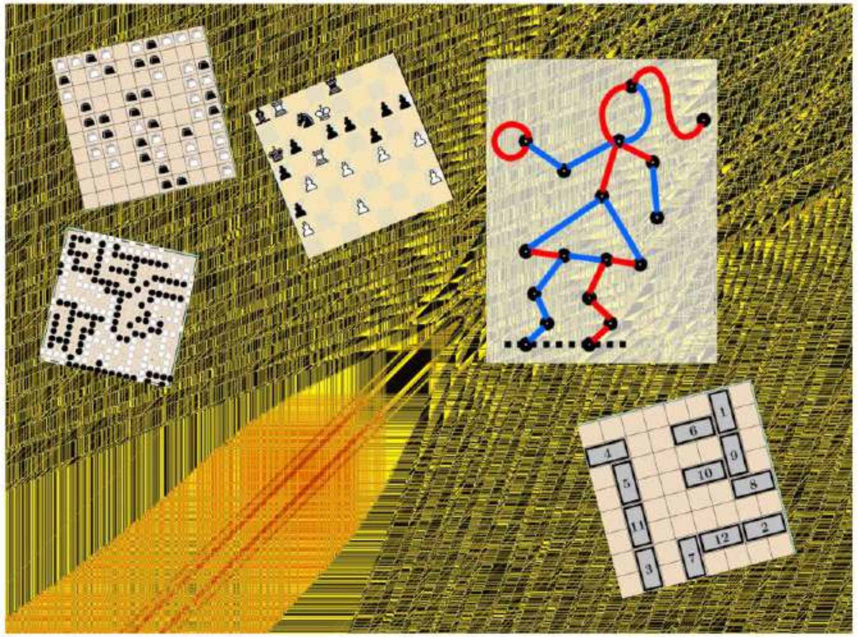
A quick journey into Combinatorial Game Theory

Games at Mumbai 2024

Combinatorial Games at Mumbai, January 21-25 2024



Carlos P. Santos, Center for Mathematics and Applications (NovaMath), FCT NOVA, Portugal

Carlos P. Santos' work is funded by national funds through the FCT, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

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Part I: Impartial Games

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Part II: Partizan Games

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Annex: Why study Combinatorial Game Theory?

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I.1: Some famous games

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I.4: How can you apply the theory?

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Part I: Impartial Games

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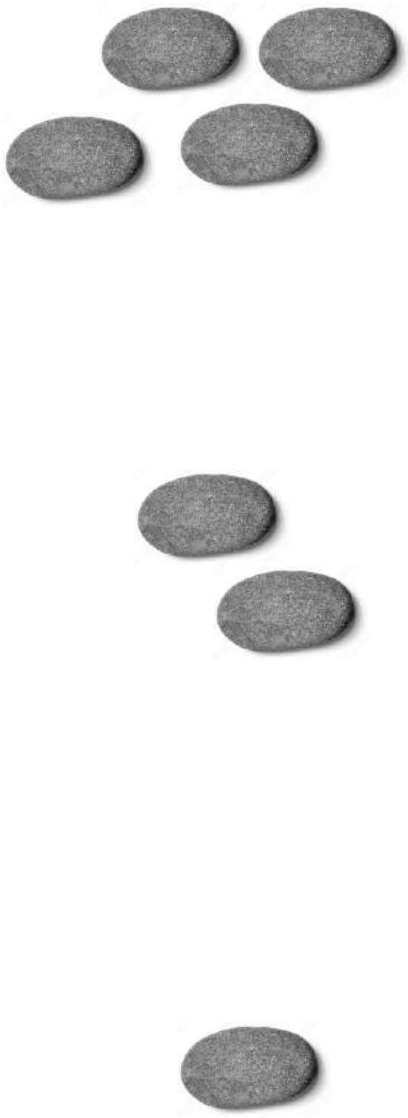
I.1: Some famous games

NIM

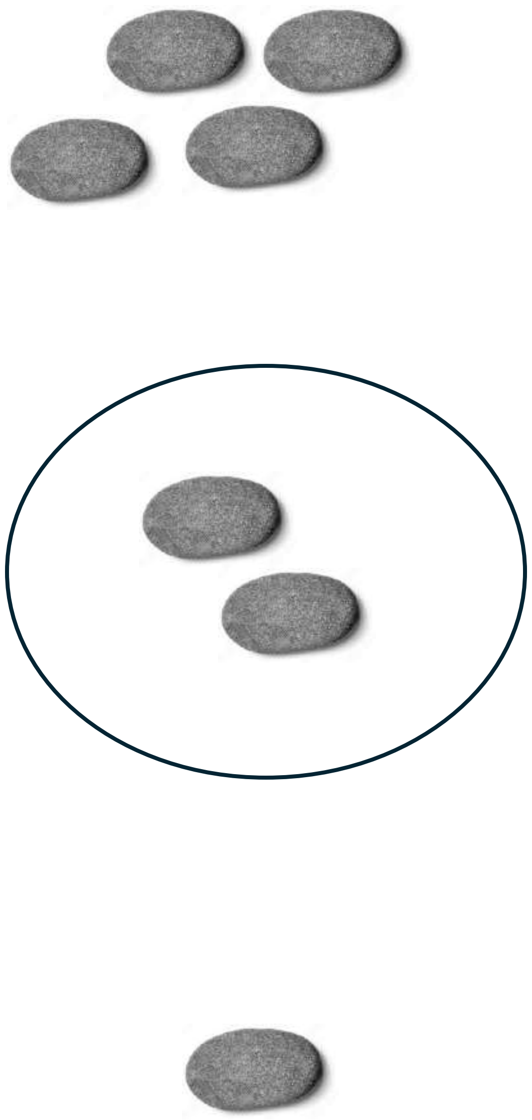
NIM

NIM



My turn

NIM



My turn

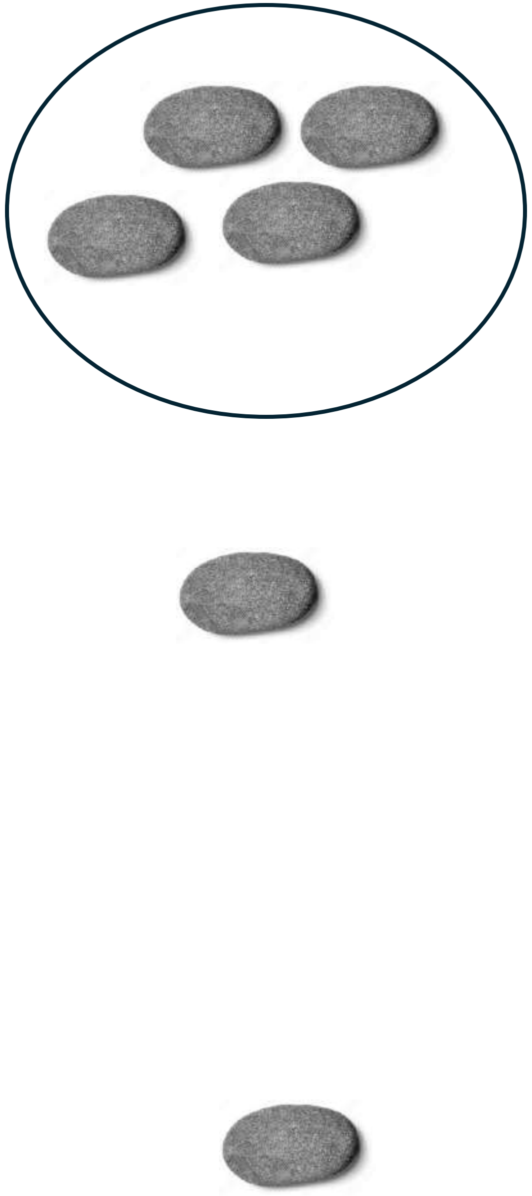
NIM

My turn

NIM

Your turn

NIM



Your turn

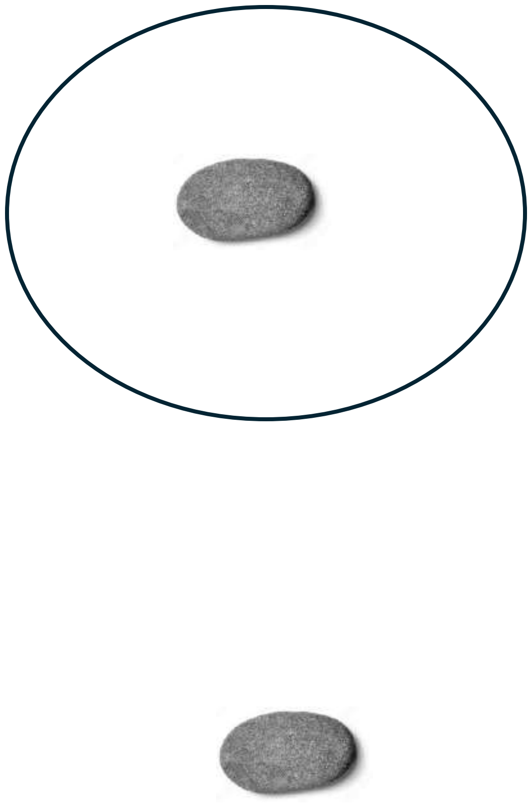
NIM

Your turn

NIM

My turn

NIM



My turn

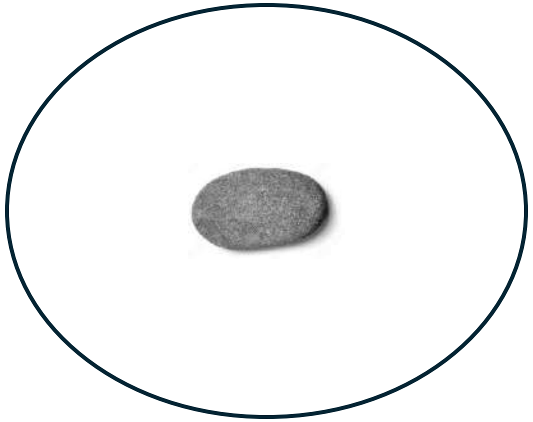
NIM

My turn

NIM

Your turn

NIM



Your turn

NIM

Your turn

NIM

My turn

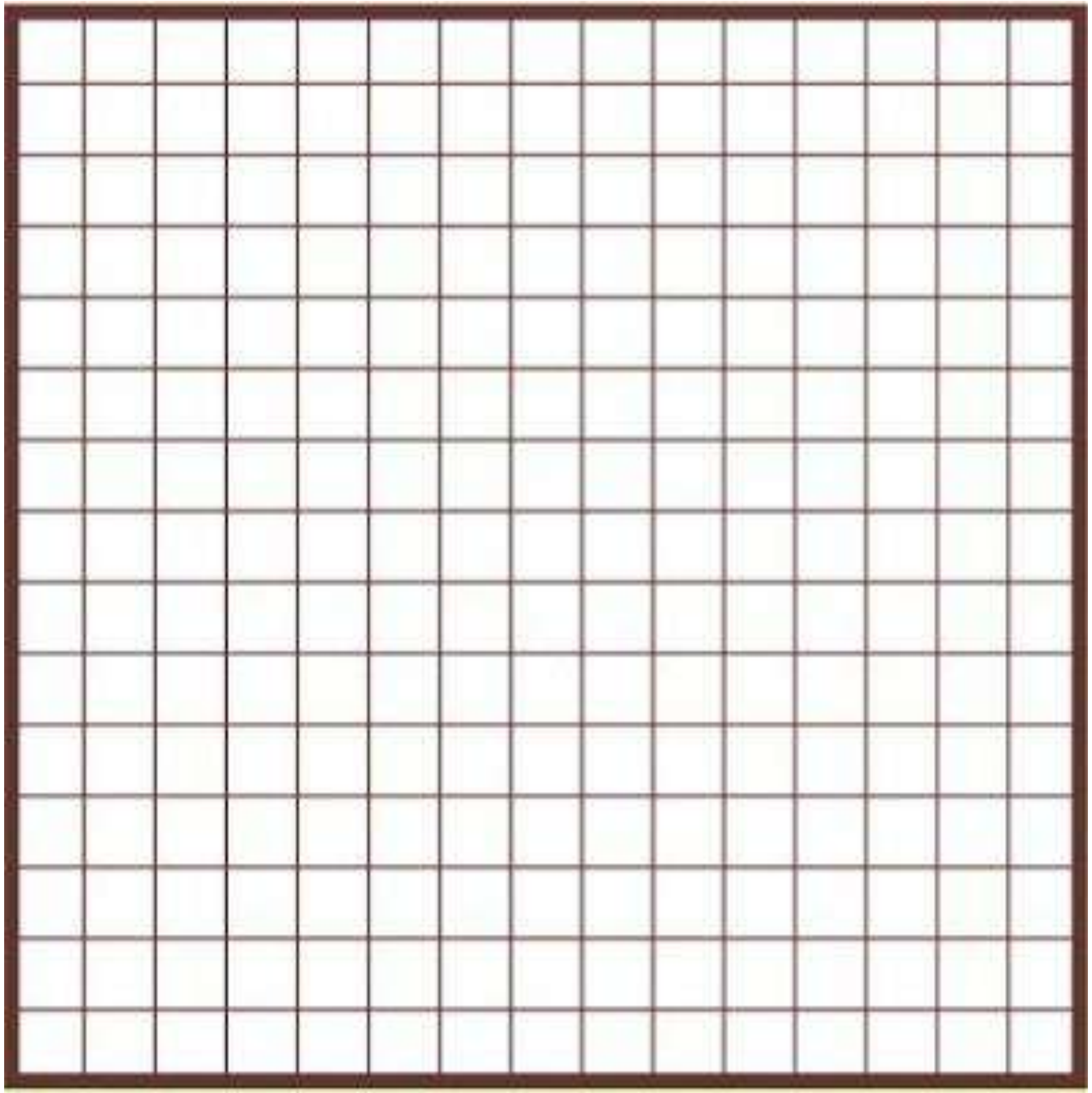
NIM

I have no moves. I lose the game.

My turn

CRAM

CRAM



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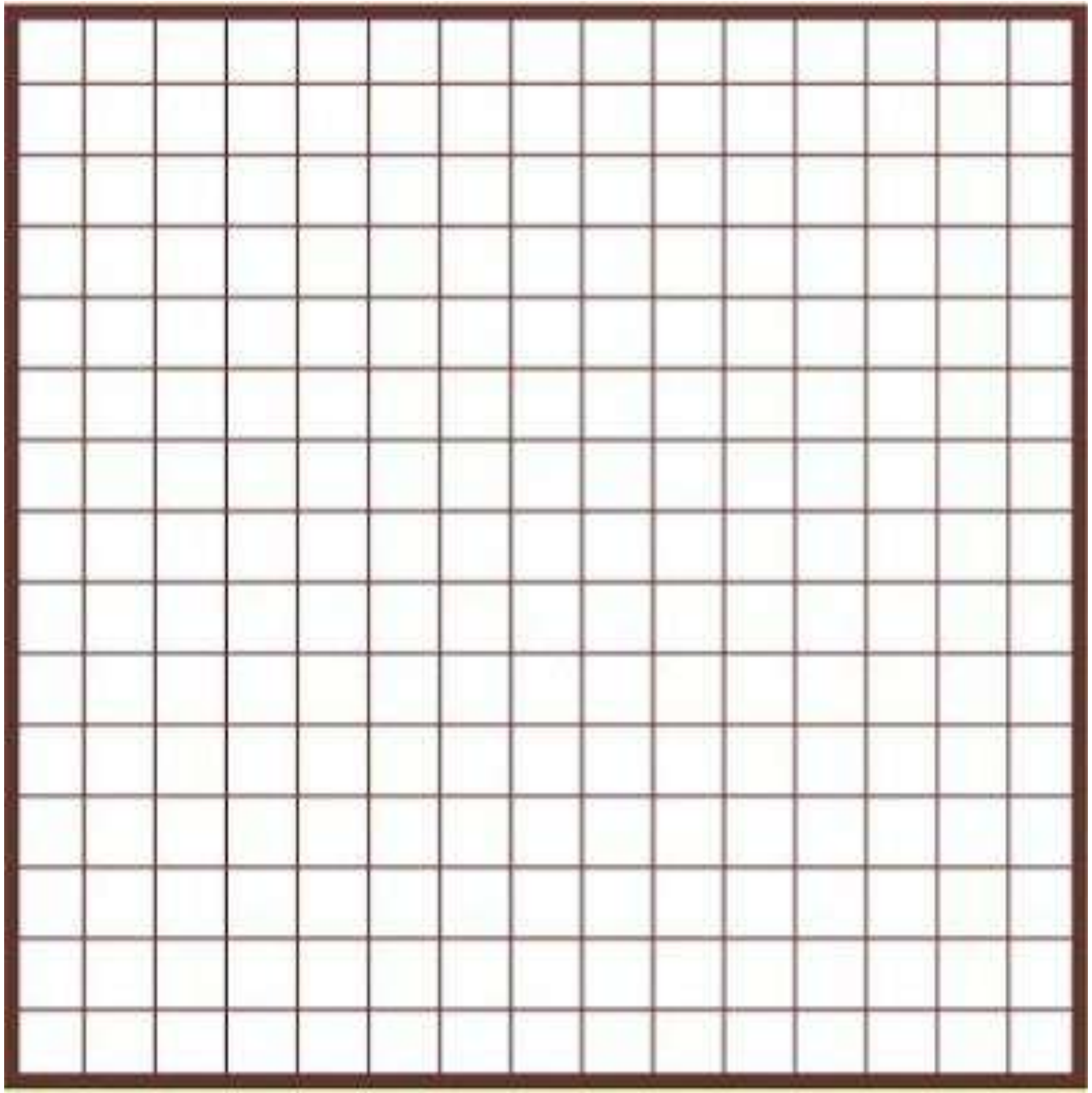
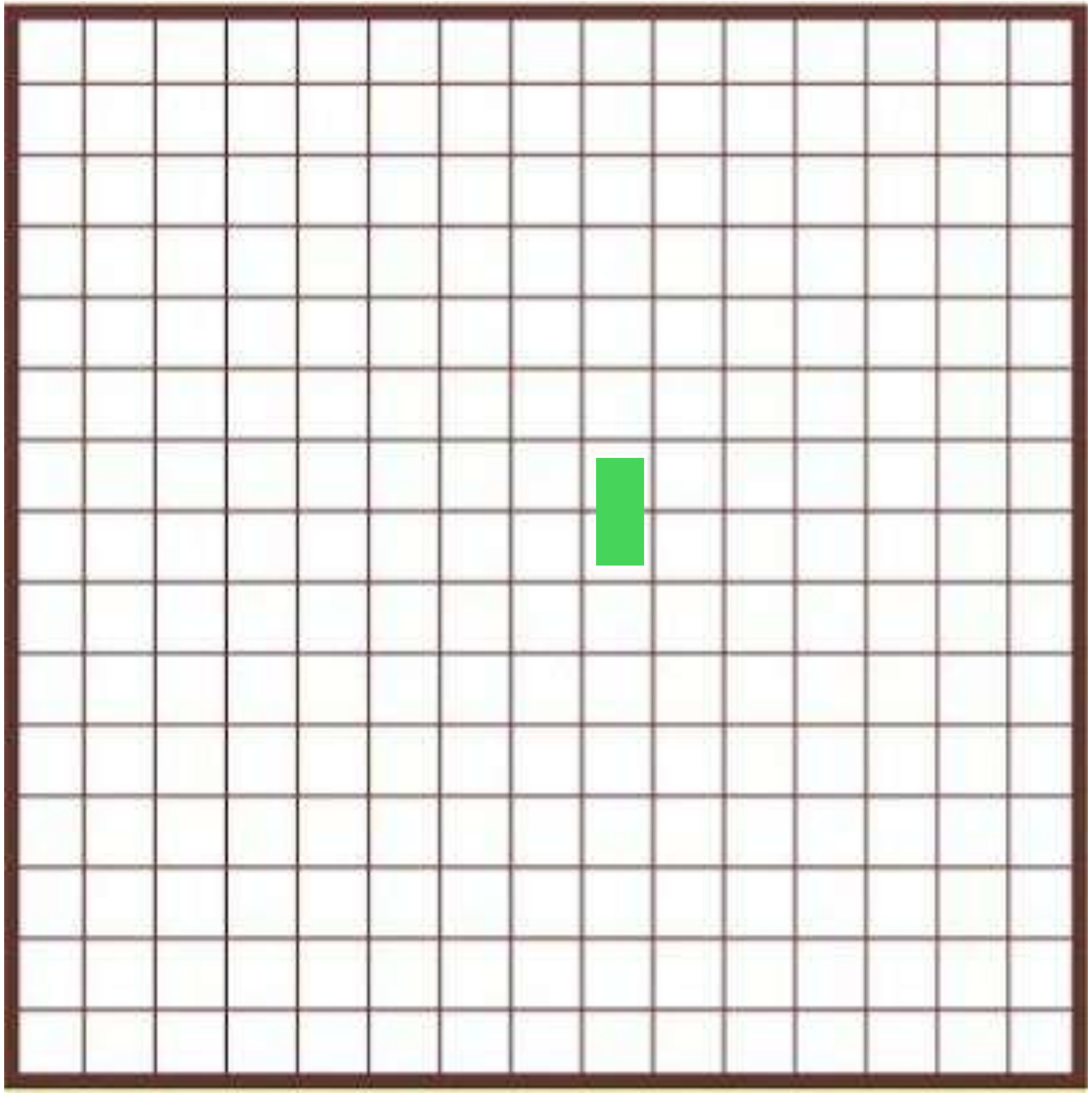


