# PhD Qualifier Exam

IEOR, IIT Bombay

July 12, 2022

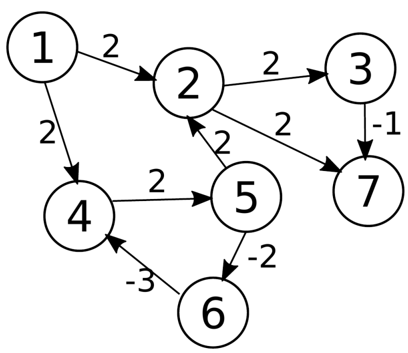
This paper has **11** questions, for a total of **100** marks. Provide clear and complete answers. If you are using some theorem/result, provide the full details of those. Your answers should be mathematically rigorous.

This is an open book and open notes exam.

Use of electronic resources (phone, tablets, laptops) is not permitted. Sharing of any information / material with anyone is permitted.

Write your answers clearly. You may make suitable assumptions if needed or if a question appears to be incomplete. Please clearly write any such assumptions.

1. (10 points) Given a directed graph of nodes (say, $1, \ldots, n$) and edges ($E$), cost of traveling on each edge, a source and a destination, we would like to find a way to start from the source and reach the destination using the least cost path. Sometimes, one can model this is a linear program (LP) with decision variables $x_{ij}$ denoting the quantity flowing on an edge going from node $i$ to $j$. Write this linear program for a general graph. Find the shortest path from node 1 to node 7 in the following graph (by any method) and check whether it is indeed the optimal solution to your LP formulation. Note that some of the costs are negative. Explain any discrepancies you see.




2. (10 points) Let $b, a_1, a_2, \cdots \in \mathbb{R}$ be given and consider the optimization problem

$$\begin{cases} \min_{x_1, x_2, \cdots \in \mathbb{R}} & \sum_{n=1}^{\infty} x_n^2 \\ \text{s.t.} & \sum_{n=1}^{\infty} a_n x_n = b. \end{cases}$$

Assuming that $b, a_i, i = 1, 2 \ldots$ are all bounded, show that the optimal value is finite and find the optimal decision.

3. Let $Q \in \mathbb{R}^n$ be a convex set that is permutation-invariant (i.e., for any $x \in Q$, any vector obtained by permuting the components of $x$ also belongs to $Q$). Let $f : Q \to \mathbb{R}$ be a convex function that is permutation-invariant (i.e., $f(x) = f(x')$ for any $x'$ obtained by permuting coordinates of $x$).

(a) (5 points) Show that if $f$ attains its minimum in $Q$, then there exists a minimizer $x^\star \in Q$ such that all coordinates of $x^\star$ are equal.

(b) (5 points) Find a minimizer of the function $h(x) := \sum_{i=1}^n x_i \log x_i$ over the unit simplex $\Delta_n := \{x \geq 0 \,:\, \sum_{i=1}^n x_i = 1\}$. You may assume that $0 \log 0 = \lim_{u \to 0+} u \log u = 0$.

4. (10 points) A food product is in the market and has two major ingredients, $A$ and $B$, which add sweetness to the product. By regulation, the proportions of $A$ and $B$, $x_A$ and $x_B$, are in ranges $[l_A, u_A]$ and $[l_B, u_B]$ for $A$ and $B$. (In reality, there may be other constraints about relative proportions also, but ignore that for the moment). The product designer has to decide on the values of $x_A$ and $x_B$, depending on the cost. The more the values of $x_A$ and $x_B$, the more is the sweetness coefficient S (note that S could be a non-linear function of $x_A$ and $x_B$ (to simplify things, you can take it as a linear increasing function), but the cost $C$ is also more (this could also be taken to be a linear function of $x_A$ and $x_B$).

   A Pareto optimal solution in this context is a feasible vector $x^*$ so that that no other feasible vector $x$ satisfies the conditions $\{S(x) > S(x^*), C(x) < C(x^*)\}$.

   The planner suggests choosing an appropriate value $\alpha$ for sweetness and finding a value of $x'$ so that sweetness is at least $\alpha$ and where cost is minimised. Show that such a solution $x'$ will always be Pareto optimal.

5. (5 points) Your parents are coming tomorrow to Mumbai by flight. From past experience, it is known that the probability of delay is $0.75$ if it rains and $0.25$ if it does not rain. From the weather forecast, you know that the probability of rain tomorrow is $0.2$. What is the probability that your parents reach on time?

