Discrete Time Markov Chains (Contd.)

Communication classes, *etc.*,

• Call state j is accessable from state i (i leads to state j) denoted as $i \to j$, if there is a finite length path from state i to state j:

$p_{ii_1} p_{i_1 i_2} \dots p_{i_n j} > 0$ for some states $i_1, i_2, \dots, i_n$. Here is a criterion for this:

• The following are true for distinct states i and j

(1) $p_{i,i} \dots p_{i_{n-1} j} > 0$ for some states $i_1, \dots i_{n-1}$

(2) $p_{ij}^{|n|} > 0$ for some n > 0

(3) $P_i(X_n = j$,for some n $> 0) > 0$

• For i $\neq j$ (1) and hence others show that $i \to j$ *iff* $P_i(\tau_j < \infty) = f_{ij} > 0$.

• Call states distinct i and j communicate if $i \to j$ & $j \to i$; denote this by $i \longleftrightarrow j$

• This relation of pairs of states is symmetric and transitive; Define $i \longleftrightarrow i$ even if $f_{ii} = 0$ so that communicating relation is also reflexive.

• This relation induces a partition of S into subsets $A_1, A_2, \dots$ called communicating classes that

(α) Every state belongs to exactly the class, say, A_k,

(β) Every comm. class A_k has at least one state,

(γ) Any pair and hence all states in each comm. class communicate,

(δ) States from distinct comm. classes do not communicate.

• $E_j[N_i] = \infty$ if i is recurrent and is accessible from j

• $E_j[N_i] < \infty$ if i is transient.

• Suppose a state of comm. class is transient (recurrent), then so is every other state of this class - 1^{st} class property theorem.

• So, a comm. class is transient (or) recurrent; later we show that there is a trichotomy.

• Set of states C is closed if $\forall i \in C$, and $j \notin C$, $p_{ij} = 0$. This implies $p_{ij}^{|n|} = 0, n \geq 1$. S is closed

• If set C is not closed, it is open.

• If S is a single class, it is called an irreducible chain.

• A recurrent class is a closed class; equivalently, open communication classes are transient.

• In contrast, not all closed classes are recurrent; give examples.

• Finite closed classes are recurrent.

• *Remark*: We will improve this result later to say that such classes are in fact positive recurrent.

• Suppose $\{X_n\}_{n \geq 0}$ is irreducible and recurrent. Then, $P(\tau_j < \infty) = 1 \, \forall j \in S$In fact, $P_i(\tau_j < \infty) = f_{ij} = 1 \, \forall$ pair (i, j) .

• Transient classes are either infinite (or) open (or) both.

• A finite transient class has to be open.

• Infinite transient class can be closed; for example, Random walk on Z with $P \neq 1/2$.

• Recall discussion about possible nature of tpm with these possibilities in mind.

• As finite closed classes are recurrent and open classes are transient, some unresolved cases are in countable systems.

• A classic example is Polya's result: $1-D$ and $2-D$ SRWs are recurrent; SRWs on higher dimensions are transient. We did all these results.

Invariant measures and invariant distributions:

• $\lambda = \{\lambda_i : \lambda_i \geq 0, i \in s\}$ is an invariant measure if $\lambda = \lambda P$. It is an invariant distribution if $\sum_{i \in s} \lambda_i = 1$, also.

• Let λ be as invariant distribution of $\{X_n\}_{n\geq 0}$. There if $\{X_n\}_{n\geq 0}$ is Markov (λ, P) then $\{X_n\}_{n\geq 0}$ is a stationary process.

• Invariant measures may exist (or) may not exist. They are not unique also, if they exist. We try to understand this.

• Let $X_0 \sim \lambda$ for some distribution λ and let σ be a stopping time. Define, $\lambda_j(\sigma) := P_\lambda(X_\sigma = j)$, $\mu_j(0) := E_\lambda\Big[\sum_0^{\sigma-1} I_{\{X_n=j\}}\Big]$, and $\mu_j(1) := E_\lambda\Big[\sum_1^{\sigma} I_{\{X_n=j\}}\Big]$. Then,

$$1)\ \underline{\lambda} + \underline{\mu(1)} = \underline{\mu}(0) + \underline{\lambda}(\sigma) \qquad 2)\underline{\mu}(1) = \underline{\mu}(0)P.$$

• We can generate stationary/invariant measures: pick a recurrent state i, set

$$v_j^{(i)} := E_i\Big[\sum_0^{\tau_i-1} I_{\{X_n=j\}}\Big] = \sum_0^{\infty} P_i(X_n = j, \tau_i > n).$$

Then $\{v_j^{(i)}\}_{j\in s}$ is an invariant measure.