My turn

CRAM



My turn

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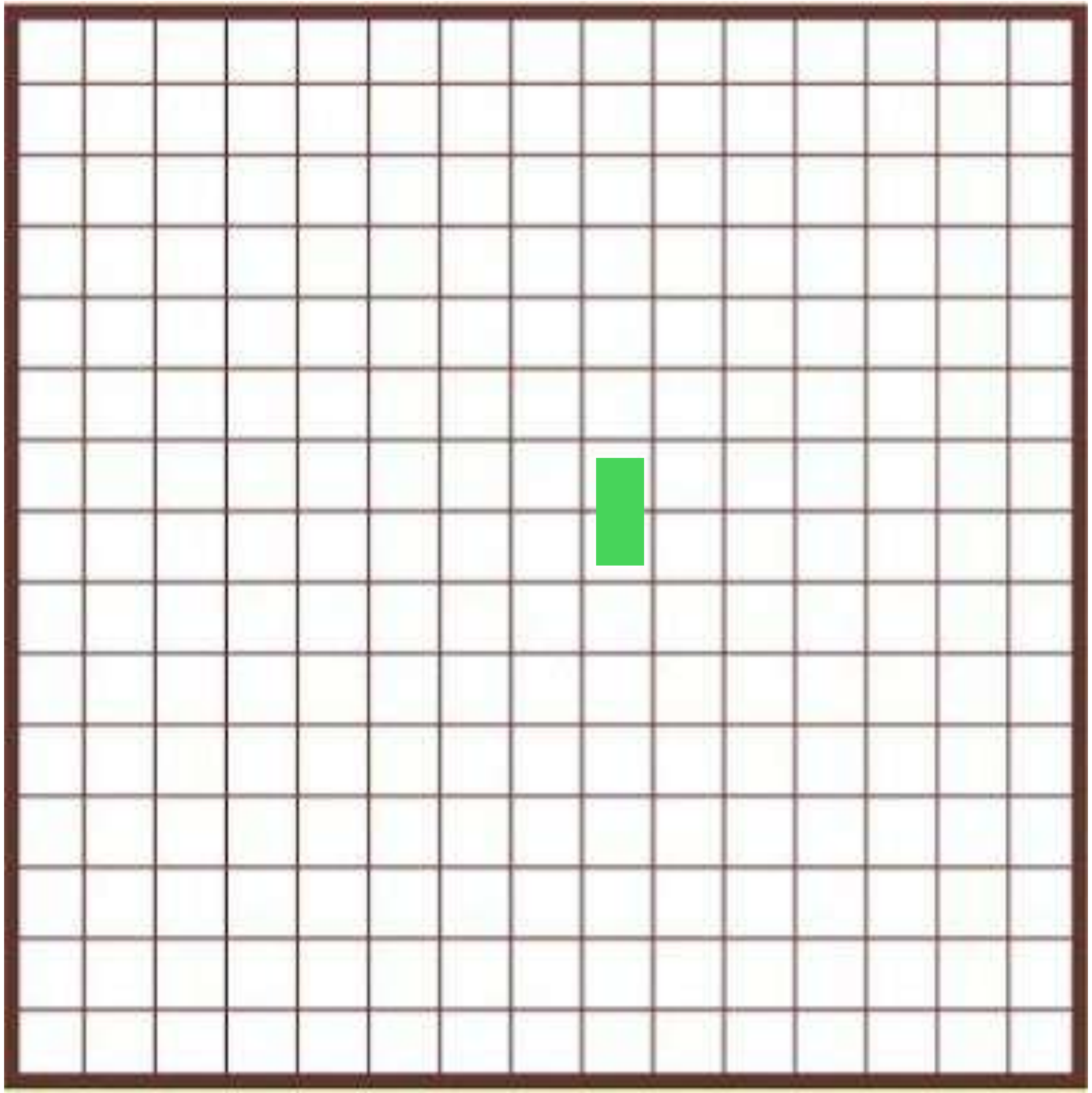

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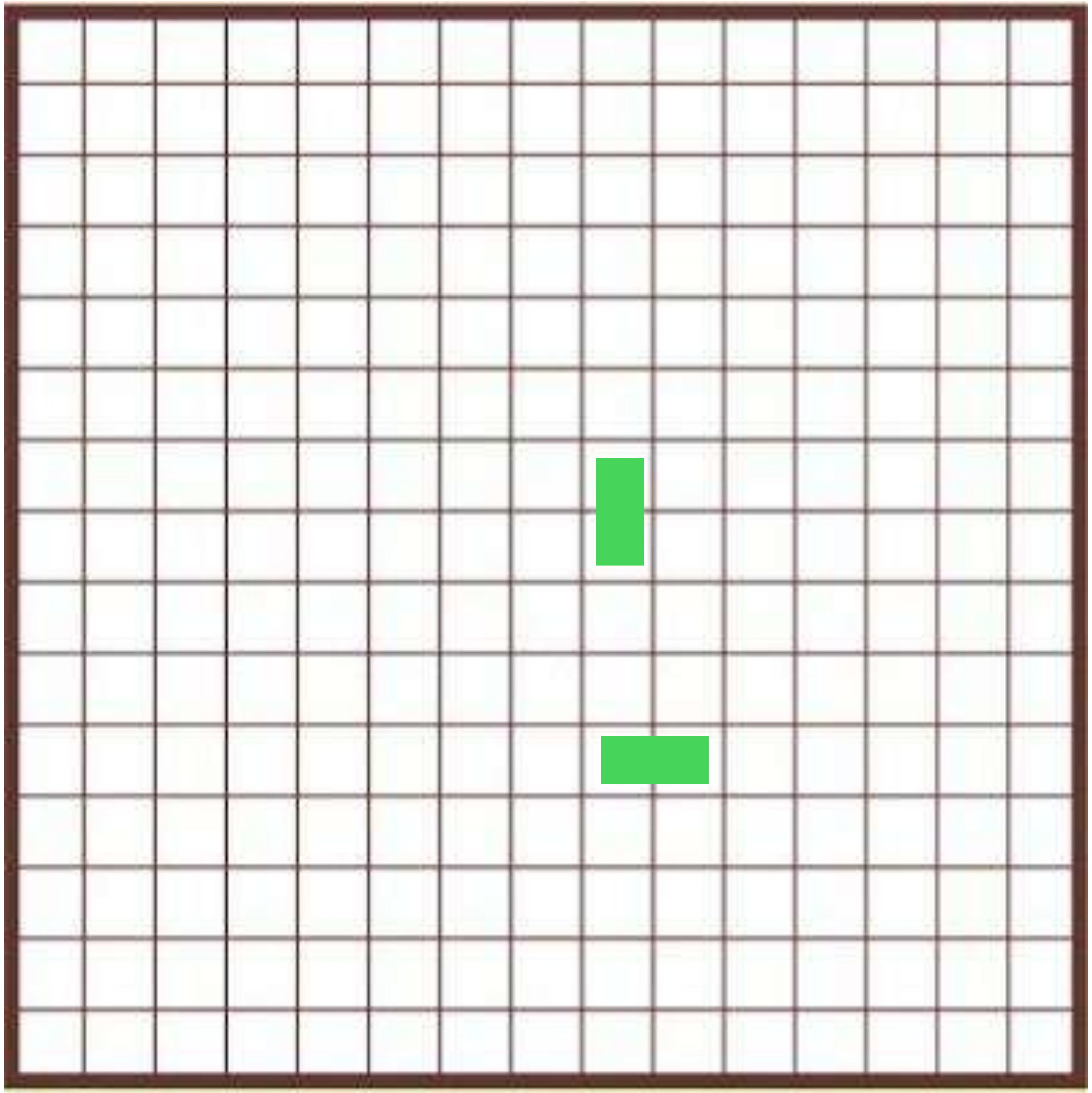

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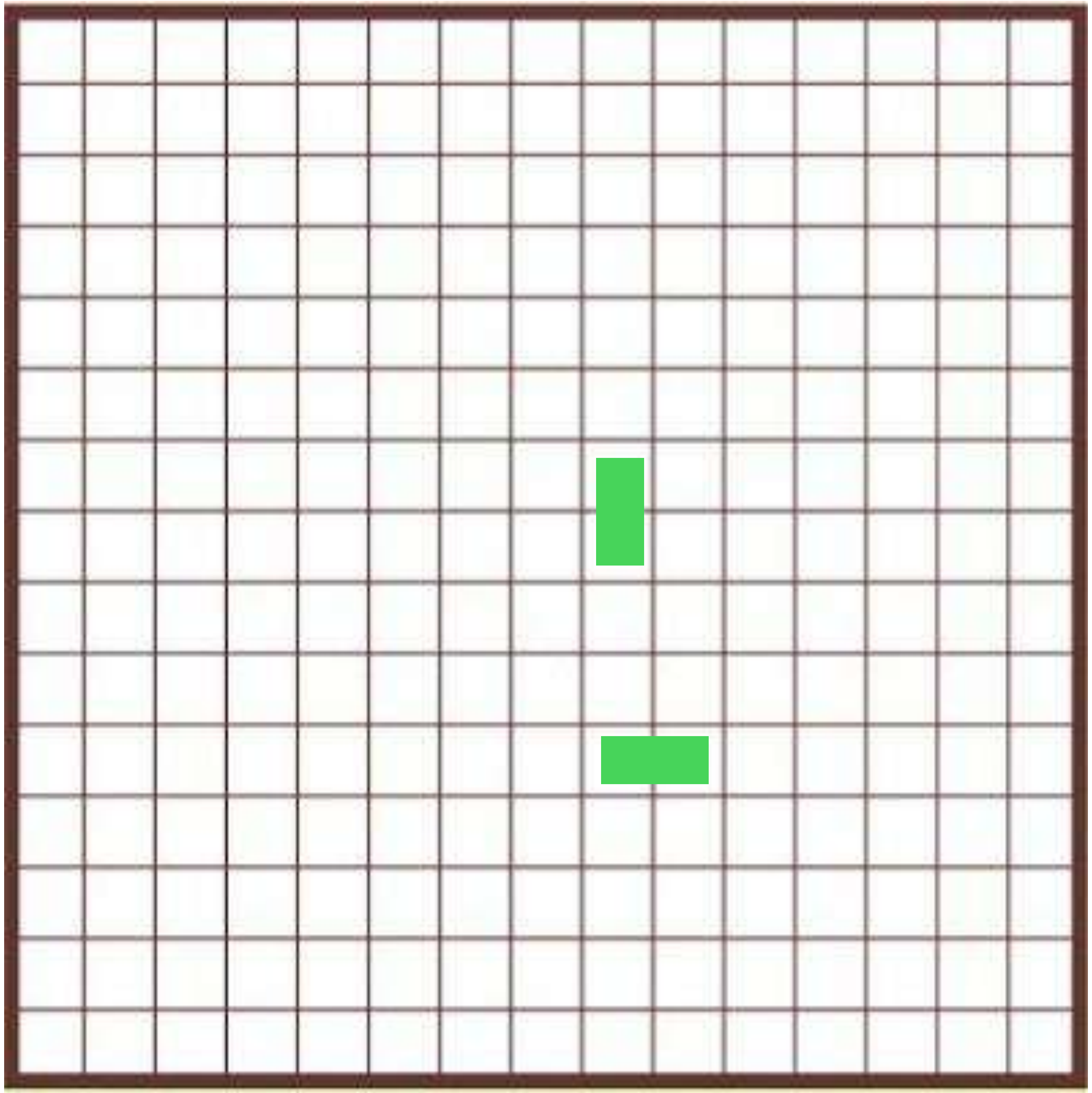

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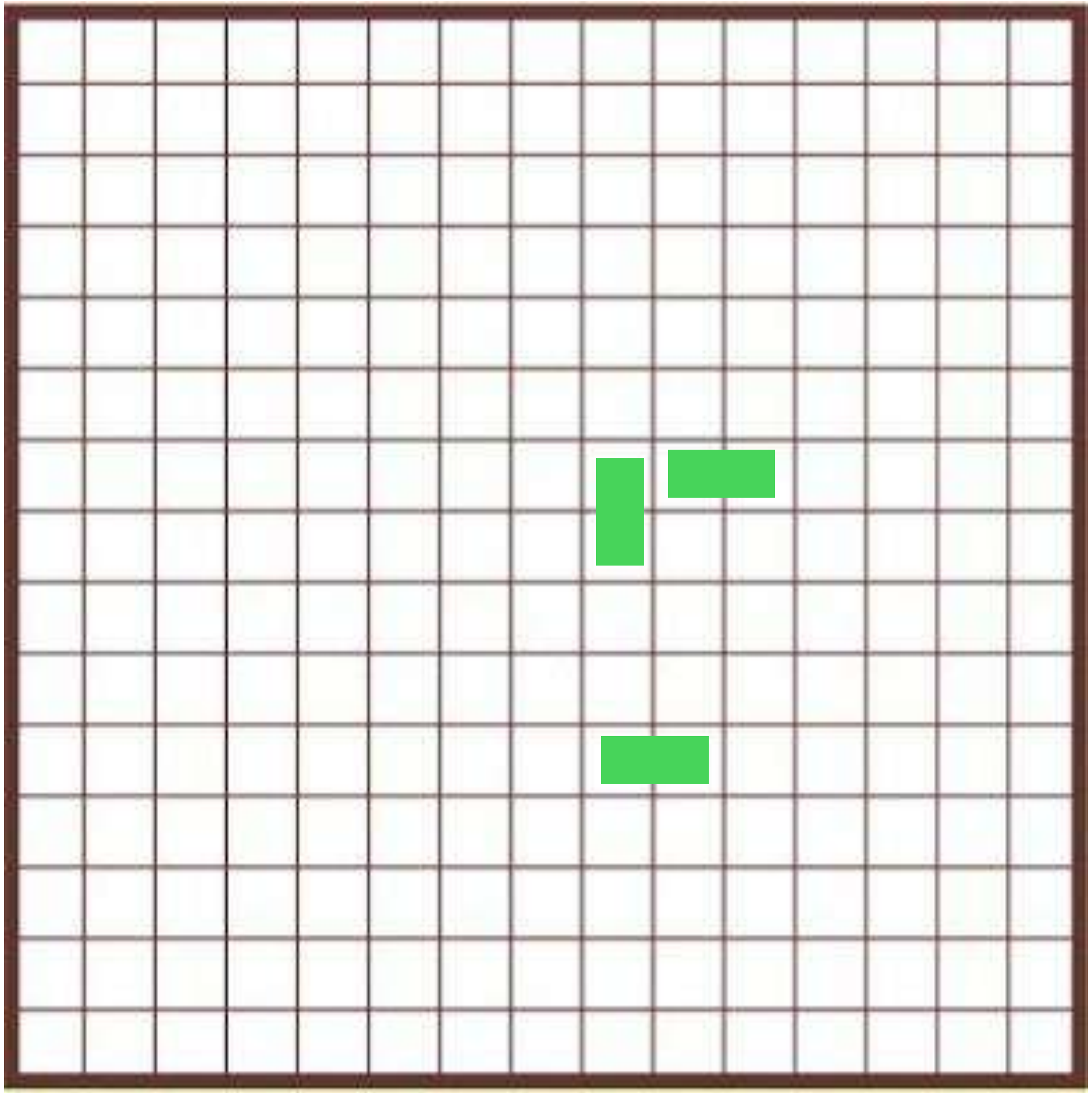

