Discrete Time Markov Chains

The setting:

• Notation: S is a countable state space ; P is a $|S| \times |S|$ matrix such that 1) $p_{ij} \geq 0$ 2) $\sum_{J \in S} p_{ij} = 1 \ \forall \ i \in S$, *i.e.*, a stochastic one step transition matrix; λ is a (prob) distribution on S *i.e.*, $\lambda_j \geq 0$ and $\sum_{J \in S} \lambda_j = 1$

• Process $\{X_n\}_{n \geq 0}$ is a S valued DTMC with initial distribution λ and one step transition probability matrix P if 1) $X_0 \sim \lambda$ 2) $P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1} \ldots X_0 = i_0)$ $= P(X_{n+1} = j | X_n = i) = p_{ij}$ for states $j, i, i_{n-1} \ldots i_0$ and $n \geq 0$; call this Markov property.

• (I^{st} Characterization) $\{X_n\}_{n \geq 0}$ is a DTMC with $X_0 \sim \lambda$ and tpm P iff it has a certain conditional independence.

• (II^{nd} Characterization and fdds of DTMC) $\{X_n\}_{n \geq 0}$ is an S valued DTMC with initial distribution λ iff $P(X_n = i_n, X_{n-1} = i_{n-1} \ldots X_0 = i_0) = \lambda_{q_i} p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n}$ for $n \geq 0$ and states $i_0, \ldots i_n$ where P is a $|S| \times |S|$ stochastic matrix and λ is a distribution on S.

• One can spot DTMC when they are driven by noise: Let Z_{n+1} be conditionally independent of $Z_n \ldots Z_1, X_n \ldots X_0$ given X_n, *i.e.*,

$P(Z_{n+1} = k | X_n = i, X_{n-1} = i_{n-1}, \ldots X_0 = i_0, Z_n = k_n \ldots Z_1 = k_1) = P(Z_{n+1} = k | X_n = i), \ n \geq 1$, with the condition prob on RHS independent of n.

Let $f : S \times F \to S$ where F is domain of Z_n. Then, $X_{n+1} = f(X_n, Z_{n+1}), n \geq 0$ is a DTMC with X_0 as initial state and $p_{ij} = P(f(i, z) = j | X_0 = i)$.

Remark : Recall the implicit use of σ fields in the proof done is class.

Exercise : Show that $P(X_{n+k_l} = j_l, X_{n+k_{l-1}} = j_{l-1} \ldots X_{n+k_1} = j_1 | X_n = i, \ldots X_0 = i_0)$ $= P(X_{n+k_l} = j_l, \ldots X_{n+k_1} = j_1 | X_n = i)$ for states $i, j_1, j_2 \ldots j_l$ and positive integers $n, l, k_1 \ldots, k_l \geq 0$.

• Simulating a DTMC: Map the state space, if required, to $\{0, 1, 2, \ldots\}$ and let $\{U_n\}_{n \geq 0}$ be the *i.i.d.* with U_0 uniform on $(0, 1)$. Set $X_0 = \sum_k 1_{\{U_0 \in [\sum_{i=0}^{k-1} \alpha_i \sum_{i=0}^{k} \alpha_i]\}}$ and

$X_{n+1} = \sum j 1_{\{U_{n+1} \in [\sum_{k=0}^{j-1} p_{X_{n,k}} \sum_{k=0}^{j} p_{X_{n,k}}]\}}$ for $n \geq 0$.

By above theorem $\{X_n\}_{n \geq 0}$ is a DTMC with $X_0 \sim \lambda$ and tpm P.

• Aggregation of states: Let $\{X_n\}_{n \geq 0}$ be a DTMC on S with P as tpm. Suppose $\{A_k\}$ partition S and define a process $\{\hat{X}_n\}_{n \geq 0}$ on $\hat{S} = \hat{1}, \hat{2}, \ldots$ by $\hat{X}_n = \hat{k}_n$ iff $X_n \in A_k, \ n \geq 0$.

Then, $\{\hat{X}_n\}_{n \geq 0}$ is a DTMC for any initial dist. λ of $\{X_n\}_{n \geq 0}$ iff $\sum_{j \in A_l} p_{ij}$ is independent of $i \in A_k, \forall k, l$, in which case, $\hat{p}_{\hat{k}, \hat{l}} = \sum_{j \in A_l} p_{ij}$ for any $i \in A_k$ is tpm of $\{\hat{X}_n\}_{n \geq 0}$.

