ABSTRACTS, COMBINATORIAL GAMES AT MUMBAI, 22-25 JAN 2024

1. A quick journey into Combinatorial Game Theory, by Carlos P. dos Santos, Center for Mathematics and Applications (NovaMath), FCT NOVA, Portugal

In the early years of the 20th century, C. Bouton presented a famous analysis of the classic game of NIM. Later, in the 1930s, independently, R. Sprague and P. Grundy outlined the method for analyzing impartial games, where allowable moves depend only on the position and not on which of the two players is making the move. Even later, in the 1970s, E. Berlekamp, J. Conway, and R. Guy delved into partizan games, with positions where one of the players may have options that the opponent does not. These three developments led to the establishment of a completely new mathematical subject: Combinatorial Game Theory. The aim of the talk is to provide a survey of the development of this mathematical field, focusing solely on the normal-play winning convention, which establishes that the player who runs out of moves loses. The speaker will complement the talk with numerous examples from game practice and various considerations.

2. Graph Nim games on graphs with 4 edges, by Moumanti Podder, Indian Institute of Science Education and Research

The Graph Nim game is played on a weight graph $G = (V, E)$, in which each edge $\{u, v\} \in E$ has been assigned a positive integer weight $w_0(\{u, v\})$. The two players, henceforth referred to as P_1 and P_2, take turns in making moves, where a move involves choosing a vertex $v \in G$ and removing a non-negative integer weight from the edge $\{u, v\}$, for each $\{u, v\} \in E$, such that the resulting / remaining weight on the edge $\{u, v\}$ is a non-negative integer. A player must make sure that the *total* weight she removes during any of her turns is a strictly positive integer. The player who is unable to make a move loses. This work focuses on explicitly characterizing the winning and losing positions for the Graph Nim game played on all possible graphs with 4 edges. Of particular interest is the analysis involved for two of these graphs, and these will be demonstrated during the talk.

3. How to lose at combinatorial games (or at least draw)?, by Aaron Siegel

I will present a cursory overview of two major aspects of combinatorial game theory: misère play, in which the object is to avoid making the last move, inverting the usual convention; and loopy games, in which repetition is permitted. Both topics will be discussed primarily in the context of impartial games.

4. Partizan Tokens, by Harhsvardhan Agarwal, Indian Institute of Technology Bombay

We investigate a partisan combinatorial game played on a two dimension board where two player take turns to place their tokens on the game board. The two players have distinct tokens - white token covers a single square on the grid while the black token occupies two orthogonally adjacent spaces. There is an additional constraint that white token cannot be placed orthogonally adjacent to another white token. Despite the seemingly disparate moves by the players, the game is unexpectedly unbiased for numerous configurations.

5. Slamming Toads and Frogs, by Aditya D. Bhat, Indian Institute of Technology Bombay

Toads and Frogs is a partisan combinatorial game invented by Richard Guy. We have a game board consisting of a strip of tiles. Each player has pieces in form of Toads or Frogs lined up from each side of the game board initially. On their turn, a player may choose to slide their piece in one or more empty tiles ahead. If confronted by an opponent piece, the player may choose to leap over that piece. A Player wins once all the pieces have crossed the board or no more moves are possible for the other player. We have improvised a modified version of this game where instead of leaping over, the player moves the opponent players pieces beyond them, in a move called a "Slam". Thus, this version is called Slamming Toads and Frogs. This ruleset modifications leads to a variety of changes in the game tree, possible game positions and game values. We share these briefly.

6. Impartial Game ruleset: Space Quest, by Deepankar Sehra, Indian Institute of Technology,Bombay

Space Quest is a captivating board game that challenges players to navigate a dynamic grid strategically. The game unfolds on an (n x n) grid, where each player controls a single piece starting from the top left corner. The objective is to make successive moves by choosing connected blocks while strategically removing others from play. The catch lies in the careful selection of blocks to remove, ensuring they haven't been traversed before. The game concludes when a player runs out of legitimate moves, either due to the removal of all adjacent blocks or an inability to complete a move without eliminating a block. Space Quest combines tactical thinking with spatial awareness, providing an engaging and dynamic gaming experience.