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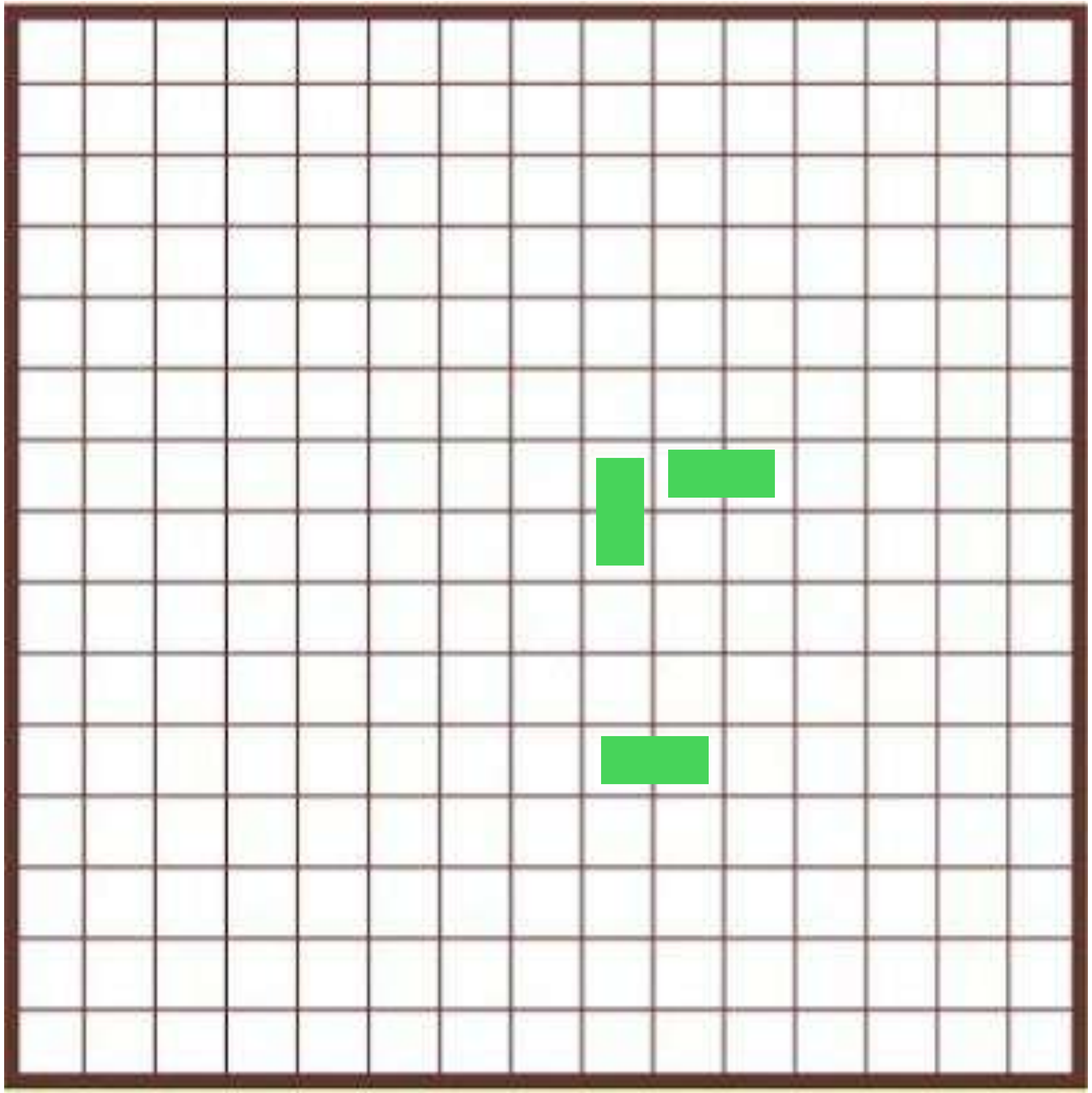

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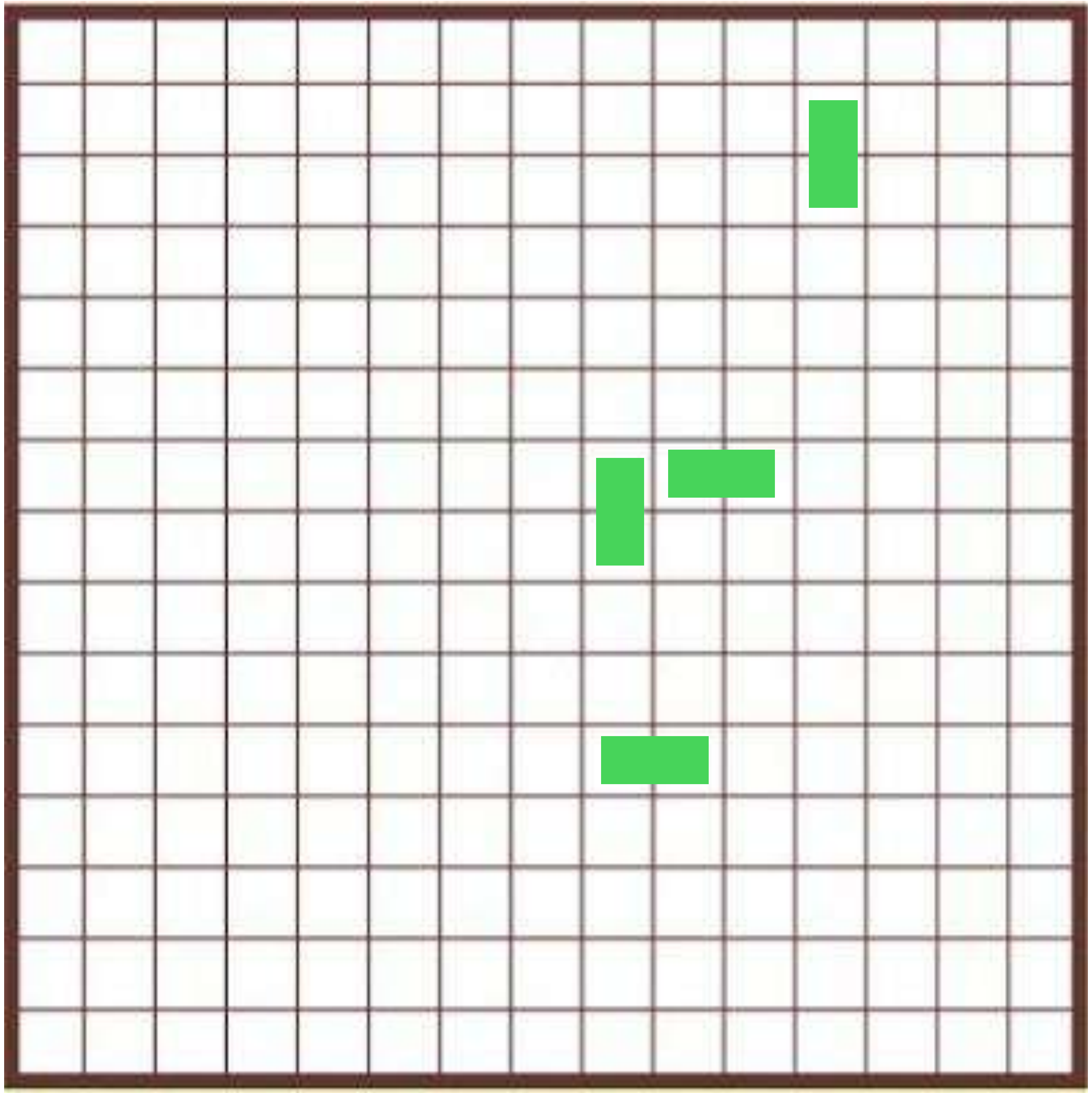

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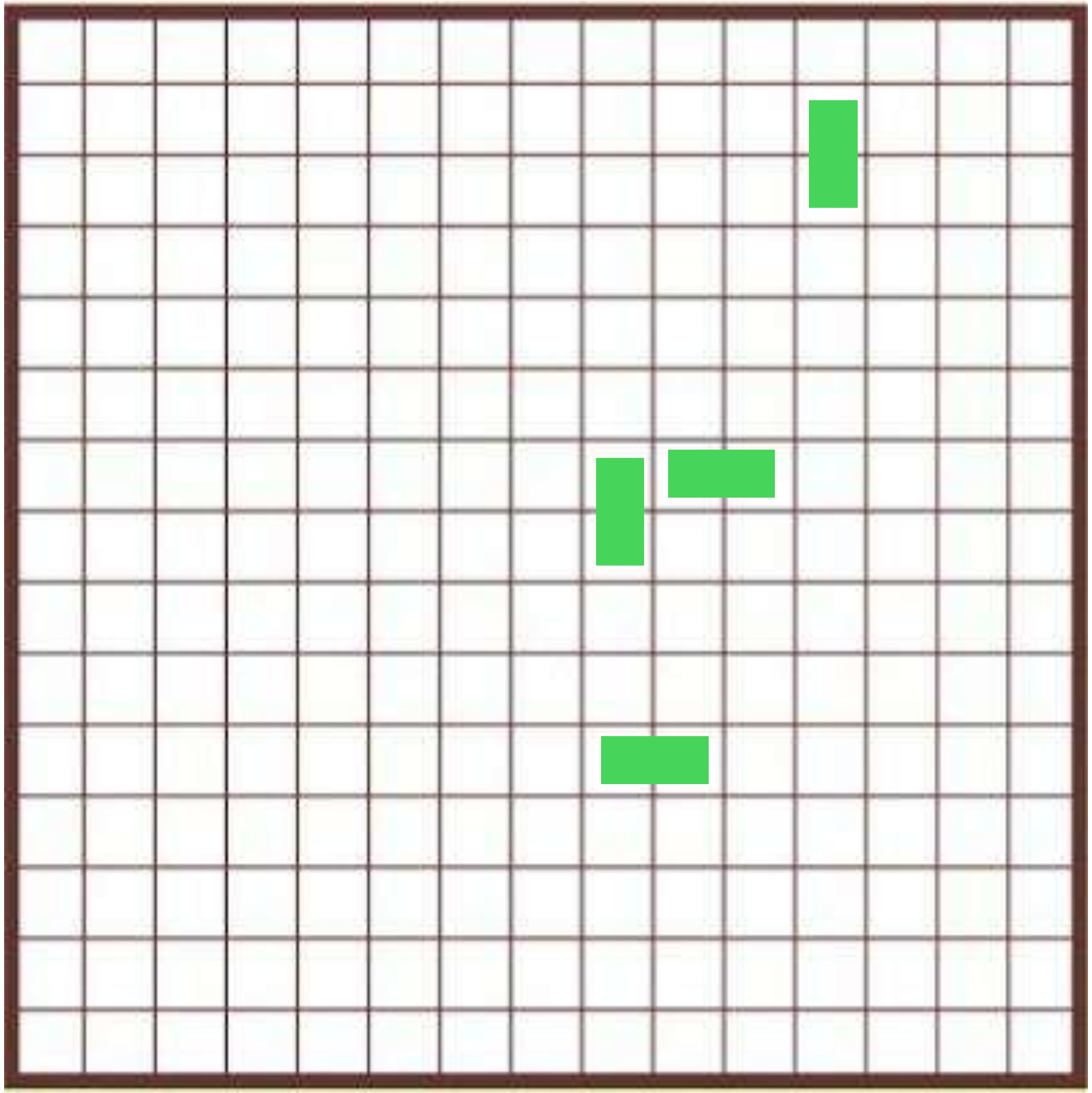

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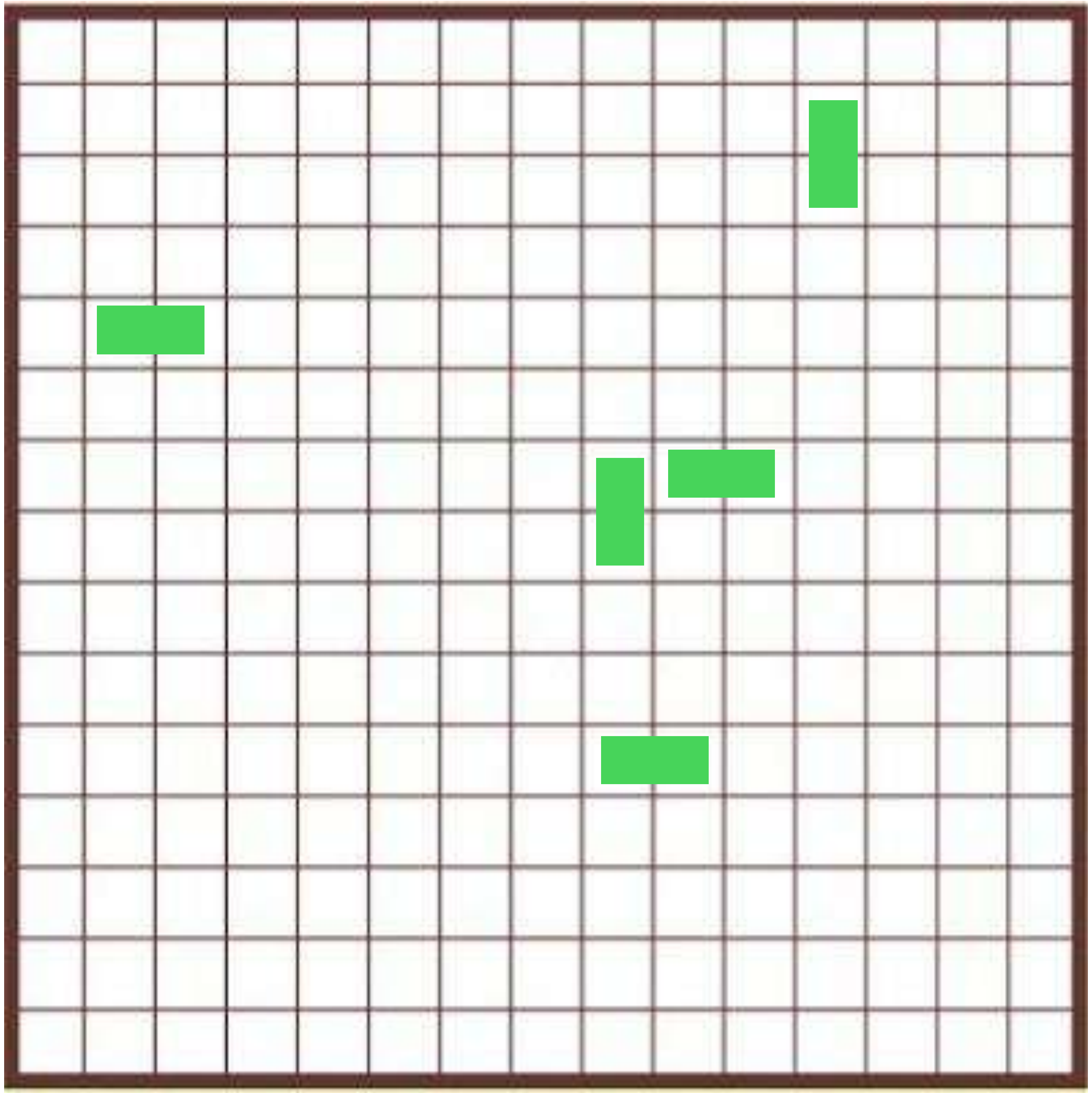

