begin{aligned} \tilde{J}(h_{T-k}, Y_{T-k}) &= r_{T-k}(h_{T-k}, Y_{T-k}) \\ +E_{Y_{T-k},h_{T-k}} &\left[\eta_{T-k+1} h_{T-k} - \sum_{j=1}^{M-1} I^j_{T-k-1} \gamma^j_{T-k+1} + \delta_{T-k+1} \right]. \end{aligned}$$

Proof: The coefficients are defined recursively to satisfy the above, and it can be verified using Table I. ■

Lemma 5. *For any $k \geq 1$, Y_{T-k}, h_{T-k} and the coefficients*

$$P\left(\mathcal{A}_{T-k+1} \Big| Y_{T-k}, h_{T-k}\right) \leq 2\exp(-(C/\rho)^2 + C/\rho), \tag{28}$$

$$E\left[\left| \sum_j c_j I^j_{T-k+1} \right| \Big| Y_{T-k}, h_{T-k} \right] \leq |\sum_j cc^j I^j_{T-k}| + cc^h h_{T-k} + cc,$$

$$E\left[\sum_j c_j I^j_{T-k+1} \Big| Y_{T-k}, h_{T-k} \right] \leq |\sum_j cc^j I^j_{T-k}| + cc^h h_{T-k} + cc. \tag{29}$$

where the coefficients on the right hand side are appropriately defined. Note that the upper bound in (28) is clearly $O(\rho)$. ■

Proof: For (28), it suffices to prove the result for the following probability for any $C > 0$:

$$p_{\mathcal{A}} := P\left(\sum_j \bar{e}^j I^j_{T-k-1} > \frac{C}{\rho} \Big| Y_{T-k}, h_{T-k} \right)$$

Let

$$\mu := \sum_j e^j I^j_{T-k-1} + e^h h_{T-k} = E\left[\sum_j \bar{e}^j I^j_{t-2} \Big| Y_{T-1}, h_{T-1} \right]$$

represent the conditional mean. Thus we need to prove that for a Gaussian random with mean μ the above implication:

$$\begin{aligned} p_{\mathcal{A}} &= \int_{C/\rho}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\omega^2}\right) \frac{dt}{\sqrt{2\pi\omega^2}} \\ &= \int_{(C/\rho-\mu)/\omega}^{\infty} \exp\left(-\frac{t^2}{2}\right) \frac{dt}{\sqrt{2\pi}} \\ &\leq 2\exp\left(-\frac{(C/\rho-\mu)^2}{w^2}\right) \\ &= 2\exp(-(C/\rho)^2)\exp(-\mu^2 + 2\mu C/\rho) \leq 2\exp(-(C/\rho)^2 + C/\rho). \end{aligned}$$

For (29), first observe that $\sum_j c_j I^j_{T-k+1}$ conditioned on (Y_{T-k}, h_{T-k}) is a Gaussian random variable and let its conditional expected value be:

$$\tilde{\mu} = E\left[\sum_j c_j I^j_{T-k+1} \Big| Y_{T-k}, h_{T-k} \right]$$

and note further that $\tilde{\mu}$ is linear in (Y_{T-k}, h_{T-k}). Let ω be the corresponding variance. Using the above definitions (without loss of generality when $\tilde{\mu} > 0$),

$$\begin{aligned} E\left[\left| \sum_j c_j I^j_{T-k+1} \right| \Big| Y_{T-k}, h_{T-k} \right] &= \int_{-\infty}^{\infty} |t| \exp\left(-\frac{(t-\tilde{\mu})^2}{2\omega^2}\right) \frac{dt}{\sqrt{2\pi\omega^2}} \\ &= \int_{-\infty}^{\infty} |t+\tilde{\mu}| \exp\left(-\frac{t^2}{2\omega^2}\right) \frac{dt}{\sqrt{2\pi\omega^2}} \\ &\leq 2\int_0^{\infty} t \exp\left(-\frac{t^2}{2\omega^2}\right) \frac{dt}{\sqrt{2\pi\omega^2}} + \tilde{\mu}. \end{aligned}$$ ■

Theorem 6. *Let g be function of the following type:*

$$g(h,\mathbf{Y}) = \bar{z}h - \hat{z}h_{-1} + \sum_{j=1}^{M-2} I^j \bar{z}^j + \alpha\left(1-\Phi\left(h\tilde{z} - \hat{\tilde{z}}h_{-1} + \sum_{j=1}^{M-1}\tilde{z}^j I^j\right)\right) + \delta \text{ with } \mathbf{Y} = [h_{-1}, I^1, I^2, \ldots, I^M],$$

with $\bar{z} > 0$, $\alpha > 0$ and $\tilde{z} > 0$. Consider the following optimization problem:

$$g^*(\mathbf{Y}) := \min_{h\in[0,\bar{h}]} g(h,\mathbf{Y}) \text{ and } h^* := \arg\min_h g(h,\mathbf{Y}).$$