7. 3-Annihilation, by Joshi Tanmay Ajay, Indian Institute of Technology Bombay

3-Annihilation is a simple combinatorial game where each player has pieces assigned to them, which are arranged in a linear configuration. In each turn, the current player has a choice to remove a piece assigned to them. In all subsequent arrangements, all groups of pieces of the same kind which have a size of at least 3, are annihilated/removed. We conjecture that the game only has integer values or infinitesimally close to integer values.

8. Nuclear Battleship and Surround, by Vedang Gupta, IIT Bombay

We present two grid-board rulesets - Nuclear Battleship and Surround. Nuclear Battleship is a perfect information game inspired by the classic game Battleship. We present some simple positions and discuss the temperature of certain positions. Next, we discuss the infinitesimal canonical forms and atomic weight of positions of Surround, an all-small ruleset. We also present a winning strategy when Surround is restricted to a strip board.

9. Fork positions and 2-dimensional Toppling Dominoes, by Anjali Bhagat, Indian Institute of Technology Bombay

Toppling Dominoes is known as a one-dimensional combinatorial game where the dominoes are arranged in a straight line. This project introduces fork positions where the dominoes are placed in a 2-dimensional plane. We define the rules for 2-dimensional fork positions in Toppling Dominoes. We explore how fork positions are different from 1-dimensional Toppling Dominoes. We prove that doubling a single domino to make the position two-dimensional favors the player whose domino was doubled. The game values become incomparable in the case of the neutral green domino. We also prove that when we make two 1-dimensional games into a fork position again, it favors the player whose domino was used to make the fork. This is joint work with Larsson.

10. Combinatorial Game Computational Complexity Basics, by Kyle Burke, Florida Southern College

This talk covers the basics of the computational complexity of short combinatorial games, including QBF, PSPACE, and reductions. We will focus on questions of winnability of games. Examples of PSPACE-complete games will be shown, including Directed Geography. No prior understanding of Combinatorial Game Theory or computational complexity is required.

11. The temperatures of Robin Hood game, by Ankita Dargad, Indian Institute of Technology Bombay

The partizan ruleset ROBIN HOOD is an instance of WEALTH NIM (see arXiv:2005.06326) with one heap in which players cannot remove more tokens than their own wealth and they may diminish the other player's wealth by the number of tokens removed. Long before the era of Pingala, within a village, two tribes led by chiefs disputed over unclaimed pieces of land for farming, which led to a war. The chiefs established rules favouring the first arrival: the tribe reaching first on a piece of land can grab what they get and proportionally diminish the other tribe's strength, and return back. The other tribe wants to take revenge and they reach first on the land the next day. Both groups aimed to weaken the opponent and claim as much land as possible which led to a heated situation. This scenario mirrors the ROBIN HOOD ruleset. The thermographs of game instances exhibit only 2 different shapes for large heap sizes, one has increasing heat with increasing heap size and the other one has a constant heat. Curiously, this bifurcation is related to Pingala (Fibonacci) sequences and the Golden Ratio. This is joint work with Balachandran and Larsson.

12. Bidding combinatorial games, by Prem Kant, Indian Institute of Technology Bombay

In 2-player Combinatorial Game Theory (CGT), normal play is the convention where a player who cannot move loses. We extend the alternating play convention to infinitely many game instances, by means of discrete Richman auctions (Develin et al. 2010, Larsson et al. 2021, Lazarus et al. 1996). In this context we manage to generalize the notion of a perfect play outcome, and moreover we find an exact characterization of outcome feasibility. This is joint work with Larsson, Upasany and Rai.