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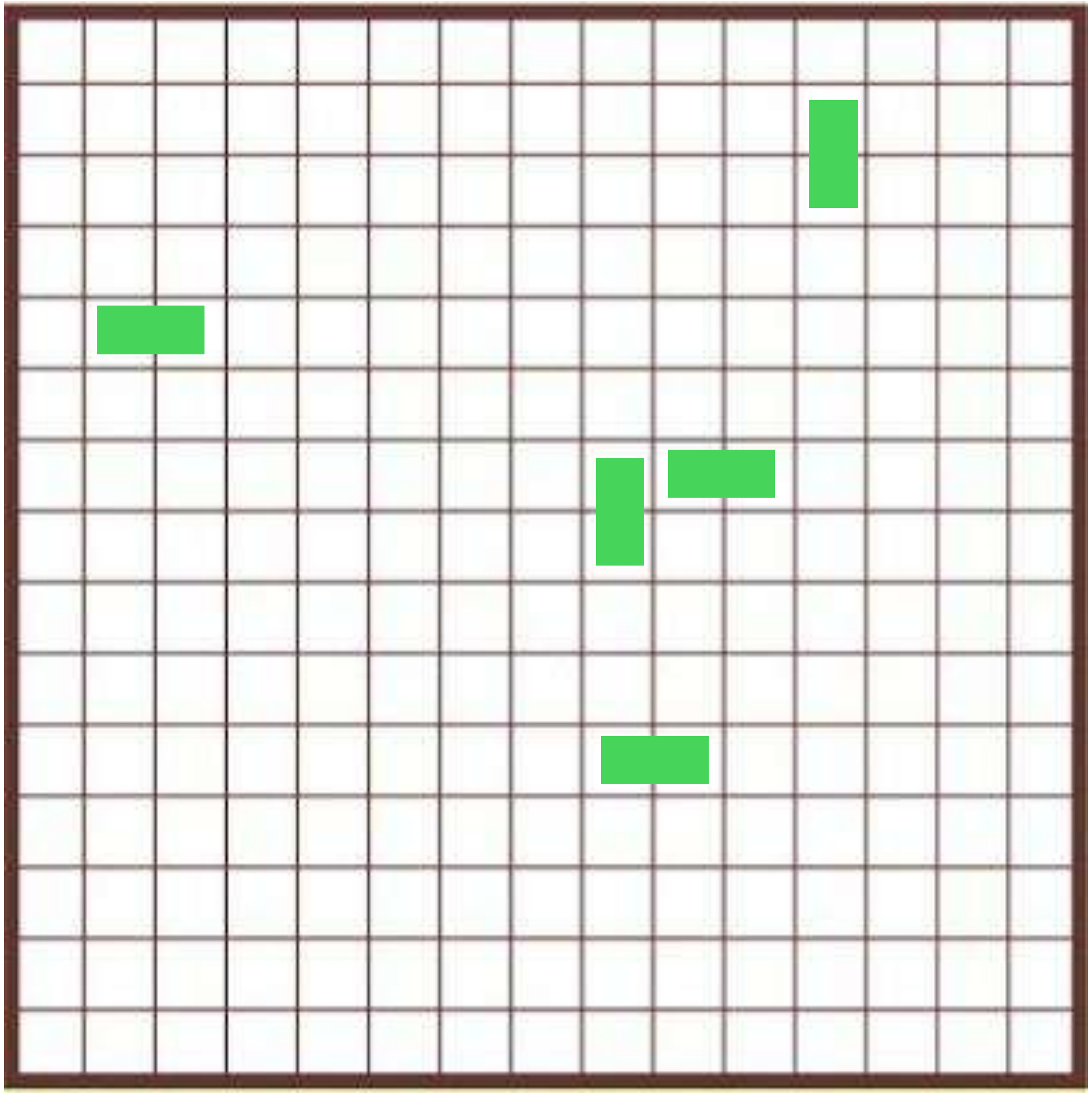

And so on…

WYTHOFF’S QUEENS

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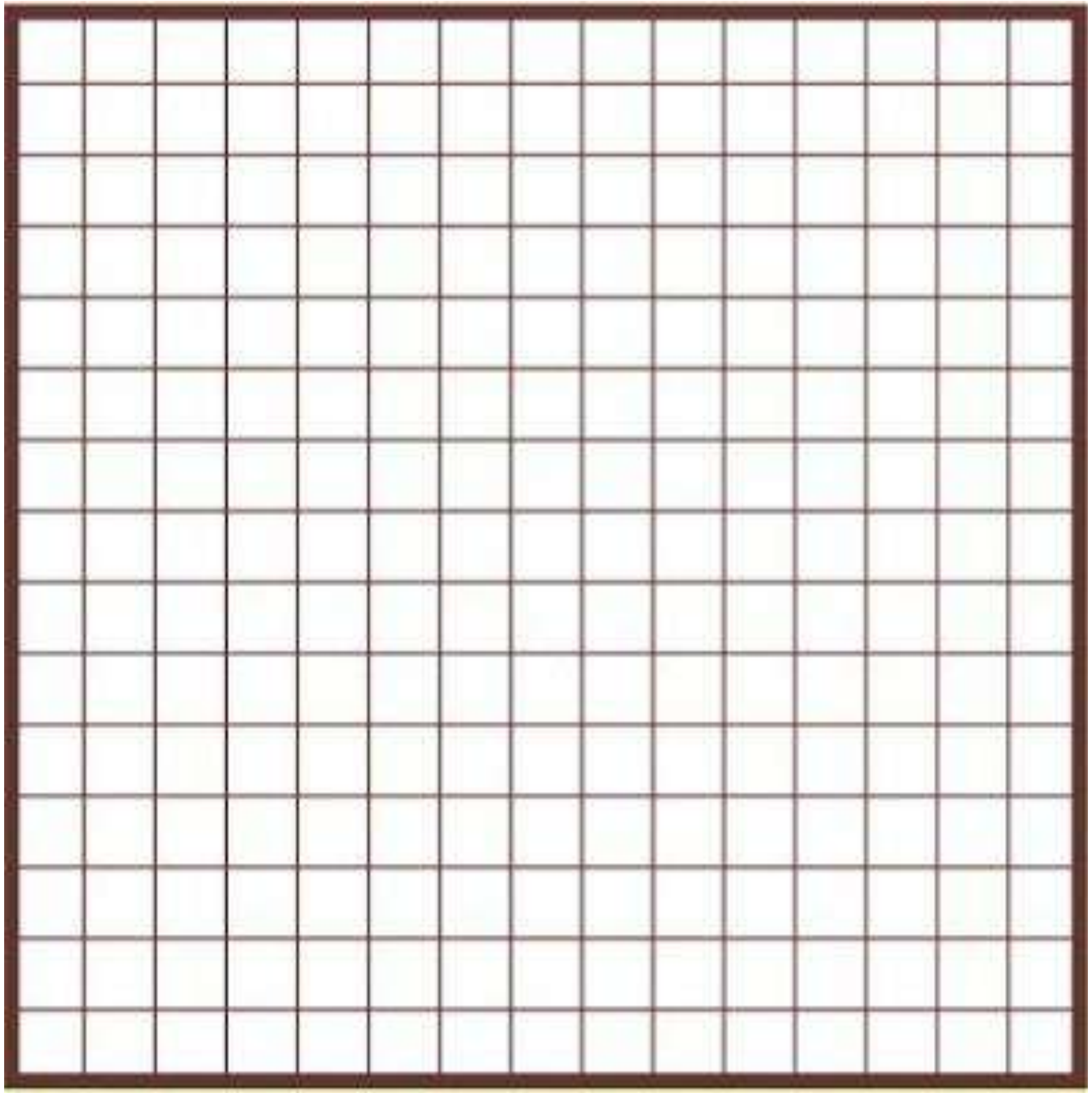

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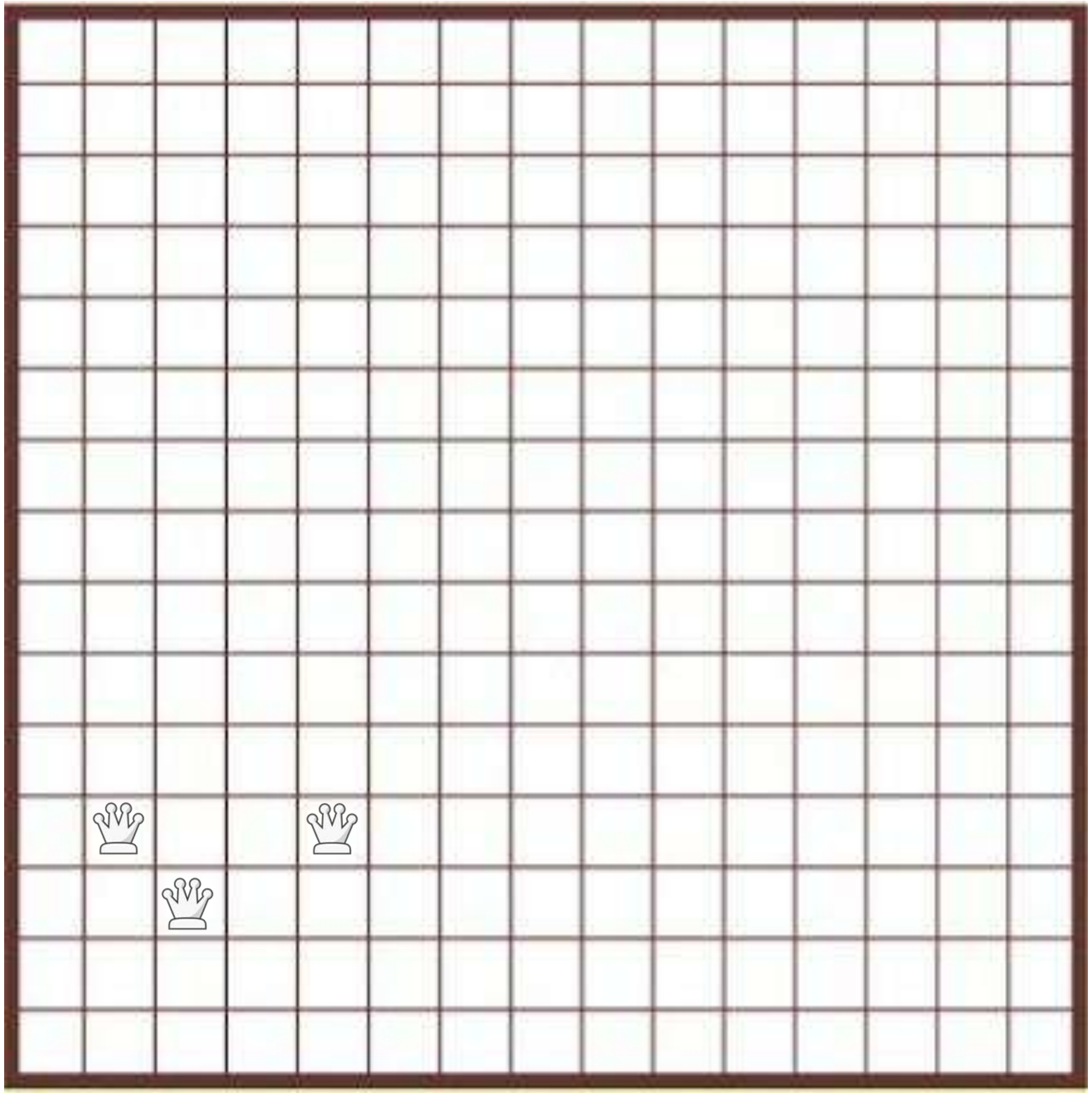

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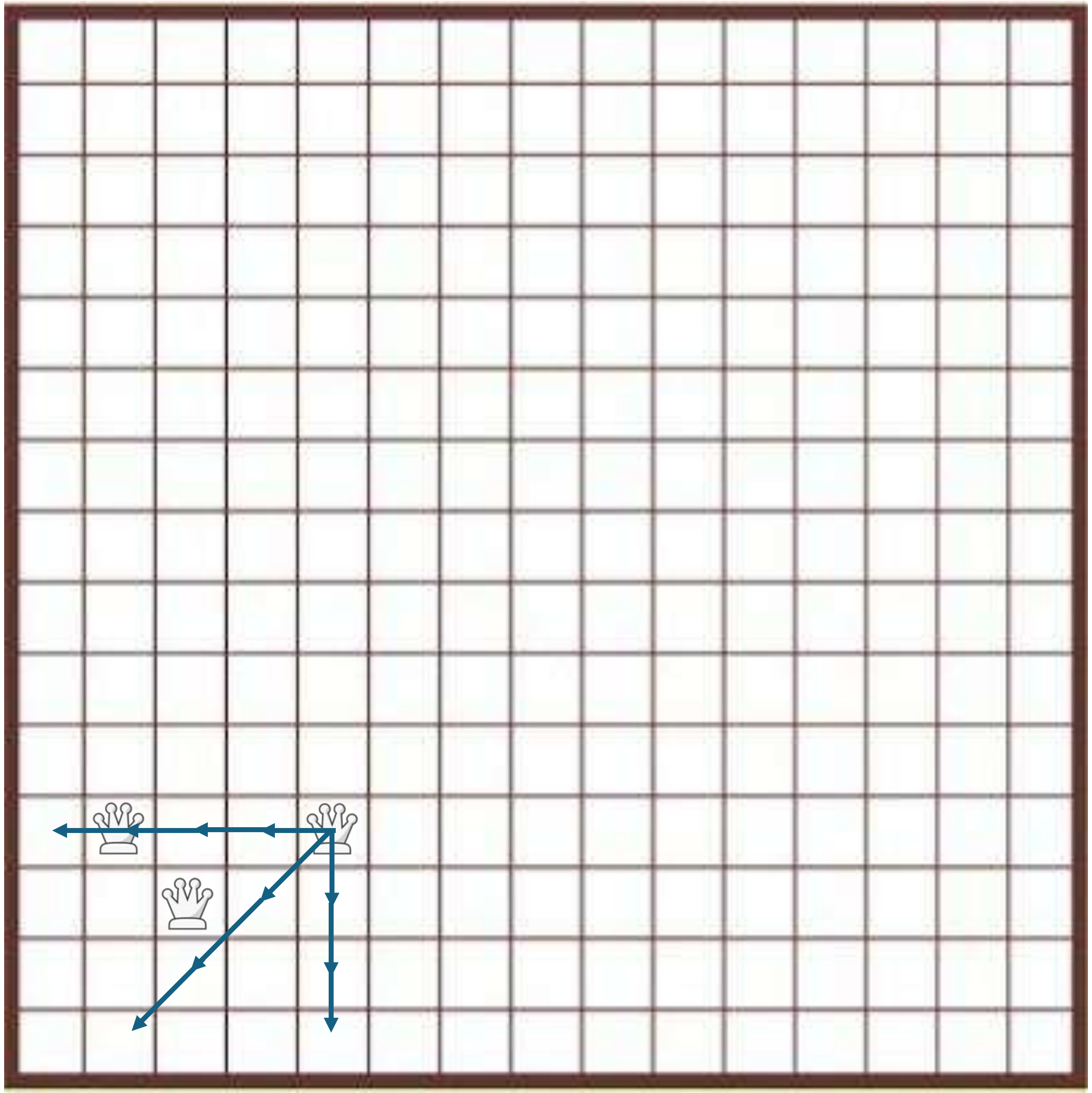

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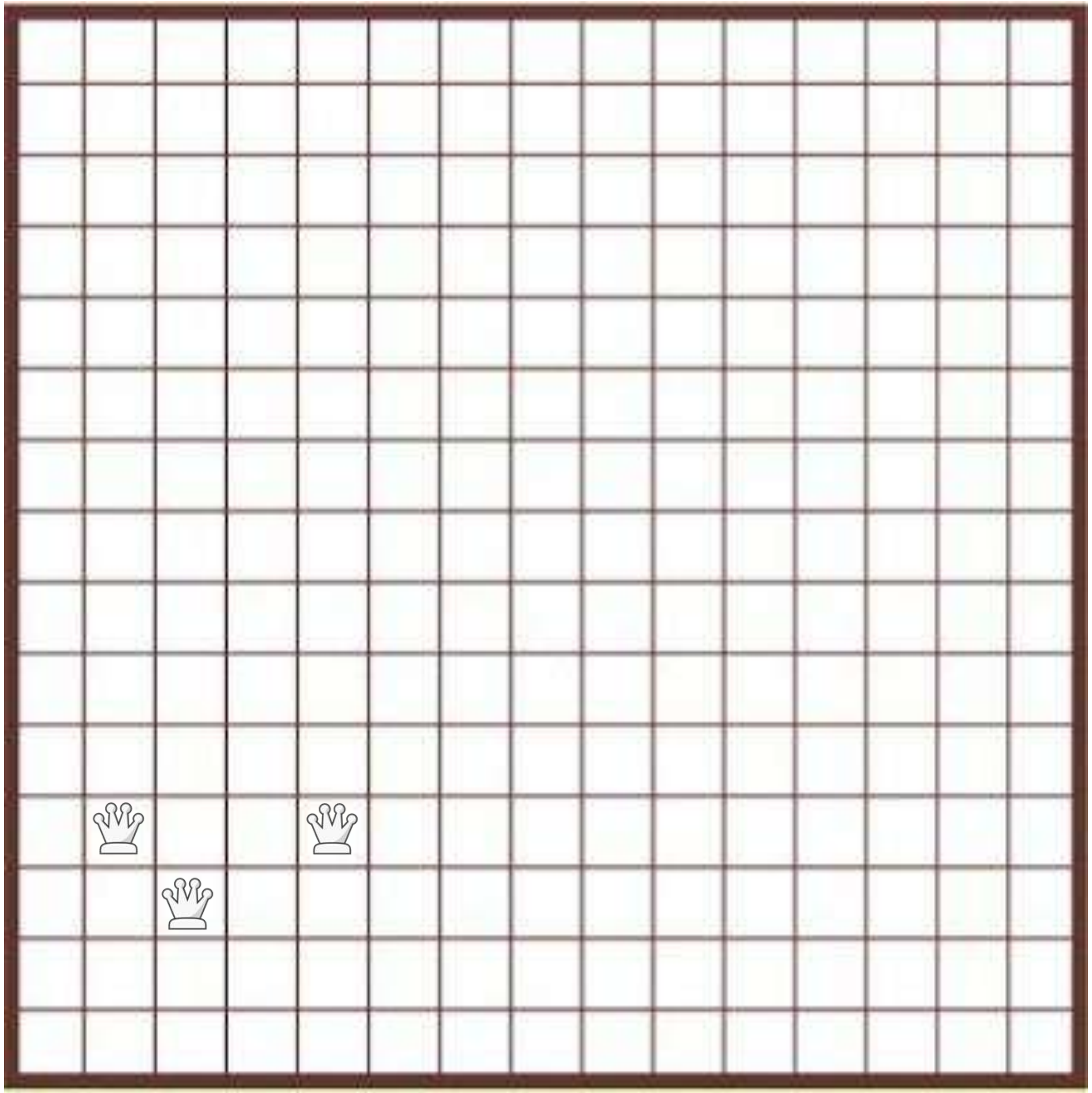

