Preliminaries

The aim of this course is to present some basic stochastic models that have found and continue to find use in modeling dynamic systems that evolve in a random environment. Given this nature, it is not suprising that these are many examples. However, in these type of basic courses, that are technique based, motivating examples are *not* the focus. Rather, the aim is to cover some basic techniques, with a couple of examples just to introduce/illustrate a topic; specific applications of these techniques will be the focus of other courses and hopefully, you will come across many publications while you are writing your thesis. Incidently, to understand those publications, one needs to be familiar with the language and part of the role of these basic courses is to provide some such language.

A remark about the pre-requisites: There is no denying the fact that to have some (technical) idea of stochastic processes, one need to be conversant with measure-theoretic probability theory; hence, one need to be familiar with measure theory and for this, basic concepts of real analysis .... This chain of backward integration can go on; however, one should start some where, to take a pragmatic viewpoint. It is a beauty of probability theory that some of its essential concepts are simultaneously insightful and are also relatively easy to present (without an appeal to measure theory); this, one can do as long as sample space is countable. An immediate corollary is that discrete time stochastic processes with values on a countable state space do not require measure theory for justification; a prime example of such processes are DTMCs!

In the teaching note on SLLN, we take the oppurtunity of presenting its proof to see in action some very basic probabilistic ideas like Borel-Cantelli lemma, *etc.* With this customary opening sentences, we move on:

- A quick review of calculus based probability theory. Independence of two events in terms of their associated σ fields is emphasized. This leads to a similar concept of independence of two or more random variables, which is useful later.
- Definition of stochastic process; Sample path viewpoint.
- Just as for a random variable its distribution or law is quite important and for N random variables their joint distribution is important, so are finite dimensional distribution functions, *fdds* for a process.

Let $\{X_t\}_{t\in T}$ be a collection of random variables on a probability l space (ω, t, P). For any vector $t := (t_1, t_2, \dots, t_n)$ containing a finite number of members of the indexing set T, associate the joint distribution:

$$F_t(x) := P(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n)$$

Then $\{F_t\}$, as t now ranges over all vector of members of T of any finite length is called *fdds* of $\{X_t\}_{t\in T}$.

These *fdds* satisfy:

1.

$$F_{(t_1,t_2,\dots,t_n,t_{n+1})}(x_1, x_2, \dots, x_{n+1}) \quad \xrightarrow{x_{n+1}\to\infty} \quad F_{(t_1,t_2,\dots,t_n)}(x_1, x_2, \dots, x_n).$$

2. If π is a permutation of $(1, 2, \dots, n)$ and πy denotes the vector

$$\pi y = (y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(n)})$$

for any vector y, then

$$F_{\pi t}(\pi x) = F_t(x) \qquad \forall\, x, t, \pi \text{ and } n.$$

Given a process $\{X_t\}_{t \in T}$ then its *fdds* satisfy above; converse is also true:

Kolmogorov Consistency Theorem

Let T be any set and suppose that for each vector $t := (t_1, t_2, \ldots, t_n)$ containing members of T and of finite length, there corresponds to a joint distribution function F_t. If $\{F_t\}$ collection satisfies the above *Kolmogorov consistency conditions*, then there is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a collection $\{X_t\}_{t \in T}$ of random variables on this space such that $\{F_t\}$ is the set of *fdds* of $\{X_t\}_{t \in T}$.