13. Games and optimization on random structures, by Johan Wästlund, Chalmers University of Technology, Sweden

Combinatorial game theory is often regarded as an esoteric field not strongly connected to "mainstream" mathematics. In this talk I discuss a few examples where combinatorial games shed light on problems that on the surface don't seem to have anything to do with games. The examples come from optimization on random graphs and graphs with random edge-weights. In particular we discuss minimum cost matching and traveling salesman problems (TSP), and the independence number of a random tree.

14. Ruleset for 1-dimensional subtraction games using Beatty sequence, Tirthankar Taraknath Adhikari, IIT Bombay

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15. The strategy stealing argument, Satvik

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16. Subtraction games in more than one dimension, by Indrajit Saha, Kyushu University, Japan

We study two-player impartial vector subtraction games, on tuples of nonnegative integers (Golomb 1966) with finite rulesets, and solve all two-move games. Through multiple computer visualizations of outcomes of two-dimensional rulesets, we observe that they tend to partition the game board into periodic mosaics on very few regions/segments, which can depend on the number of moves in a ruleset. For example, we have found a five-move ruleset with an outcome segmentation into six semi-infinite slices. In this spirit, we develop a coloring automaton that paints P-positions in segments of the game board, independent of game play. Moreover, we prove that games in two dimensions have row/column eventually periodic outcomes. Several regularity conjectures are provided. Through visualizations of some rulesets, we pose open problems on the generic hardness of games in two dimensions. We conjecture that not all games have regular outcomes. This is joint work with Larsson.

17. Some open problems in CGT: Not much background required!, by Carlos P. dos Santos, Center for Mathematics and Applications (NovaMath), FCT NOVA, Portugal

In this talk, I will briefly discuss some open problems, both old and recent. I will focus on open problems that do not require an extensive technical background and can be analyzed by any mathematics student. However, please note that these are open problems! As such, their solution will inevitably demand sophisticated thinking.

18. Games and puzzles with automatic proofs, by Gandhar Joshi, The Open University, UK

Automatic sequences are obtained by playing little games (in a sense) with an integer representation of computer's natural numbers (0,1,2 onwards). We discuss that with an example of the famous 'Thue–Morse' sequence. We then see how an automatic sequence can completely solve a well-known puzzle called the 'Tower of Hanoi'. Keeping with the theme, we discuss a 2-player chess-based game—'Wythoff Nim' and its connection with automatic sequences. This work not mine, but my PhD research intersects with the use of automatic theorem-proving software 'Walnut'. The inspiration to share this comes from my supervisor Dan Rust, who studied a modification of the Wythoff Nim a few years ago (Link: https://doi.org/10.1007/s00182-022-00824-1).

19. Enforce operation of the disjunctive sum and the continued conjunctive sum, by Shun-ichi Kimura, Hiroshima University, Japan

To combine two rulesets G and H, one standard way is the disjoint sum, where the player chooses one of the rulesets, G or H, and then play it. There are also many other methods to combine two rulesets, see Chapter 14 of "On Numbers and Games" by Conway (Chapter title: How to play Several Games at Once in a Dozen Different Ways), and one important example for us is the Continued Conjunctive sum, where the player must play both G and H as far as there are some options for both of them, but the player can skip one of them if it has no options.

Then for the product set of the phase sets of G and H, we have two rulesets, namely the disjunctive sum and the continued conjunctive sum. When one phase set has two rulesets, one can combine these two rulesets by enforce operator: The opponent enforce the player to play according to only one of the ruleset

Odawara observed that this enforce operation of disjunctive sum and continued conjunctive sum can produce interesting set of $\mathcal{P}$-positions, even when G and H are very simple. One particular example we carefully studied is the case both G and H are the restricted one pile nim, where the player can take $i \in \{1, 2, \cdots, n\}$ tokens with $n \in \mathbb{Z}_{>0}$, where the set $\{1, 2, \cdots, n\}$ is called the set of the removable numbers. The $\mathcal{P}$-positions for the disjunctive sum is $\{(x, y) \mid |x - y| \text{ is a multiple of } n + 1\}$

and the $\mathcal{P}$-positions for the continued conjunctive sum is $\{(x, y) \mid \mathrm{Max}(x, y)$ is divisible by $n + 1\}$. Both are fairly simple, but once they are combined by the enforce operator, the set of $\mathcal{P}$-positions becomes very complicated (but beautiful).