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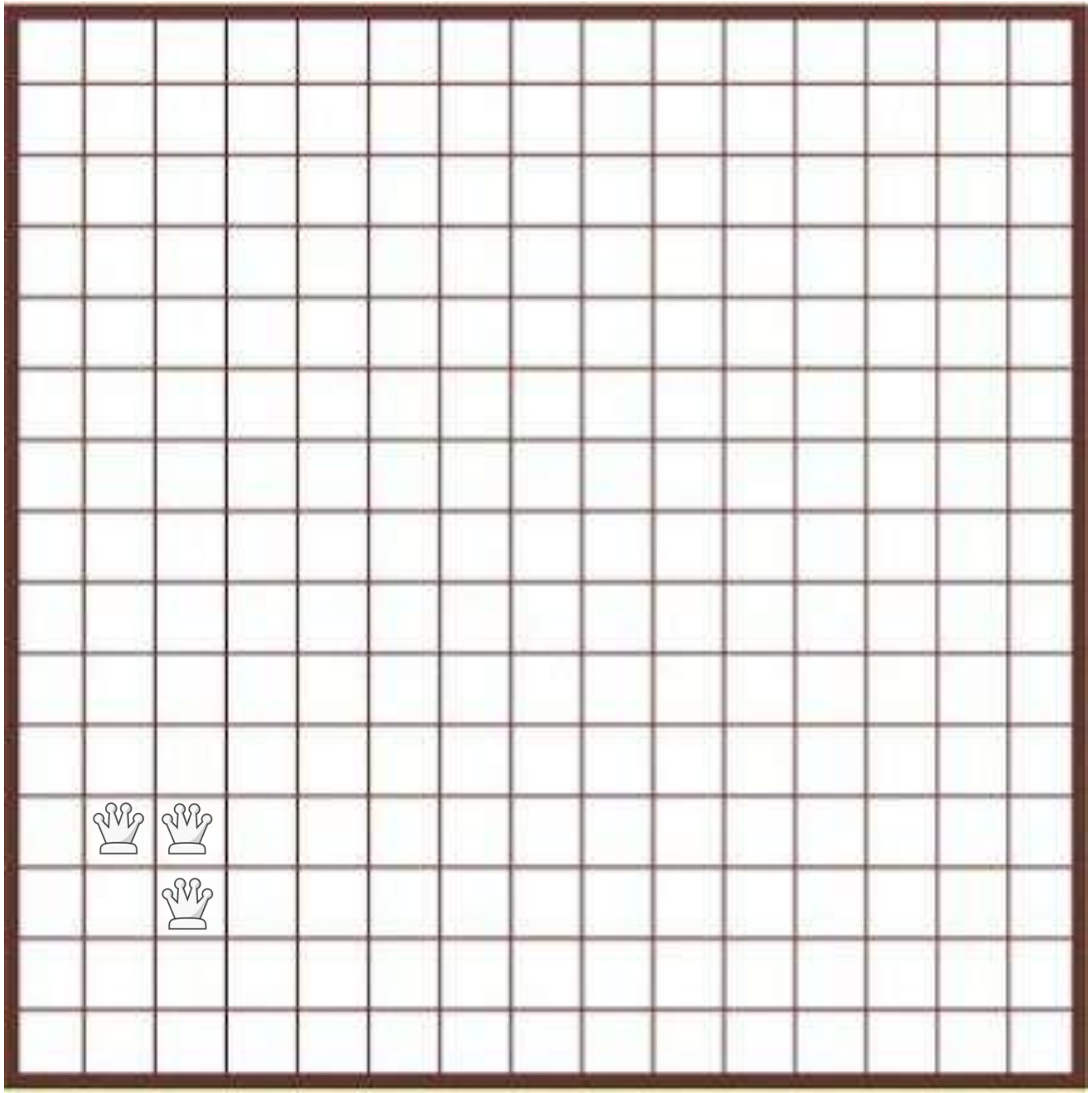

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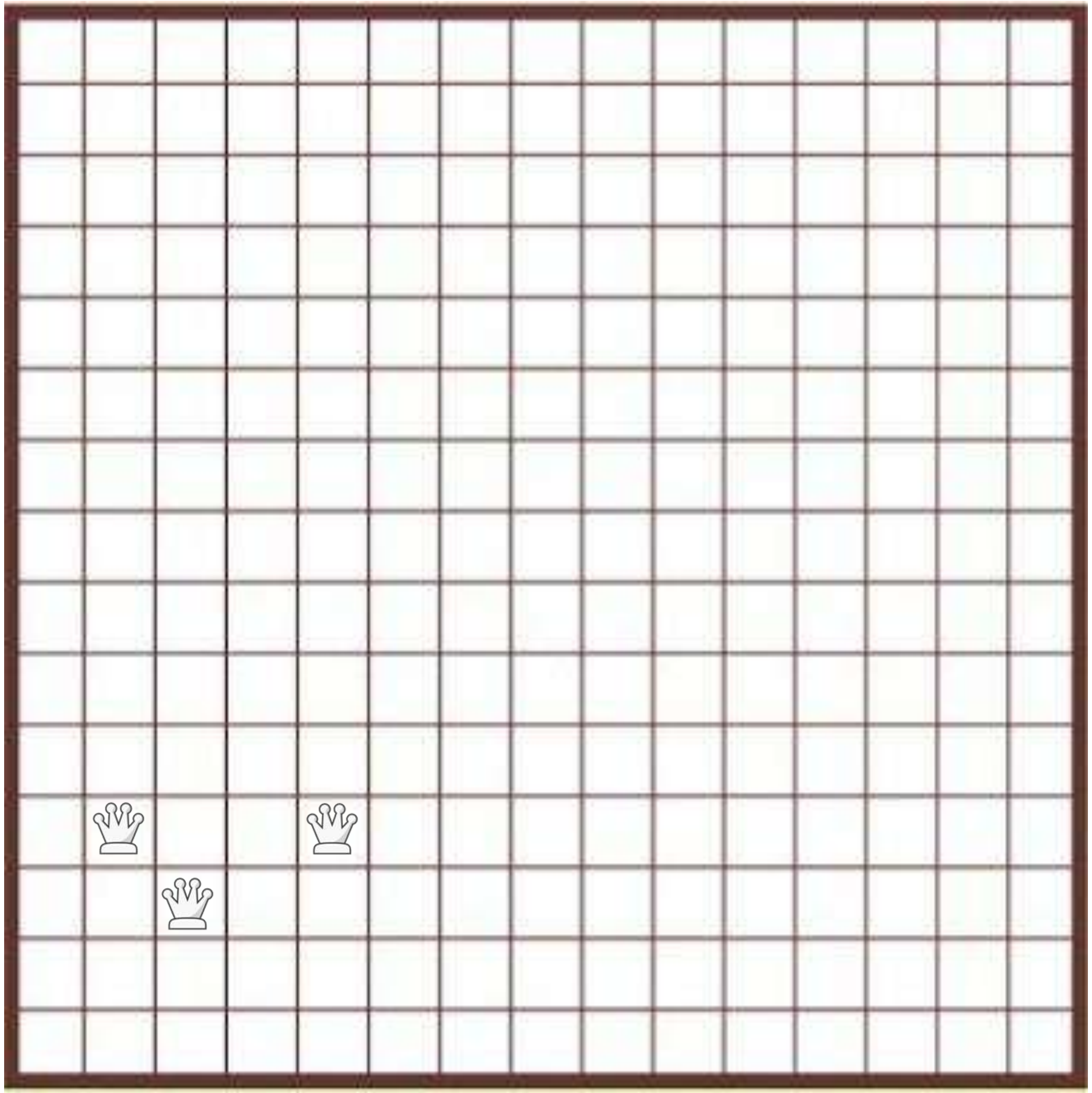
WYTHOFF’S QUEENS



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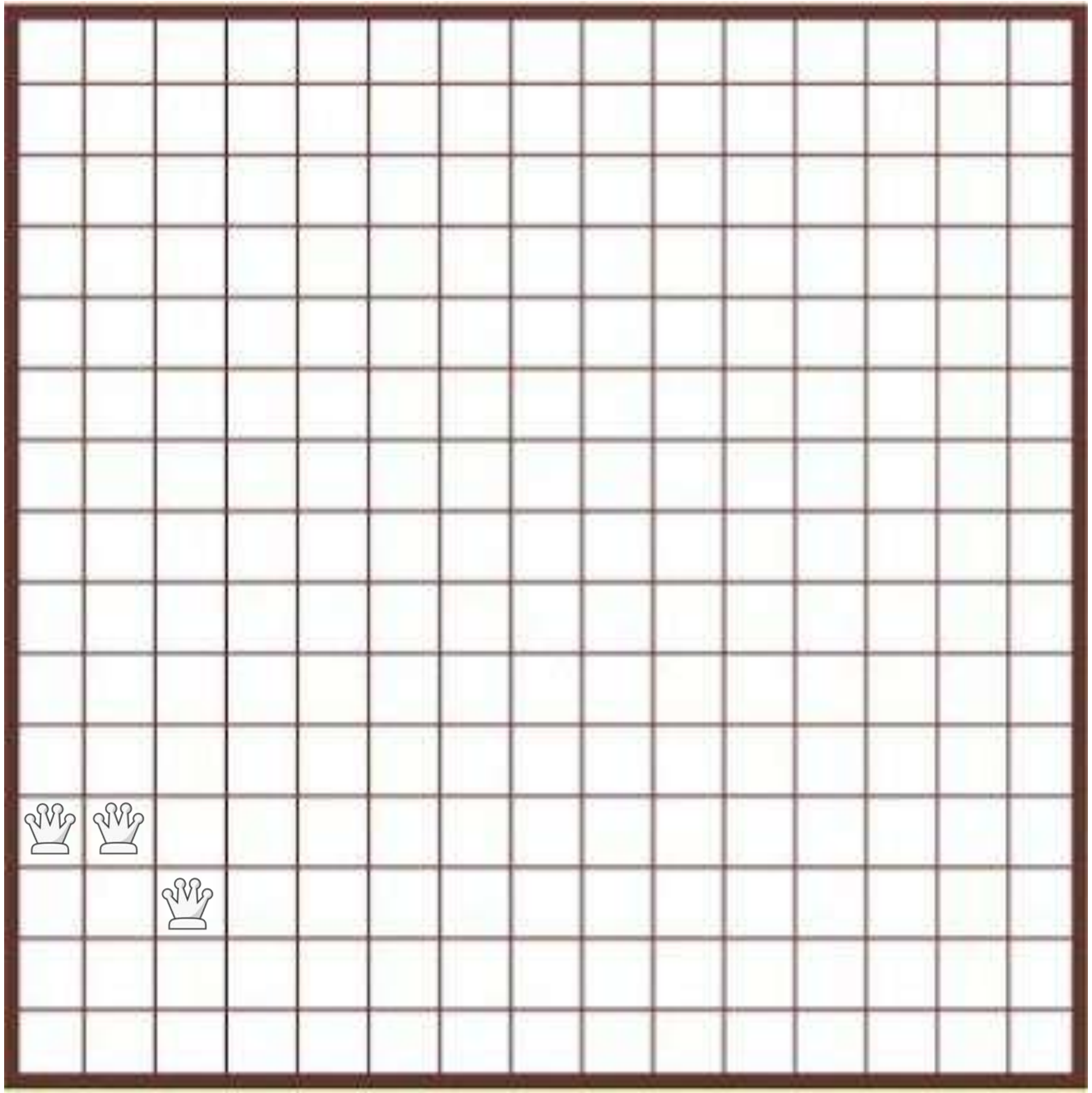

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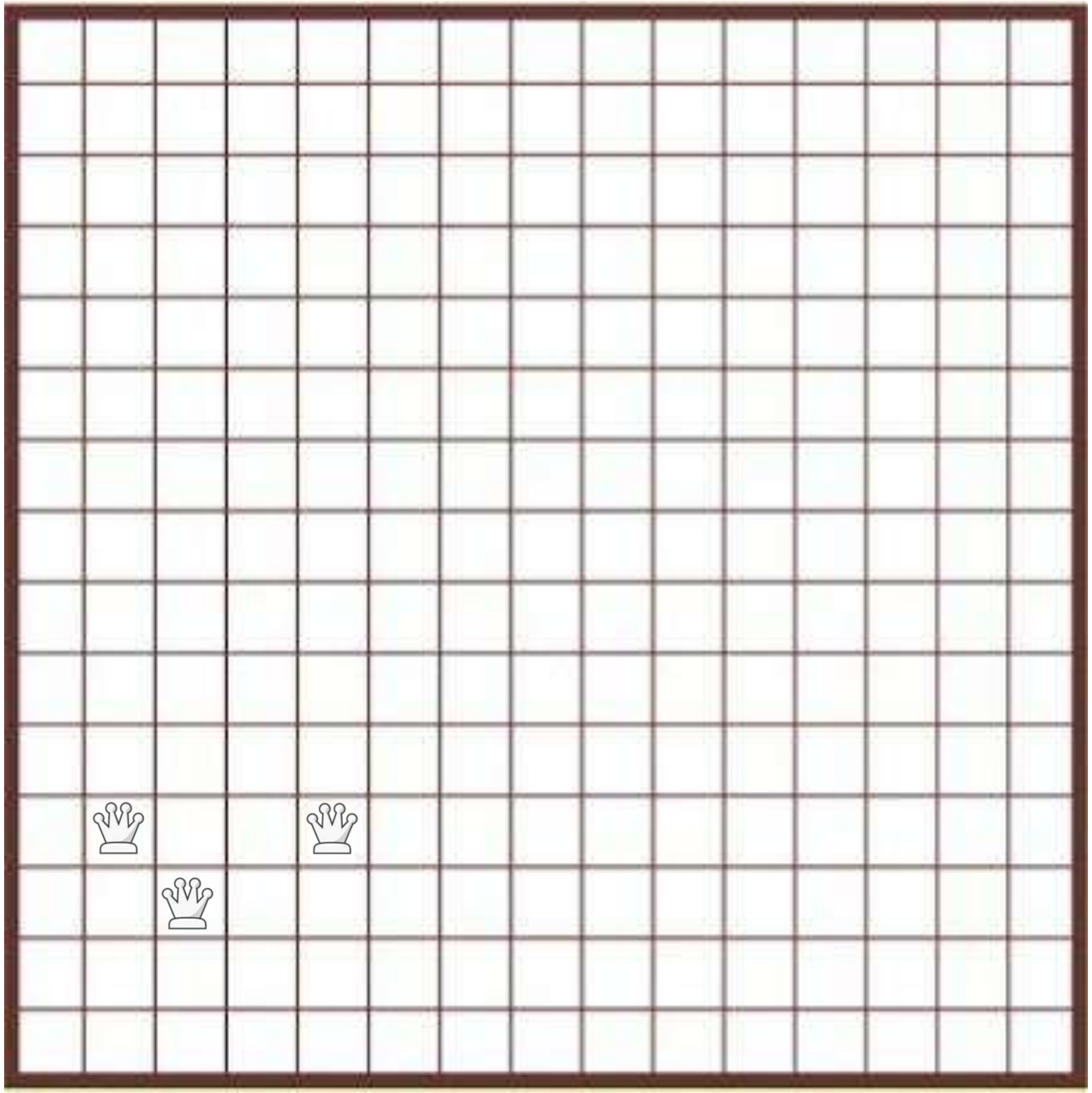

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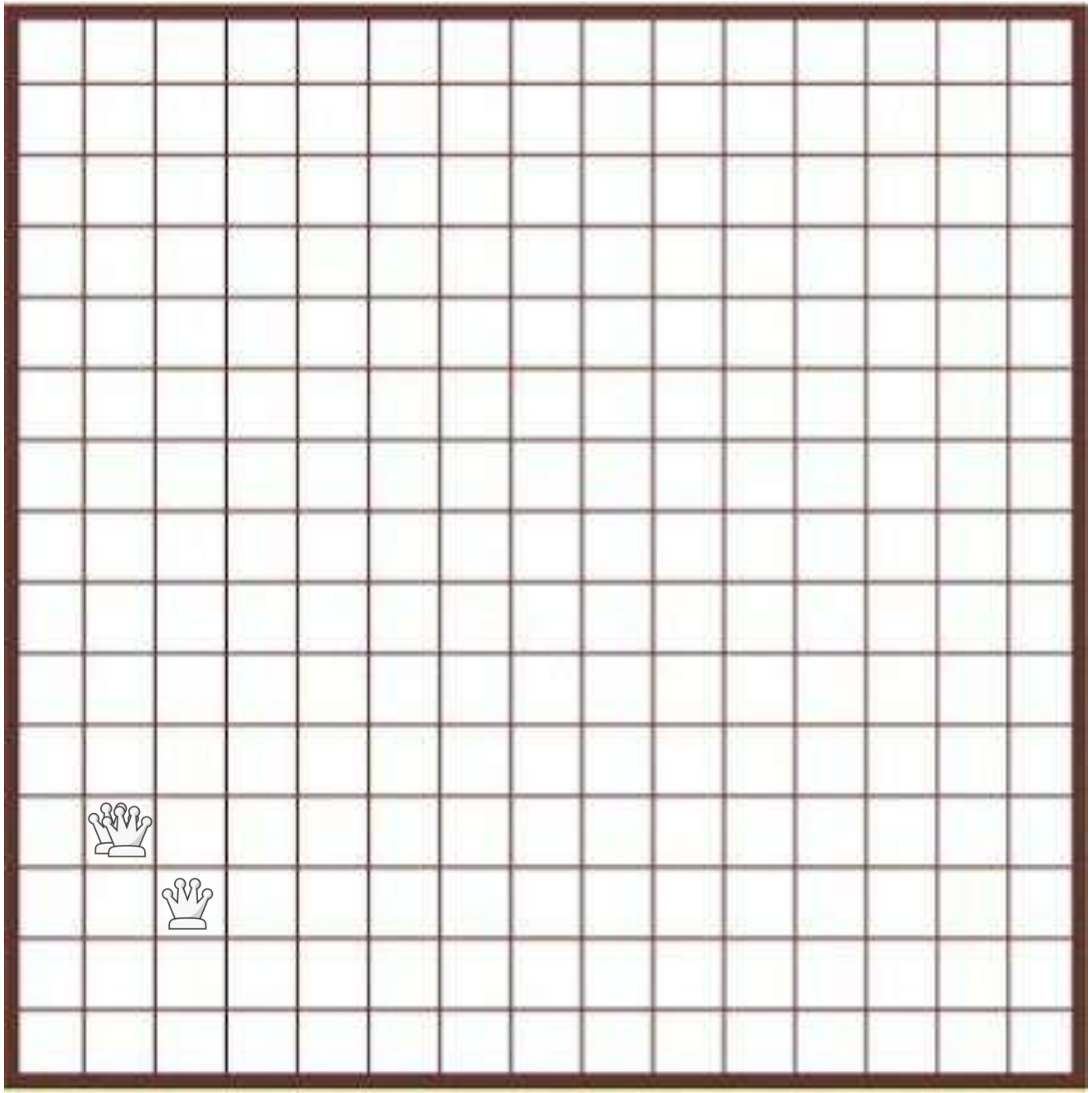