• Now that invariant measures exist we have this technically important result: If v is a measure such that $vP \leq v$ with $v_i = 1$, then, $v_j \geq v_j^{(i)}$, $\forall j \in S$.

• For an irreducible and recurrent chain, there exists a stationary measure $\underline{v}$ such that $v_j \in (0, \infty)$, $j \in S$ and $\underline{v}$ is unique upto scalars multipliers.

• Stationary measure $\underline{v}^{(i)}$ is such that $|\underline{v}^{(i)}| = \sum_{j\in S} v^{(i)} = E_i(\tau_i)$

• Call a recurrent state i, positive recurrent if $E_i(\tau_i) < \infty$; otherwise call it a null recurrent state.

• Immediately, we can now improve earlier dichotomy as: In an irreducible chain, the following trichotomy exists. All states are either (1) transient (or) (2) positive recurrent (or) (3) null recurrent. This is an exclusive *or* statement.

• An irreducible chain is positive recurrent *iff* there is a solution to $\pi = \pi P$ with $\sum \pi_i = 1$ and $\pi_i \geq 0$. If so, $\pi_i = \frac{1}{E_i[\tau_i]} > 0$, $i \in S$.

• (Example): 1-D RW is recurrent if $p = \frac{1}{2}$. We also saw that it admits no invariant distribution. So by above it has to be a null recurrent chain.

• A finite state chain that is irreducible is positive recurrent.

• **Ergodic Theorem for DTMCs**: Let $\{X_n\}_{n\geq 0}$ be an irreducible positive recurrent chain with stationary distribution π and let $f : S \to \mathbb{R}$ be such that $\sum |f(i)|\pi_i < \infty$. Then for any initial distribution μ,

$$lim_N \frac{1}{N}\sum_{k=0}^{N} f(x_k) = \sum_{i\in S} f(i)\pi_i \ \ P_\mu \ a.s.$$

Absorbtion Probabilities, *etc*:

• Let T and C be disjoint subsets of S ($T \cup C = S$ (or) $T \cup C \neq S$) and for every $j \in T$ define: $x_j(n) = P(x_n \in C, x_{n-1} \in T \ldots, x_1 \in T | x_0 = j)$ and $x_j : \sum x_j(n)$

• If $T \cup C = S$ then, x_j is the probability that the chain ever visits C and if $T \cup C \neq S$) then x_j is the probability that chain visits C before any state not in either C or T, given $X_0 = j \in T$; with $(T \cup C)$ as taboo states, x_j is then the taboo probability.

• Let $y_i(n) = P(x_n \in T, x_{n-1} \in T, \ldots, x_1 \in T | x_0 = j)$ and $y_j = lim\ y_j(n)$ is the probability that chain never leaves T, given $x_0 = j \in T$.

• 1) $\{y_i\}$ as above, are maximal solution to $y_j = \sum_{i\in T} p_{ji} y_i$ that is bounded by 1.

2) Either $y_j = 0 \quad \forall j \in T$ (or) $\sup_{i\in T} y_i = 1$

3) In the first case, $lim\ y_j(n) = 0 \quad \forall j \in T$, *iff* $\{y_j = 0\}_{j\in T}$ is the unique bounded solution of $y_j = \sum_{i\in T} p_{ji} y_i$, $i \in T$

• Probabilities x_j is the unique bounded solution to $x_j = \sum_{i \in C} p_{ji} + \sum_{i \in T} p_{ji} x_j \ \forall \ j \in T$.

• If there is no unique solution in above, then, x_j is the minimal non-negative solution to $x_j = \sum_{i \in C} p_{ji} + \sum_{i \in T} p_{ji} x_i \ \forall \ j \in T$.

• From the above characterization of taboo probabilities, we have: The set of probabilities $\{x_i\}$ is the unique bounded to $x_j = \sum_{i \in C} p_{ji} + \sum_{i \in T} p_{ji} x_i \ \forall \ j \in T$ if either 1) T is a finite set of transient states, 2) T is a proper subset of a recurrent communication class.

• Using the above, we have this important criterion for transience: For an irreducible DTMC on $\{0, 1, 2, \cdots\}$ all states are transient *iff* the system of linear equations, $y_i = \sum_1 p_{ji} y_i, \ j = 1, 2, \cdots$, has a non-zero bounded solution.