In particular when $n = 2$, the set of $\mathcal{P}$-positions is too complicated, and we don't know how to mathematically describe it. There appears many other examples with interesting set of $\mathcal{P}$-positions, at least experimentally, and we showcase the gallery of these examples.

20. Yama Nim, Triangular Nim, and their Wythoff Variations, by Takahiro Yamashita, Hiroshima University, Japan

Yama Nim is a two heaps Nim game introduced in my Master Thesis, where the player takes at least 2 tokens from one heap, and return 1 token to the other heap. Triangular Nim is a generalization, where the player takes at least 2 tokens from one heap, and return at least one token to the other heap, so that the total number of tokens in the heaps decrease strictly. In this talk, I will talk about their winning strategies, Nim values(Grundy numbers), and the Wythoff variations of these games. Particularly interesting are two topics. The first one is Nim values of Triangular Nim. Most positions of the Nim values of Triangular Nim can be written sinple, but some of them are complicated and interesting. For example, the positions that the Nim values are equal to 1 are $(x, y) \in \{(2, 4), (4, 6), (6, 8), (8, 10), (10, 12), \ldots\}$ under the assumption $x \leq y$. The second one is Triangular Nim with the Wythoff variation, where in addition to the Triangular Nim moves, the player is allowed to take tokens from both heaps, say i tokens from the first heap and j tokens from the other, under some restrictions for i and j. For example when we force $i = j > 0$, then $(x, y) \in Z \geq 0$ with $x \leq y$ is in the P-position if and only if $(x, y) \in \{(0, 0), (0, 1), (1, 3), (3, 6), (6, 10), (10, 15), \ldots\}$, namely the winning strategy is described by triangular numbers. In other restrictions, there are examples where the P-positions are described by the square numbers, pentagonal numbers, geometric progressions, Mersenne numbers, and so on. This is a joint work with Professor Shun-ichi Kimura.

21. Computational Complexity of Turning Tiles, by Hirotaka Ono, Nagoya University, Japan

In combinatorial game theory, the winning player for a position in normal play is analyzed and characterized via algebraic operations. Such analyses define a value for each position, called a game value. A game (ruleset) is called universal if any game value is achievable in some position in a play of the game. Although the universality of a game implies that the ruleset is rich enough (i.e., sufficiently complex), it does not immediately imply that the game is intractable in the sense of computational complexity. We prove that the universal game Turning Tiles is PSPACE-complete. We also give other positive and negative results on the computational complexity of Turning Tiles.

22. Diving into intricacies of Pixel Pummel, by Hridank Garodia, Dhirubhai Ambani International School

The ruleset, called "Pixel Pummel", is a directional reversi type game, but it has significant differences. I have solved the ruleset for 1xN and found some solutions to 2x2 and onwards. It is very interesting since the ruleset has a natural way of avoiding mimicking in even by even boards such as 2xN.

23. On some variations of Wythof's game, Geremias Polanco, Smith College, Northampton, MA, USA

Wythoff's game starts with two piles, labeled A and B, of finitely many tokens. Two players alternate, and at each turn the current player may either
(a) remove any positive number of tokens from a single pile, possibly the entire pile,
(b) or remove an equal number of tokens from piles A and B simultaneously.
The winning positions of this game are the pairs $([an], [bn])$, where $a = (1+\sqrt{5})/2$, and $b = a+1$. In this talk we discuss some new and old results on different variations of this game including the forward problem (Given a variation of the Wythoff's game, find the pair of sequences that are solution to this game), and also the inverse problem for Beatty sequences (Given a pair of complementary Beatty sequence, find a Wytthoff's type game with solution the given pair of sequences).