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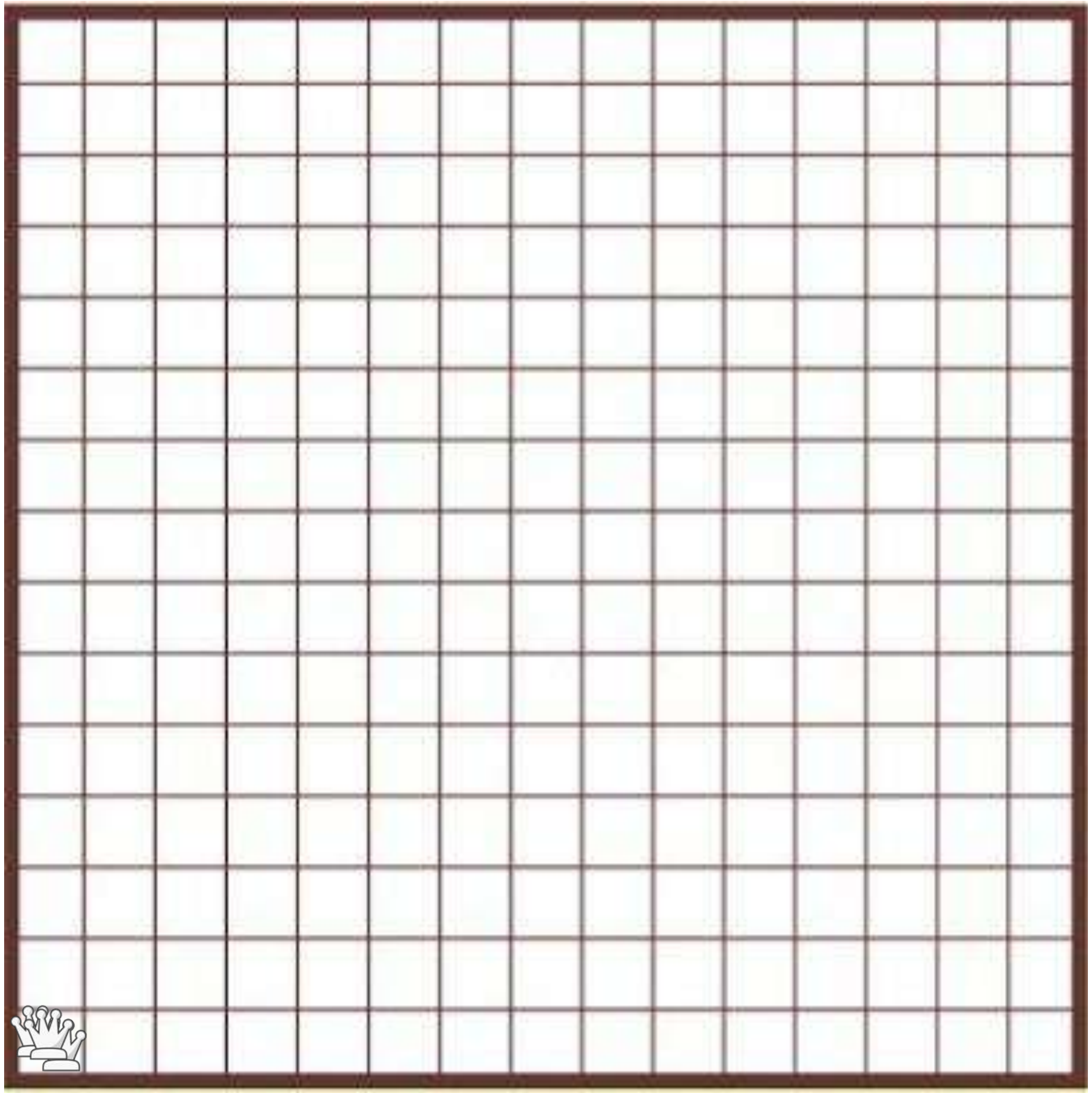


Terminal position

Part I: Impartial Games

I.1: Some famous games
I.2: Contribution of Charles Bouton (1902)

Charles Leonard Bouton

Born	April 25, 1869 St. Louis
Died	February 20, 1922 (aged 52) Cambridge
Resting place	Mount Auburn Cemetery
Nationality	United States of America
Occupation(s)	mathematician, university teacher

NIM, A GAME WITH A COMPLETE MATHEMATICAL THEORY.

By Charles L. Bouton.

The game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory.* The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

1. Description of the Game. The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:—The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

It is the writer's purpose to prove that if one of the players, say A, can leave one of a certain set of numbers upon the table, and after that plays without mistake, the other player, B, *cannot* win. Such a set of numbers will be called a *safe combination*. In outline the proof consists in showing that if A leaves a safe combination on the table, B at his next move cannot leave a safe combination, and whatever B may draw, A at his next move can again leave a safe combination. The piles are then reduced, A always leaving a safe combination, and B never doing so, and A must eventually take the last counter (or counters).

2. Its Theory. A *safe combination* is determined as follows: Write the number of the counters in each pile in the binary scale of notation,† and

* The modification of the game given in §6 was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

† For example, the number 9, written in this notation, will appear as

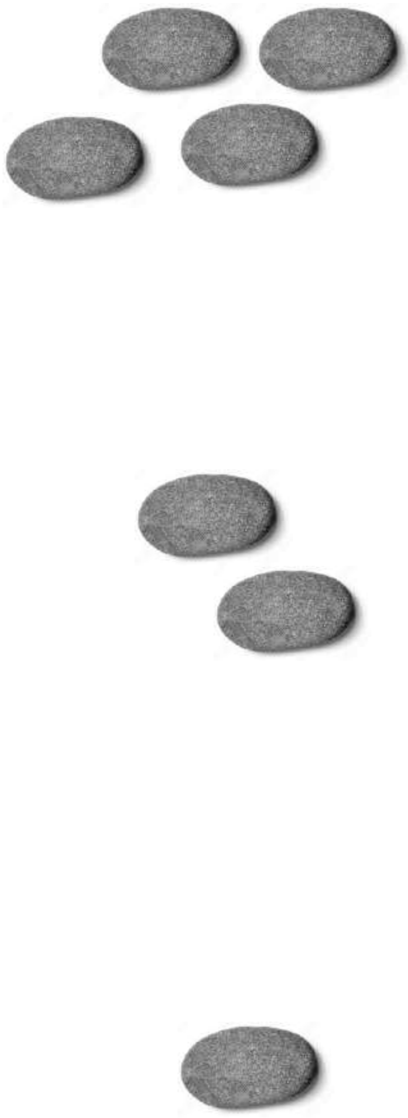
$$1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 1001.$$

(35)

1902

NIM

NIM



4	2	1

NIM

4 →

4	2	1
1	0	0

NIM

4 →

2 →

4	2	1
1	0	0
	1	0

NIM

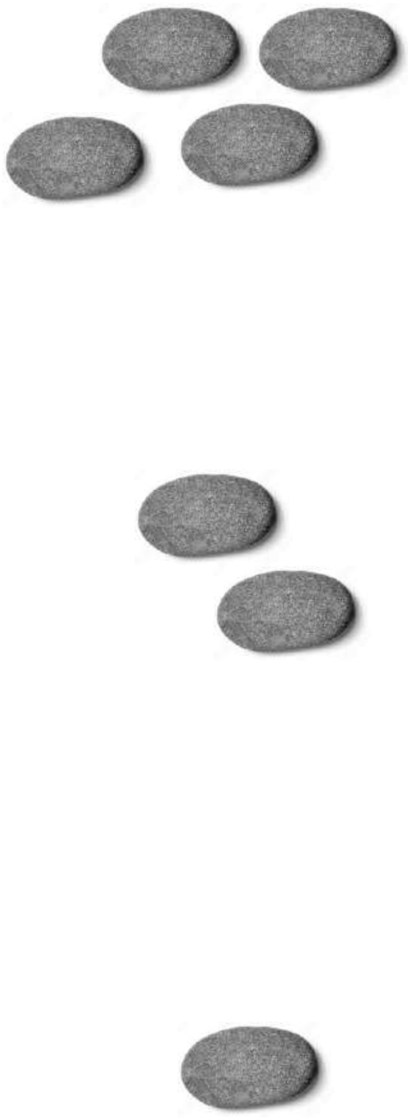
4 →

2 →

1 →

4	2	1
1	0	0
	1	0
		1

NIM



	4	2	1	
4	1	0	0	
2		1	0	
1			1	
⊕	1	1	1	7

NIM

	4	2	1	
4 →	1	0	0	
2 →		1	0	
1 →			1	
⊕	1	1	1	7

My turn

NIM

	4	2	1	
3 →		1	1	
2 →		1	0	
1 →			1	
⊕		0	0	0

NIM

	4	2	1	
3 →		1	1	
2 →		1	0	
1 →			1	
⊕		0	0	0

Your turn

NIM

	4	2	1	
3 →		1	1	
2 →		1	0	
0 →			0	
⊕			1	1

NIM

3 →

2 →

	4	2	1	
3		1	1	
2		1	0	
⊕			1	1

NIM

3 →

2 →

	4	2	1	
		1	1	
		1	0	
⊕			1	1

My turn

NIM

2 →

2 →

	4	2	1	
		1	0	
		1	0	
⊕		0	0	0

NIM

2 →

2 →

	4	2	1	
		1	0	
		1	0	
⊕		0	0	0

Your turn

NIM

		4	2	1	
0 →				0	
2 →			1	0	
	⊕		1	0	2

NIM

2 →

	4	2	1	
		1	0	
⊕		1	0	2

NIM

2 →

	4	2	1	
		1	0	
⊕		1	0	2

My turn

NIM

	4	2	1	
$\oplus$			0	0

NIM

I win!

	4	2	1	
⊕			0	0

Why does it work?

1) Whenever the NIM sum results in zero, **if there are** available moves, **any move results in a NIM sum different than zero**.

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(look for the left-most place value).

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If it is your turn and the NIM sum is zero, you should be sad. You are lost.

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(look for the left-most place value).

If the NIM sum is zero, the position is a ***P*-position**.

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(look for the left-most place value).

If the NIM sum is zero, the position is a ***P*-position**.

If the NIM sum is not zero, the position is an ***N*-position**.

POSSIBLE OUTCOMES

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I.1: Some famous games
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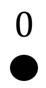
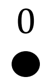
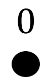
What can be appropriate abstract **game forms**?


0

$$0 = \{ \mid \}$$



$0 = \{ \mid \}$



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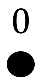
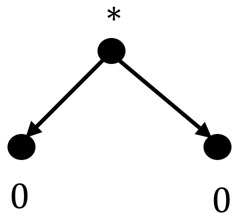
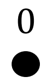
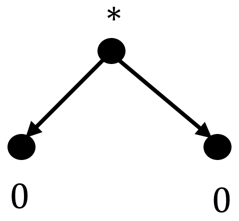




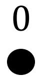
$$0 = \{\ |\ \}$$



$$* = \{0|0\}$$



$0 = \{ \,|\, \}$



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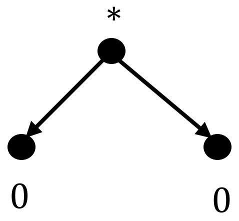

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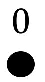

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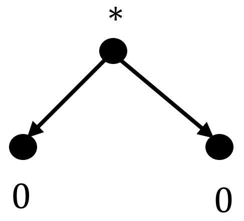

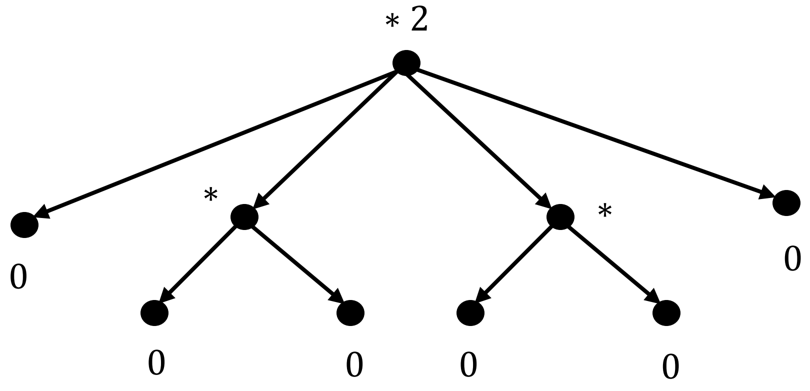

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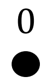
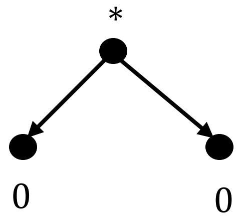
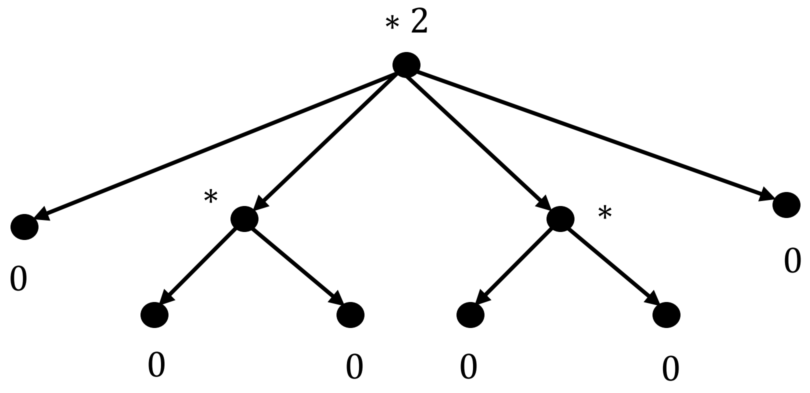


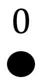
$* = \{0|0\}$



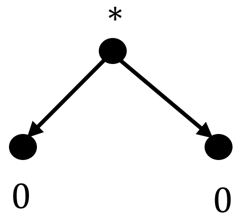
$*2 = \{0,*|0,*\}$



$0 = \{\ |\ \}$



$* = \{0|0\}$



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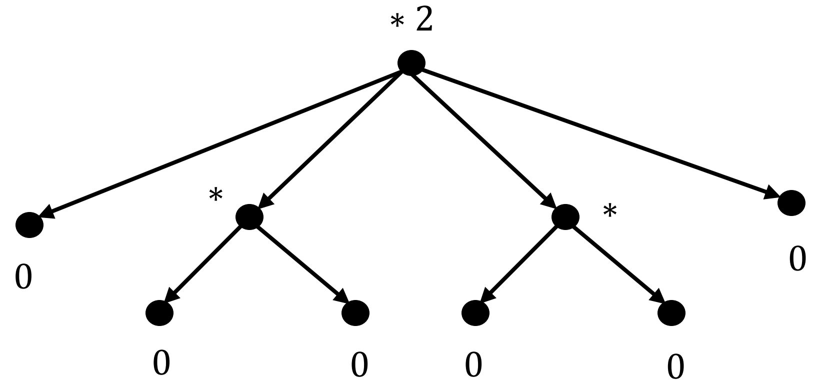
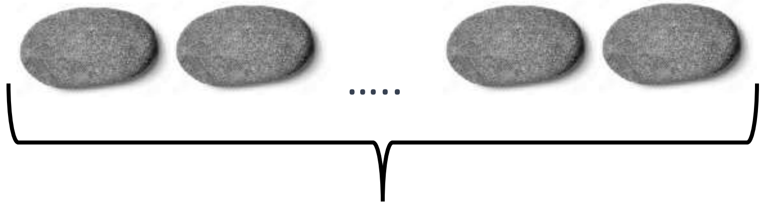
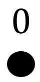