First step analysis:

• The following are first of set of results that follow from first step analysis. First step analysis itself follows by considering a one step delayed DTMC. $\{Y_n\}_{n \geq 0}$: $Y_n = X_{n+1}, n \geq 0$.

• $A \subset S$; $H_A(\omega) = inf_n \{X_n \in A\}$; $h_i^A = P(H^A \leq \infty | X_0 = i) = P_i(H_A \leq \infty)$ and $k_i^A = \sum_{n \in Z^+} n P(H_A = n) + \infty P(H^A = \infty)$ with the convention that $\infty.0 = 0$

• When A is a closed class, to be defined later, this is absorbtion probability.

• $\{h_i^A\}_{i \in S}$ is the minimal nonnegative solution to the linear system of equations

$$h_i^A = 1, \ i \in A, \quad \text{and} \quad h_i^A = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} h_j^A, \ i \notin A.$$

Minimal in componentwise sence: for any other solution $\{x_i\}_{i\in S}$ we have $x_i \geq 0$, $\forall\, i \in A$ and $x_i \geq h_i^A,\ i \in S$.

• This result will be used later in conjunction with results regarding probability of never leaving a set to obtain a criteria for recurrence and transience.

• $\{k_i^A\}_{i\in S}$ is the minimal non negative solution to the system of linear equation: $k_i = 0,\ i \in A$ and $k_i^A = 1 + \sum_{j\notin A} p_{ij} k_j^A,\ i \notin A$.

Stopping times, SMP, *etc.*,

• Recall the examples done in class where $P(x_{\tau+1} = j | x_\tau = i) = 1 \neq y_z = P(x_{n+1} = j | x_n = i)$ where τ is the last time epoch when the process is in a subset A of states.

• A r.v τ taking values $\mathbb{N} \cup \{\infty\}$ is called a stopping time w.r.t a process $\{X_n\}_{n\geq 0}$, if $\forall$ integers $m \in \mathbb{N}$, the event $\{\tau = m\} \in \xi_m$ where $\xi_m = \sigma(x_0, x_1 \ldots, x_m)$.

• When $\{X_n\}_{n\geq 0}$ is a countable state DTMC, we have that $\forall$ m $1_{\{\tau=m\}} = \psi_m(x_0, x_1 \ldots, x_m)$ for some binary valued function ψ_m of $m+1$ variables.

• Condition $\{\tau = n\} \in F_n$ is equivalent to $\{\tau \leq n\} \in F_n\ \ \forall\, n \geq 0$.

• Constants & $\tau^1 = \tau + m_0$ where m_0 is a positive integer are stopping times.

• $\tau_i = inf_{n\geq 1}\{X_n = i\}$ with $\tau_i = \infty$ if $X_n \neq i,\ \forall\, n \geq 1$ is called return time to state i; it is also a stopping time; so is the hitting time to set A.

• Successive return times, $T_1 = \tau_i, T_2, T_3, \ldots$ with $T_{r+1} = T_{r+2} = \ldots = \infty$ if only r of these are finite, are also stopping times.

• If $\tau = \infty$, then append a state (cemetry state) $\Delta \notin s$ and set $X_\infty = \Delta$. Then, $\{X_{\tau+m}\}_{n\geq 0}$ is process after/beyond τ. Also $X_0, \ldots, X_{\tau(\omega)}(\omega)$ is process before τ, equivalently represented as $\{X_{\tau\wedge n}\}_{n\geq 0}$; also called process stopped/killed at τ with value frozen that at X_τ.

• In $\{\tau = n\}, \{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\}$ for some states $i_0, i_1, \ldots i_n$ and hence for $B \in \sigma\{X_0, X_1, \ldots, X_n\}$, we have that

$$B \wedge \{\tau = n\} \in \tau_n \qquad \forall\,\text{integer } n \geq 0$$

Deposing $B \cap \{\tau < \infty\}$ into such disjoint events and interpreting Markov property on each, one has the generalization of Markov property:

• **Strong Markov Property (SMP):**

Let $\{X_n\}_{n\geq 0}$ be Markov (λ, P) on countable state space S and let τ be a stopping time w.r.t. $\{X_n\}_{n\geq 0}$. Then, conditional on $\tau < \infty$ & $X_\tau = i,$, the process after τ, $\{X_{\tau+n}\}_{n\geq 0}$ is Markov (δ_i, P) & and is independent of process before τ, $\{X_{\tau\wedge n}\}$.