6. (10 points) One particle of a certain bacteria enters inside the stomach of a person. The particle attempts to split itself into two particles. If successful (which happens with probability $p$) we have two particles, if not that particle dies. Now the same thing happens at any time: a particle among the existing ones attempts to split itself and can either split successful with probability $p$ or dies (with $1 - p$ probability). At a time only one particle attempts splitting. When all the particles are dead, one new particle enters the system. And this continues. Model this as an appropriate stochastic process and comment on the stability of the process: in particular comment on the existence of some limiting distribution if there can be one. This will depend upon $p$.

   Now suppose a new particle does not enter after all the particles are dead. Comment on the limiting distribution/behaviour of the process, if there can be one.

7. (10 points) Consider a workstation where jobs arrive according to a Poisson process with arrival rate of 10 jobs per hour. The jobs have exponentially distributed service times, where the mean service time depends on the number of jobs in the system. If the number of jobs in system is $<= 3$, then the mean service time is 12 minutes; if the number of jobs in system is 4, 5 or 6, then the mean service time is 6 minutes; and if the number of jobs in system $> 6$, then the mean service time is 4 minutes.

   Give the state–space representation of the above system. Compute the steady state probability, $P_0$, of 0 (zero) jobs in the system. What is the probability that the workstation operates at the maximum service rate?

8. (10 points) For a random variable $X$ identify $\arg\min_{a \in \mathbb{R}} \mathbb{E}[(X - a)^2]$. Interpret this. Write any assumptions you made.

   Consider a bivariate random variable $(X, Y)$ with probability mass function, $\mathbb{P}(X = 1, Y = 1) = 0.4$, $\mathbb{P}(X = 1, Y = 2) = 0.3$, $\mathbb{P}(X = 2, Y = 2) = 0.2$, $\mathbb{P}(X = 2, Y = 1) = 0.1$. You are told that the event $(Y = 1)$ has occurred. What is your best estimate for value of $X$ when the error criteria is MSE (Mean Squared Error)? You need not write the justification, but indicate the intuition.

For the above pair of random variables $(X, Y)$, compute $\int_{\{Y=1\}} \mathbb{E}[X|Y] \mathrm{d}\mathbb{P}$ without explicitly evaluating this integral, but by evaluating some other integral. Moreover, state (need not prove) the general result that you are using here.

9. (5 points) A farmer who grows apple trees supplies a local supermarket daily with 50 apple baskets where each basket contains 50 apples. Generally the farmer separates out rotten apples from good apples and then supplies the baskets. But as days pass by, the farmer becomes greedy and plans the following. He replaces exactly 3 good apples with 3 rotten apples in each of the 50 baskets and supplies them to the supermarket. The supermarket owner grows suspicious of the quality of the apples, and decides on one fine day to check the quality by randomly choosing two apples from each of the 50 baskets. If the supermarket owner finds even one rotten apple, he will stop procuring from the farmer. With what probability will the supermarket continue doing business with the farmer the next day?

10. (10 points) Consider the optimization problem

$$\min_x \ f(x) = x^\top Ax + 2b^\top x \ \text{ s.t. } x^\top x \le r \tag{1}$$

where $x, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $r > 0$ is a positive integer. Suppose there exist $u \in \mathbb{R}^n$ and $\mu \ge 0$ such that the following hold:

$$u^\top u \le r, \ A + \mu I \succeq 0, \ (A + \mu I)u + b = \mathbf{0}, \ \mu(u^\top u - r) = 0. \tag{2}$$

   a. Consider $h(\theta) = \inf_{\hat{x}} L(\hat{x}, \theta)$ where $L$ is the Lagrangian of the problem (1). Show the relation between $f(u)$ and $h(\mu)$. Comment on the duality gap and about optimality of $u$.
   b. Suppose $u$ satisfies $u^\top u < r$. Using equations (2), comment on the characteristics of $\mu$, optimality of $u$ and the nature of objective function $f(x)$.

11. (10 points) Let $p \in [0, 1]$. Let $Y_1, Y_2, \ldots, Y_n$ denote $n$ independent random variables such that $Y_i \in \{0, 1\}$ and let $P[Y_i = 1] = p$. Define $Y = \sum_{i=1}^n Y_i$. Let $q \in [0, 1 - p]$. Show the following:

   a. $P[Y \ge (p + q)n] \le \frac{E[e^{\lambda Y}]}{e^{\lambda(p+q)n}}$ for any $\lambda > 0$.
   b. Show that the right hand side in the above inequality is minimized when $e^\lambda = \frac{(1-p)(p+q)}{p(1-p-q)}$.