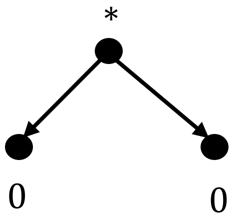


n stones

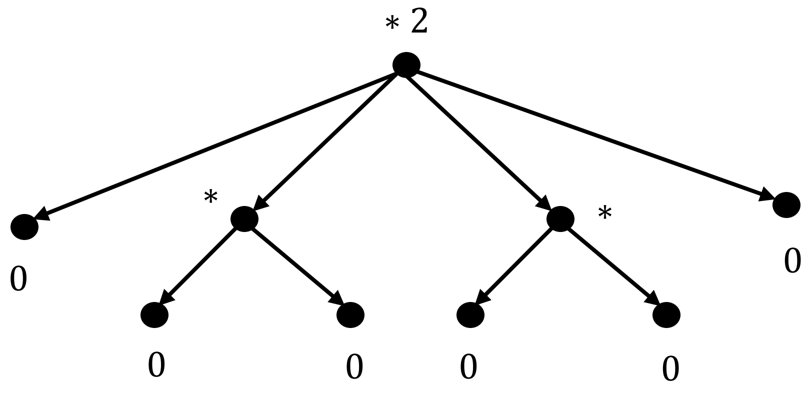
$$0 = \{\ |\ \}$$

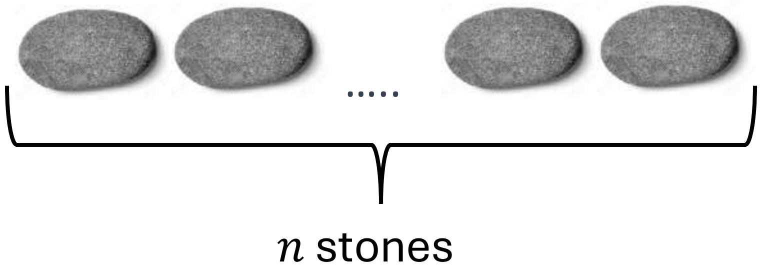


$$* = \{0|0\}$$



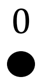
$$* 2 = \{0,*|0,*\}$$



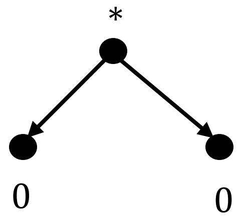


$$* n = \{0,*,\ldots,*\ (n-1)|0,*,\ldots.,*\ (n-1)\}$$

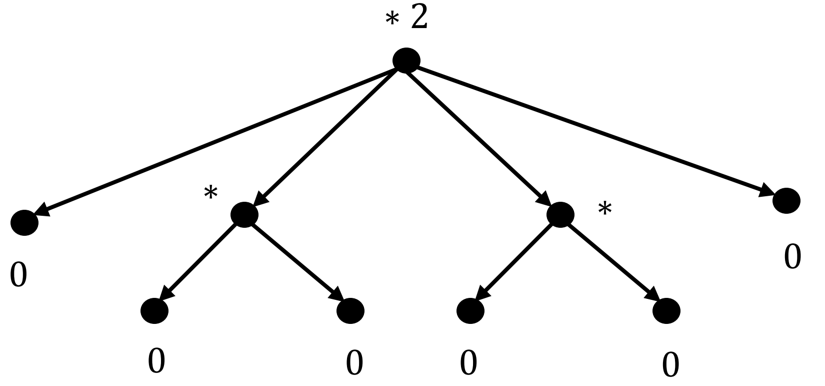
$0 = \{\ |\ \}$

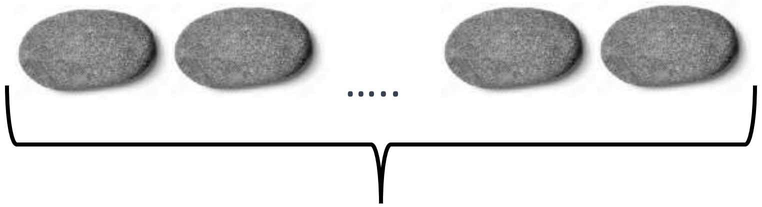


$* = \{0|0\}$



$*\,2 = \{0,*|0,*\}$





n stones

$*\,n = \{0,*,\ldots,*\,(n-1)|0,*,\ldots.,*\,(n-1)\}$

NIMBERS

$$G = \{G^{\mathrm{L}} \mid G^{\mathrm{R}}\}$$

How can one formalize a situation where there is more than one pile (**disjoint components, disjunctive sum**)?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$+$

$*$

* + *

{*+* | *+* }

$*2$ $+$ $*$

$\{*+* \quad | *+* \quad \}$

0 + *

$\{* + *, 0 + * \mid * + *, 0 + * \}$

$*2$ $+$ $*$

$\{*+*, 0+* \mid *+*, 0+* \}$

$*2$ $+$ 0

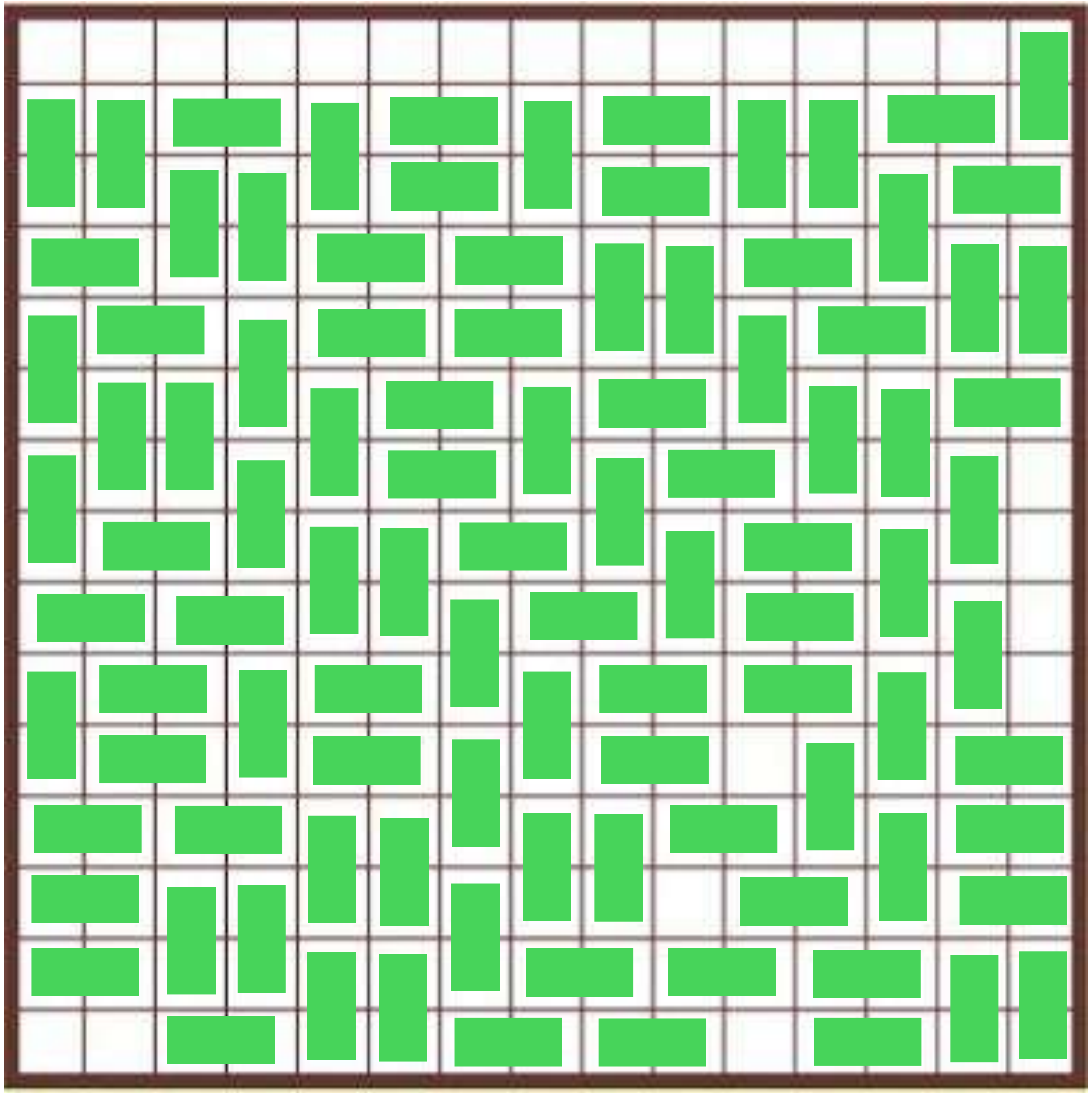
$\{*+*, 0+*, *2+0 \mid *+*, 0+*, *2+0\}$

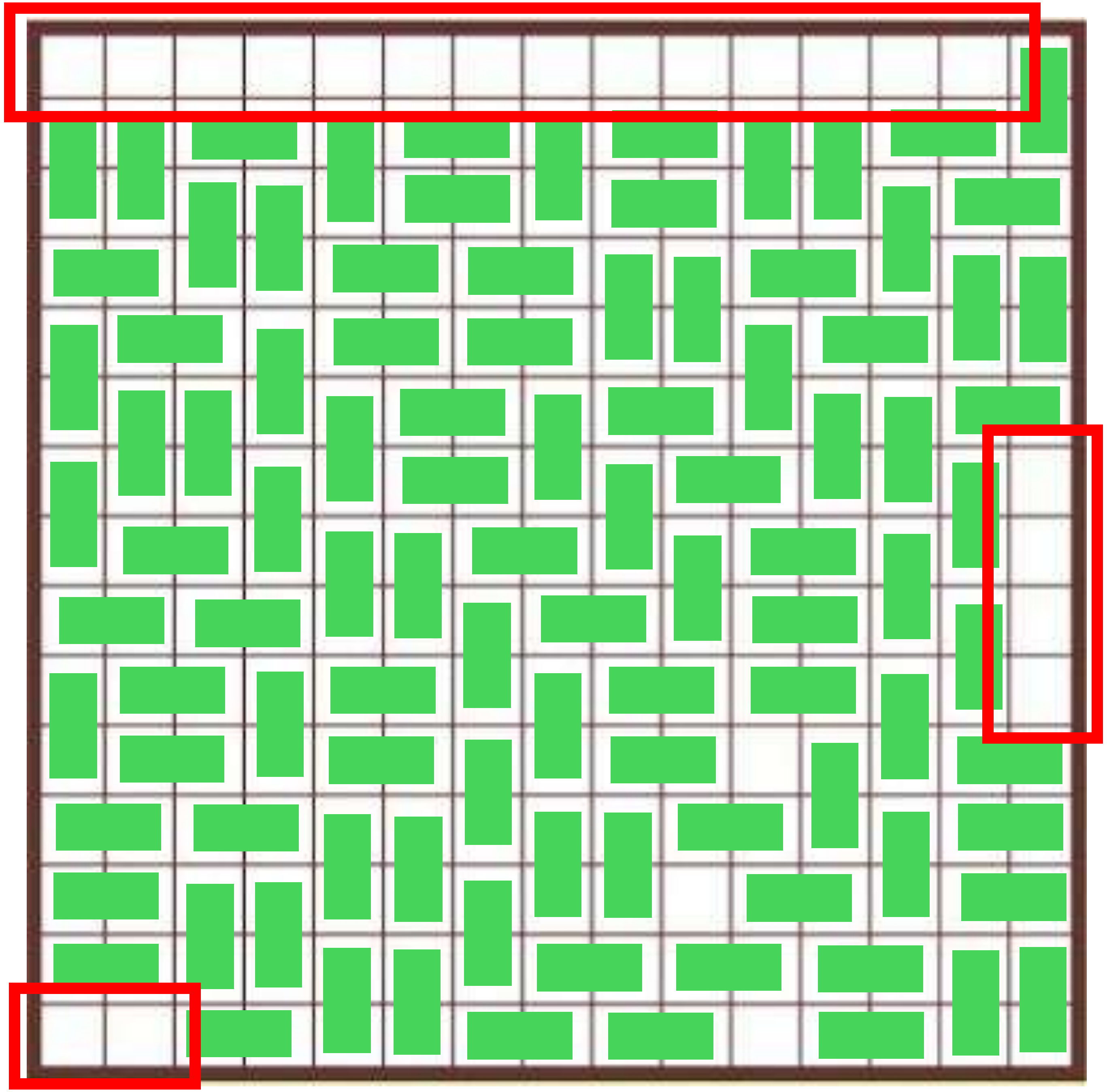
$G + H$

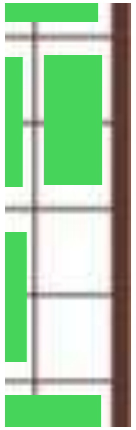
$$G + H = \{G^{\mathrm{L}} | G^{\mathrm{R}}\} + \{H^{\mathrm{L}} | H^{\mathrm{R}}\}$$

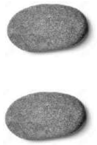
$$G + H = \{G^{\mathrm{L}} \mid G^{\mathrm{R}}\} + \left\{H^{\mathrm{L}} \mid H^{\mathrm{R}}\right\} = \{G^{\mathrm{L}} + H, G + H^{\mathrm{L}} \mid G^{\mathrm{R}} + H, G + H^{\mathrm{R}}\}$$

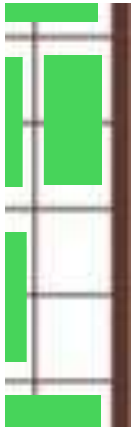
$$G + H = \{G^{\mathrm{L}} \mid G^{\mathrm{R}}\} + \left\{H^{\mathrm{L}} \mid H^{\mathrm{R}}\right\} = \{G^{\mathrm{L}} + H, G + H^{\mathrm{L}} \mid G^{\mathrm{R}} + H, G + H^{\mathrm{R}}\}$$

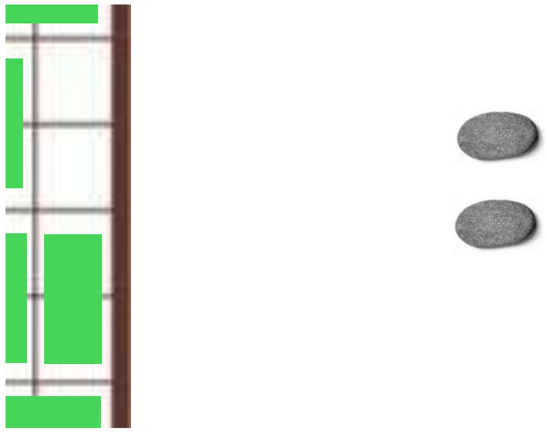


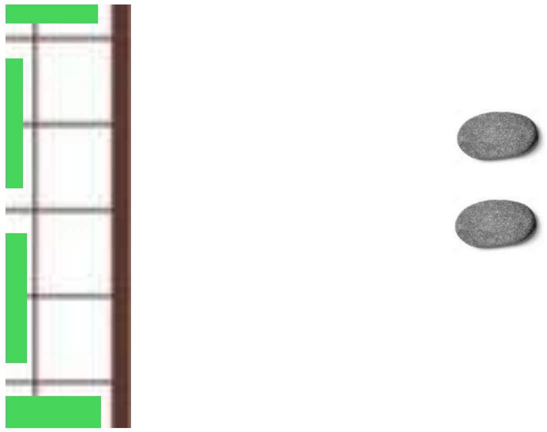


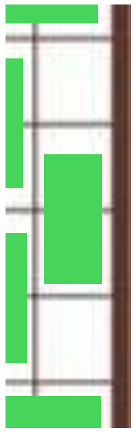


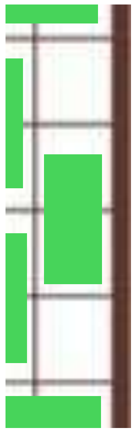


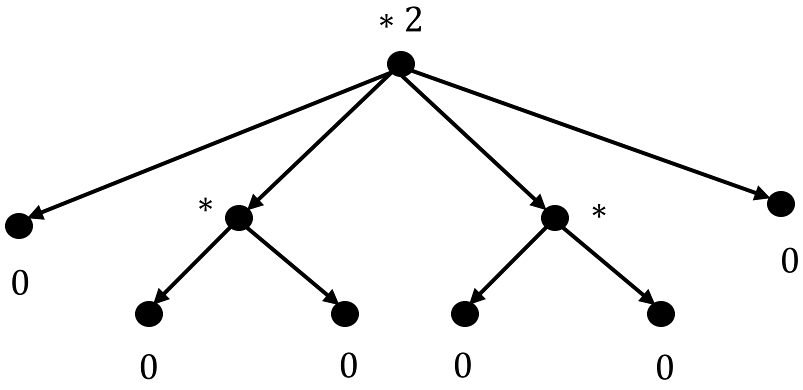








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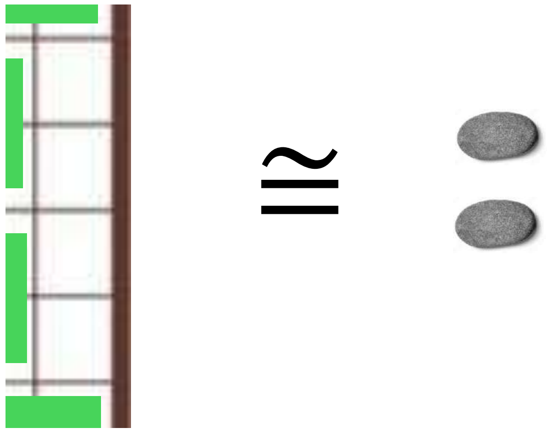
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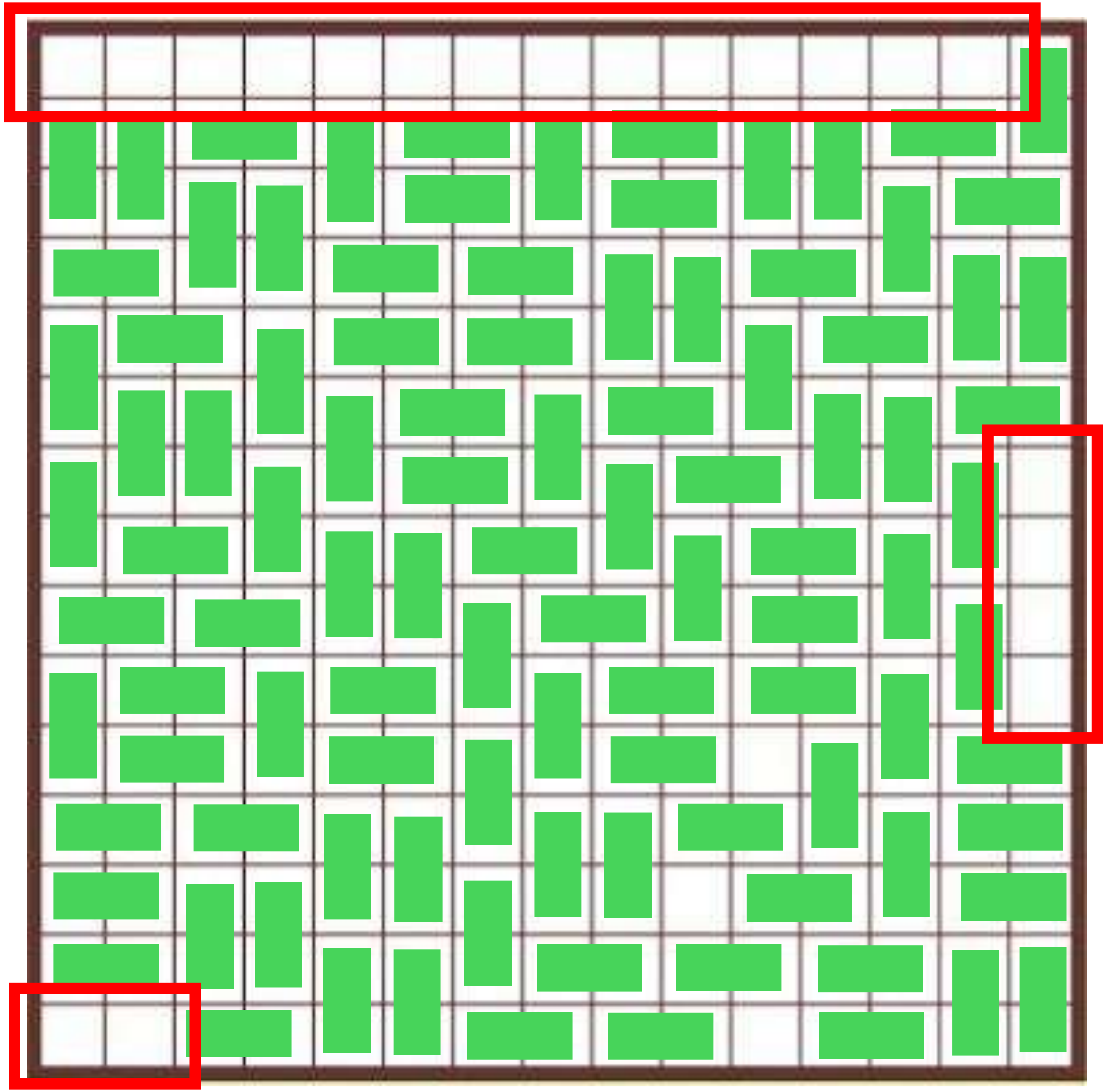