• Look at examples like: a) At $P(H_1^0 = n)$ in a RW on Z^+ with state zero as absorbtion state, b) fdds of a process where a DTMC is observed only when it takes values in $J \subseteq S$.

Recurrence and transcience

• While invoking SMP for stopping time H_i, $X_{H_i} = i$ is automatic; whether $H_i < \infty$ $a.s.$ leads to recurrence/transience idea as below:

Set, the number of visits to state i $N_i := \sum_0^\infty 1_{\{X_n = i\}}$, $\bar{N}_i := \sum_1^\infty 1_{\{X_n = i\}}$ and $f_{ji} := P_j(\tau_i < \infty)$ where, τ_i is a return time to state i.

• The distribution of $\bar{N}_i$ if $X_0 = j$ is

$$P_j(\bar{N}_i = r) = \begin{cases} f_{ii} f_{ii}^{r-1}(1 - f_{ii}) & if \quad r > 0 \\ 1 - f_{ii} & if \quad r = 0 \end{cases}$$

• The above is derived by looking at fdds of *excursions* for $r = 2, 3, \ldots,$ conditional on $\tau_i^{r-1} < \infty$, $S_i^{(r)}$ is independent of $\{X_{\tau_i^{(r-1)}} \wedge m\}$ and then,

$$P\left(S_i^{(r)} = n | \tau_i^{r-1} < \infty\right) = P_i(\tau_i = n)$$

.

• Under $X_0 = i$, $\bar{N}_i + 1 = N_i$ taking values in $1, 2, \ldots$ and hence

$$P_i(N_i > r) = f_i^r,\ r \geq 0.$$

In fact,

$$P(\bar{N}_i = r | X_0 = i, \bar{N}_i \geq 1) = f_{ii}^{r-1}(1 - f_{ii});$$

$\bar{N}$ is geometric given $X_0 = i$ and $\bar{N}_i$ restricted to $\{1, 2, \ldots\}$.

• Call state i recurrent if $P_i(\tau_i < \infty) = 1$, *i.e.*, τ_i is a proper r.v. if $X_0 = i$; otherwise it is a transient state.

• Let $i \in S$ be a given state.

1. The following are equivalent:

   α) i is a recurrent state

   β) $\bar{N}_i = \infty\, a.s.;\ X_0 = i$

   γ) $E_i[\bar{N}_i] = \sum_{n\geq 1} p_{ii}^{(n)} = \infty$

2. The following are also equivalent:

   α') i is a transient state

   β') $\bar{N}_i < \infty\, a.s.;\ X_0 = i$

   γ') $E_i[\bar{N}_i] = \sum_{n\geq 1} p_{ii}^{(n)} < \infty$

• There is a dichotomy of the nature of a state which will later be extended to a trichotomy.

• Also $P(N_i = \infty)$ is either 0 or 1, not a fraction; compare this with standard *Borel-Cantelli zero-one lemma*, here we have events that are pairwise dependent.

• **Regenerative theorem of DTMC:**

let $\{X_n\}_{n\geq 0}$ be Markov (τ_∞, P) with state 0 recurrent. For successive return time $\{\tau_0^n\}_{n\geq 0}$ with $\tau_0^0 = 0$, the pieces of trajectories/sample paths

$$\{X_{\tau_k}, X_{\tau_k+1}, \ldots, X_{\tau_{k+1}-1}\}_{k\geq 0}$$

are *i.i.d.* In particular, $S_s^{(k)}$ are *i.i.d.* (These pieces are called regeneration cycles with random time $\{\tau_0^{(k)}\}$ as regeneration epochs. More about these later.)

• One has a delayed regenerative process when starting from $j \neq i$, if $f_j(\tau_i < \infty) = 1$; this is what happens in an irreducible recurrent class.