24. Ergodicity of a generalized probabilistic cellular automaton with parity-based neighbourhoods, by Dhruv Bhasin, IISER Pune

We study a one-dimensional generalized probabilistic cellular automaton $E_{p,q}$ with universe Z, alphabet $A = \{0, 1\}$, parameters p and q such that $0 < p + q \leq 1$ and two neighbourhoods $N_0 = \{0, 1\}$ and $N_1 = 1, 2$. The state $E_{p,q}\eta(x)$ of any $x \in Z$ under the application of $E_{p,q}$ is a random variable whose probability distribution depends on the states $\eta(x + y)$ for $y \in N_i$ where i has the same parity as x. We establish ergodicity of this GPCA for various ranges of values of p and q via its connection with a suitable percolation game on a two-dimensional lattice. For these same ranges of values of p and q, we show that the above-mentioned game has probability 0 of resulting in a draw.

25. Additional Computational Complexity of Games Considerations, by Kyle Burke, Florida Southern College

This talk extends the topics of the previous talk. We will cover other questions that can be asked about games besides the general winnability question. We will cover the reachability of reduced positions, likelihood of reduced positions, and the complexity of finding game values.

26. A Stackelberg Game for Cross-channel Free-riding, by Anirban Mitra, Indian Institute of Technology Roorkee

Multi-channel retailing is an approach where similar products can be bought through different platforms (online, and offline). These channels could be either traditional brick-and-mortar (B&M) stores or online retailers (websites or apps). In today's competitive multi-channel retailing environment, retailers face the challenge of managing their pricing and promotion strategies to compete effectively with other retailers and capture the attention of increasingly informed and empowered customers. In this digital era, customers are surrounded by all the information related to a product and make informed purchase decisions. It's been widely observed that customers search for product details from a respective channel (B&M/online) by regular visits but purchase it from a different channel (online/B&M). Showrooming happens when a customer visits a physical store to collect information about a product, but then purchases it online. Webrooming, on the other hand, happens when a customer gathers information online, but then buys products in a physical store to take advantage of immediate availability or other benefits. Several survey reports indicate the co-existence of showrooming and webrooming. We model this situation as a Stackelberg game considering the power dynamics of the retailers. This model can capture both of these consumer phenomena. We further determine the Stackelberg equilibrium, which benefits the retailers for pricing decision-making. We generalize the Stackelberg game model of Basak et al. 2017 which captures only showrooming.

27. Eternal Vertex Cover Game on Graphs, by Saraswati Girish Nanoti, Indian Institute of Technology Gandhinagar

The Eternal Vertex Cover problem is a dynamic variant of the vertex cover problem. We have a two-player game in which guards are placed on some vertices of a graph. In every move, one player (the attacker) attacks an edge. In response to the attack, the second player (the defender) moves some of the guards along the edges of the graph in such a manner that at least one guard moves along the attacked edge. If such a movement is not possible, then the attacker wins. If the defender can defend the graph against an infinite sequence of attacks, then the defender wins. The minimum number of guards with which the defender has a winning strategy is called the eternal vertex cover number of the graph G. On general graphs, the computational problem of determining the minimum eternal vertex cover number is NP-hard and admits a 2-approximation algorithm and an exponential kernel. The complexity of the problem on bipartite graphs is open, as is the question of whether the problem admits a polynomial kernel. We settle both these questions by showing that Eternal Vertex Cover is NP-hard and does not admit a polynomial compression even on bipartite graphs of diameter six. We also show that the problem admits a polynomial time algorithm on the class of co-bipartite graphs. We also give a characterization of bipartite graphs satisfying the lower bound for the eternal vertex cover number.

28. Scopes of games in bitopological dynamical systems, by Santanu Acharjee, Gauhati University

Bitopological dynamical systems was introduced in 2020. It's a new area of mathematics that connects bitopological spaces and dynamical systems mainly along with several other fields. In this paper, we show scopes of games in bitopological dynamical systems from the perspective of modified homotopy theory.

29. A family of slow exact k-Nim games, Matthieu Dufour, Université du Quèbec à Montréal, Canada

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