Then there exists an unique optimizer to this problem and the optimizer is given by:

$$h^* = \begin{cases} 0 & \text{if } \hat{\tilde{z}}h_{-1} - \sum_{j=1}^{M-1} I^j \tilde{z}^j < -a \\ \bar{h} & \text{if } \bar{h}\tilde{z} - \hat{\tilde{z}}h_{-1} + \sum_{j=1}^{M-1} I^j \tilde{z}^j < a \\ \frac{1}{\tilde{z}}\left[\hat{\tilde{z}}h_{-1} - \sum_{j=1}^{M-1} I^j \tilde{z}^j + a\right], \text{ with } a := \sqrt{-2\log\left(\frac{\bar{z}\sqrt{2\pi}}{\tilde{z}\alpha}\right)} & \text{else.} \end{cases} \tag{30}$$

Further,

$$\begin{aligned} g^*(\mathbf{Y}) &= \begin{cases} g(0,\mathbf{Y}) & \text{if } \hat{\tilde{z}}h_{-1} - \sum_{j=1}^{M-1} I^j \tilde{z}^j < -a \\ g(\bar{h},\mathbf{Y}) & \text{if } \bar{h}\tilde{z} - \hat{\tilde{z}}h_{-1} + \sum_{j=1}^{M-1} I^j \tilde{z}^j < a \\ \hat{\tilde{z}}h_{-1} - \sum_{j=1}^{M-1} I^j \tilde{\psi}_j + \delta^* & \text{else, with} \end{cases} \\ \tilde{\psi}_j &= \frac{\tilde{z}^j\bar{z} - \bar{z}^j\tilde{z}}{\tilde{z}}, \qquad \delta^* = \alpha\left[1-\Phi(a)\right] + \frac{\bar{z}a}{\tilde{z}} + \delta. \end{aligned}$$

Proof: Consider the objective function and for notational simplicity rename h_{-1} as I^0 and $-\hat{z} := \bar{z}^0$ and $-\hat{\tilde{z}} := \tilde{z}^0$ and consider

$$g^*(\mathbf{Y}) = \min_{h\in[0,\bar{h}]}\left\{\bar{z}h + \sum_{j=0}^{M-2} I^j\bar{z}^j + \alpha\left(1-\Phi\left(h\tilde{z} + \sum_{j=0}^{M-1}\tilde{z}^j I^j\right)\right) + \delta\right\}. \tag{31}$$

Let h^* be the optimal policy for equation (31), i.e.,

$$\begin{aligned} h^* &\in \arg\min(g(\mathbf{Y})), \text{ and either} \\ \Rightarrow \frac{d}{dh_M}(g(\mathbf{Y}))\Big|_{h=h^*} &= 0 \text{ or } h^* \text{ is on the boundary} \end{aligned}$$

The first derivative is given by:

$$\frac{d}{dh_M}(g(\mathbf{Y})) = \bar{z} - \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{\left(h\tilde{z} + \sum_{j=0}^{M-1}\tilde{z}^j I^j\right)^2}{2}\right)\tilde{z}.$$

When $\alpha < \frac{\bar{z}\sqrt{2\pi}}{\tilde{z}}$, the first derivative is always positive and hence the optimizer is at lower boundary, i.e., $h^* = 0$. Otherwise, there exists a zero of the derivative as below (with a as defined in hypothesis (30)):

$$\left(h\tilde{z} + \sum_{j=0}^{M-1}\tilde{z}^j I^j\right)^2 = a^2, \text{ i.e., } \Rightarrow h\tilde{z} + \sum_{j=0}^{M-1}\tilde{z}^j I^j = a$$

If the LHS is bigger than a for all positive h (i.e., if $\sum_{j=0}^{M-1}\tilde{z}^j I^j > a$), then the derivative is positive for all $h \geq 0$, hence again $h^* = 0$. On the other hand if

$$\bar{h}\tilde{z} + \sum_{j=0}^{M-1}\tilde{z}^j I^j < a,$$

then the derivative is negative for all $h \in [0,\bar{h}]$, and hence $h^* = \bar{h}$. For the rest of the cases (i.e., when $a - \bar{h}\tilde{z} < \sum_{j=0}^{M-1}\tilde{z}^j I^j < a$) we have the interior optimizer

$$h^* = \frac{a - \sum_{j=0}^{M-1}\tilde{z}^j I^j}{\tilde{z}},$$

as the second derivative (at such h^*) is positive.

Substituting the above, the optimal objective function (when interior point is optimal) equals:

$$\begin{aligned} g^*(\mathbf{Y}) &= h^* \bar{z} + \sum_{j=0}^{M-2} I^j \bar{z}^j + \alpha \left(1 - \Phi \left(h^* \tilde{z} + \sum_{j=0}^{M-1} I^j \tilde{z}^j \right) \right) + \delta \\ &= \sum_{j=0}^{M-1} I^j \frac{\tilde{z}^j \bar{z} - \bar{z}^j \tilde{z}}{\tilde{z}} + \frac{\bar{z} a}{\tilde{z}} + \alpha \left(1 - \Phi \left(a \right) \right) + \delta \\ &= \sum_{j=0}^{M-1} I^j \tilde{\psi}_j + \delta^*, \text{ where,} \end{aligned}$$

$\{\tilde{\psi}_j\}$, and δ^* are as defined in hypothesis. ■