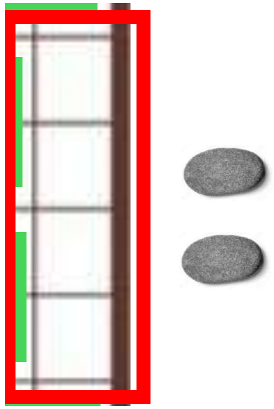

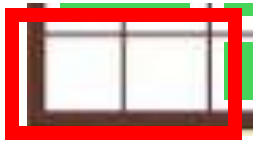

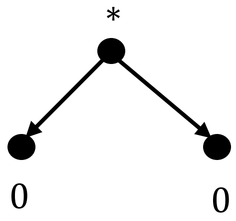
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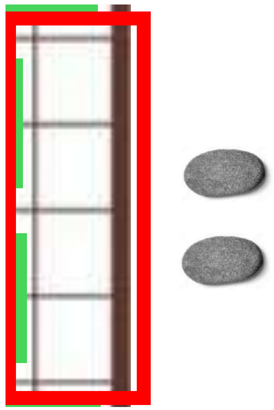

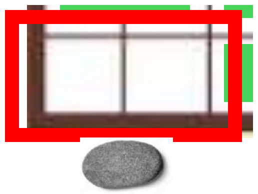

Is **«being isomorphic»** the only way to **«be equal»**?

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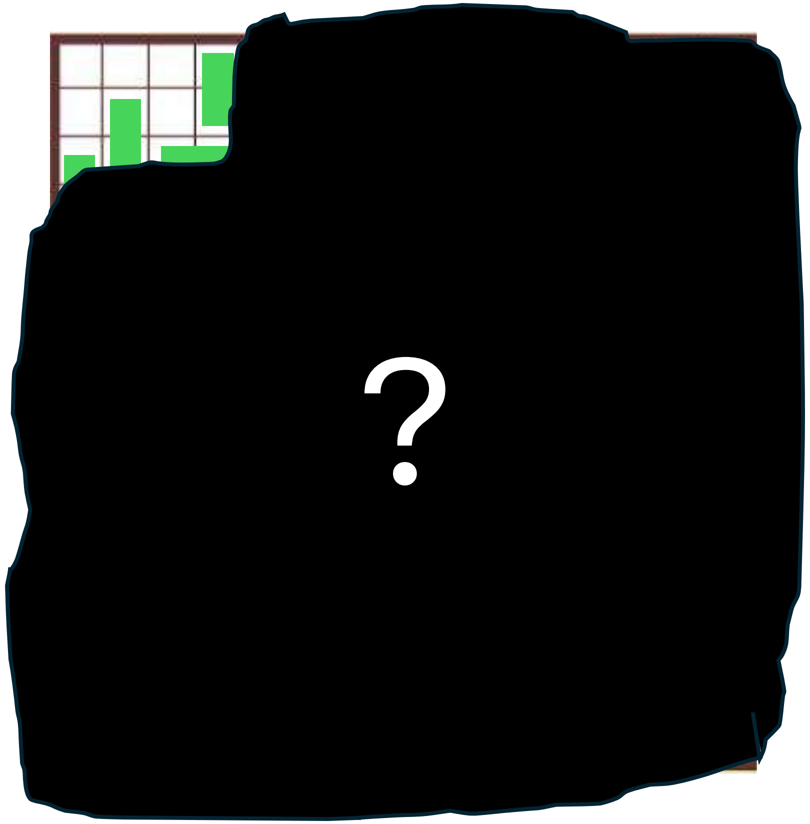
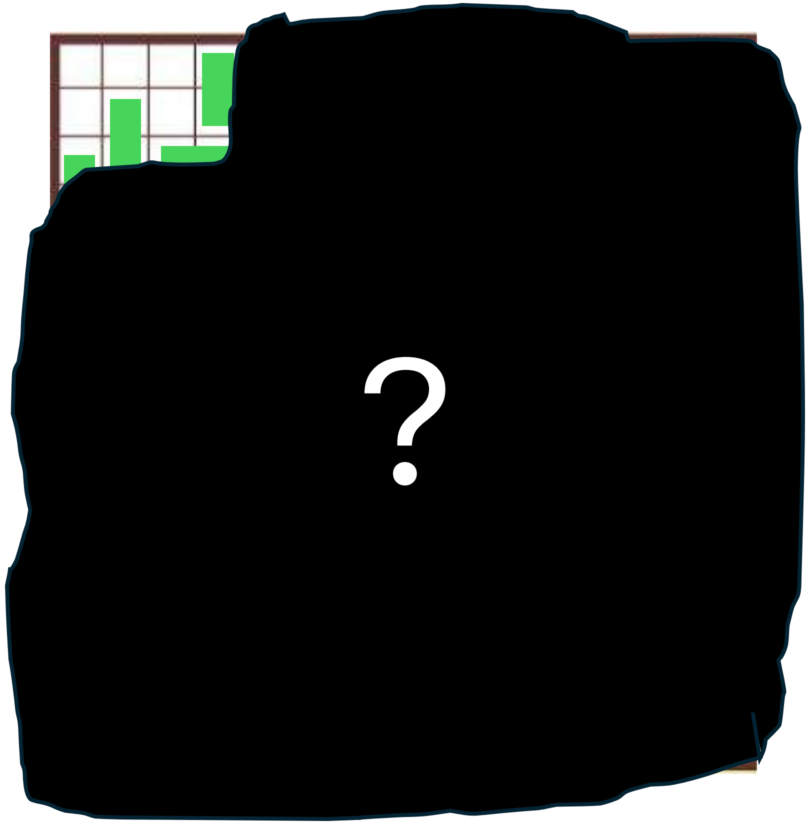
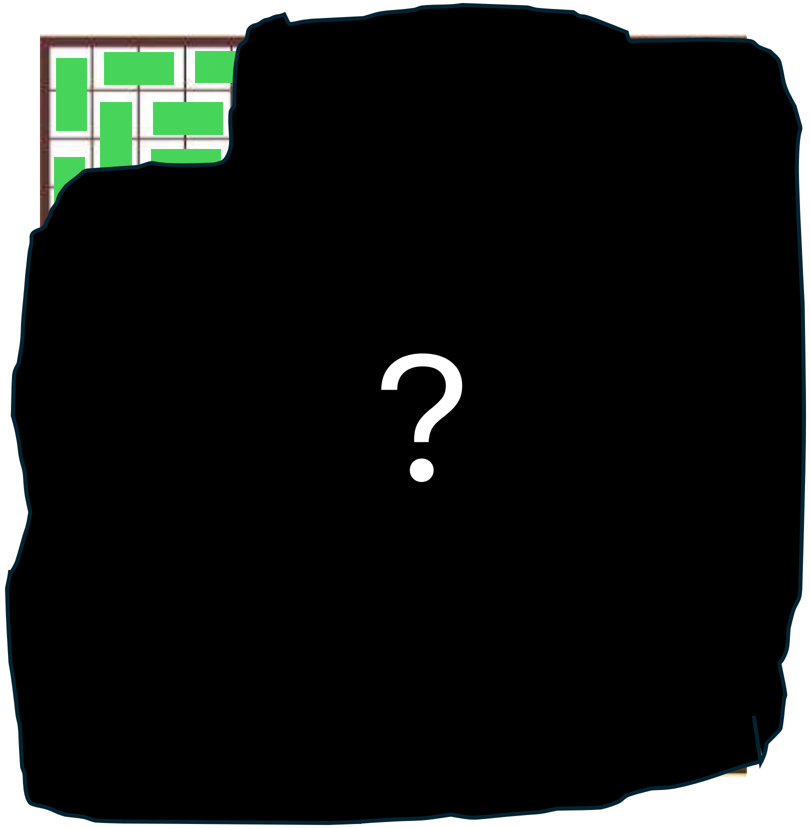
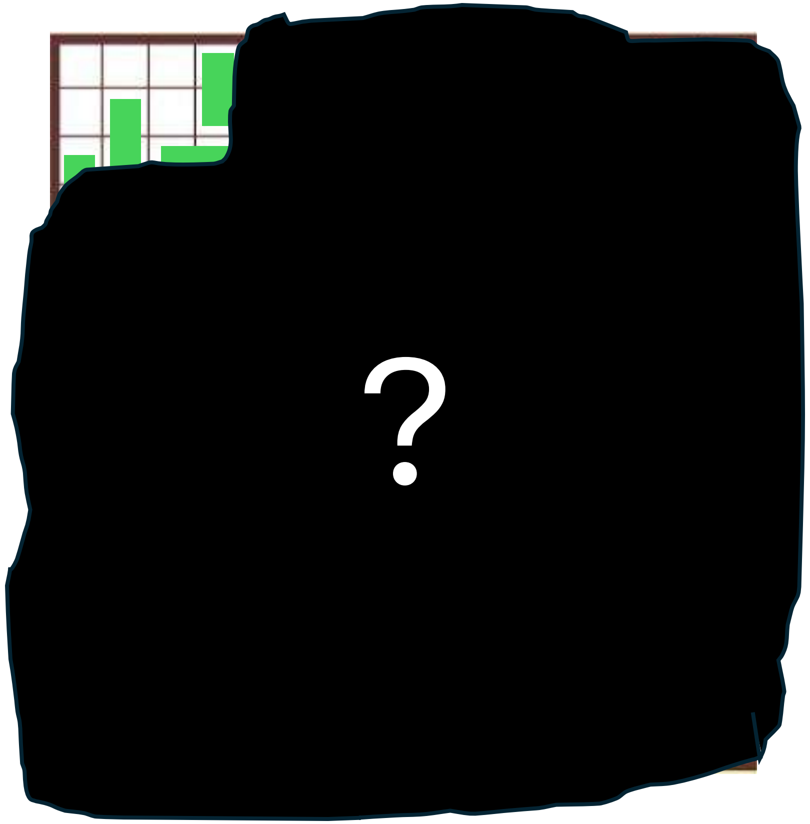
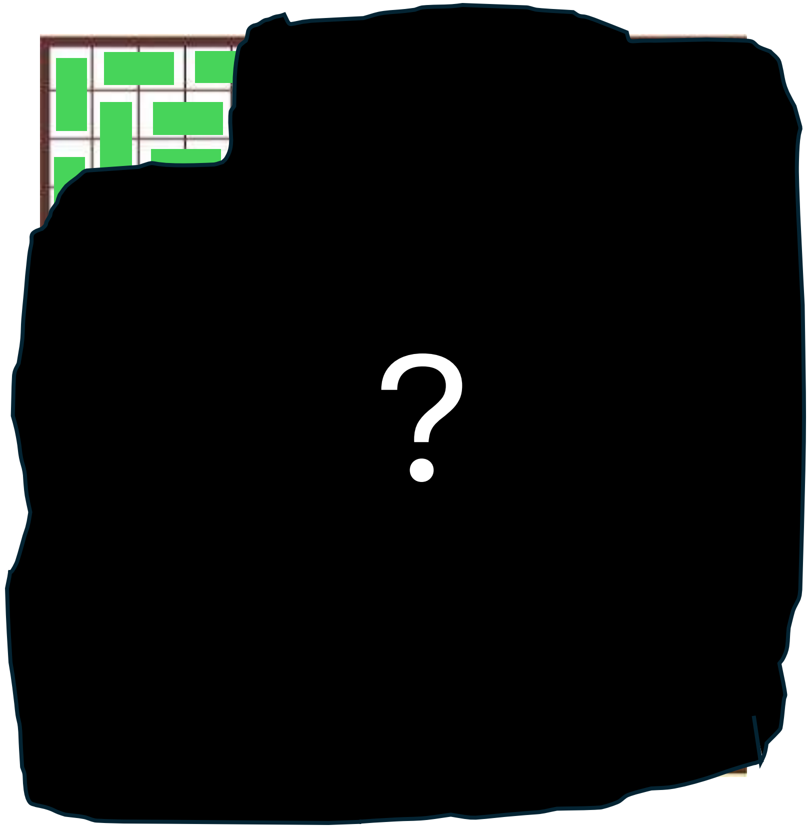
In terms of **game practice**, when should two components be considered equal?











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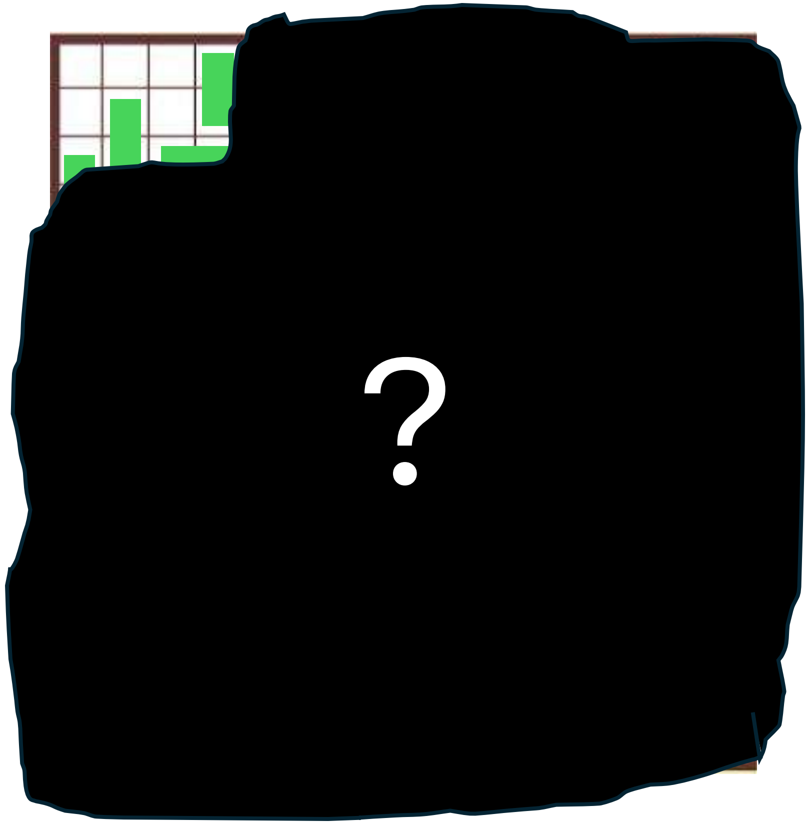

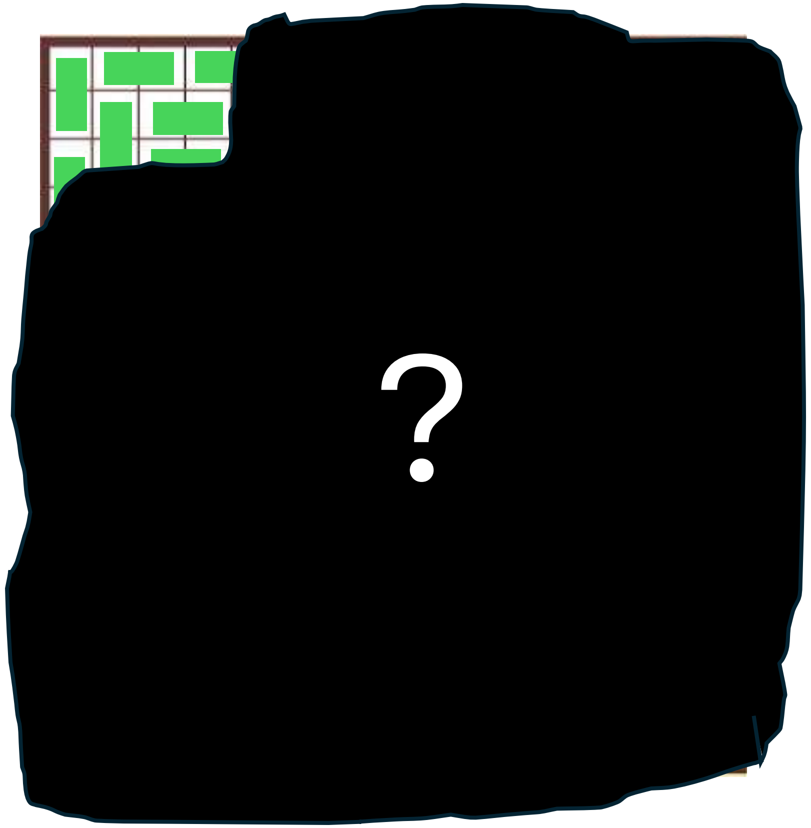

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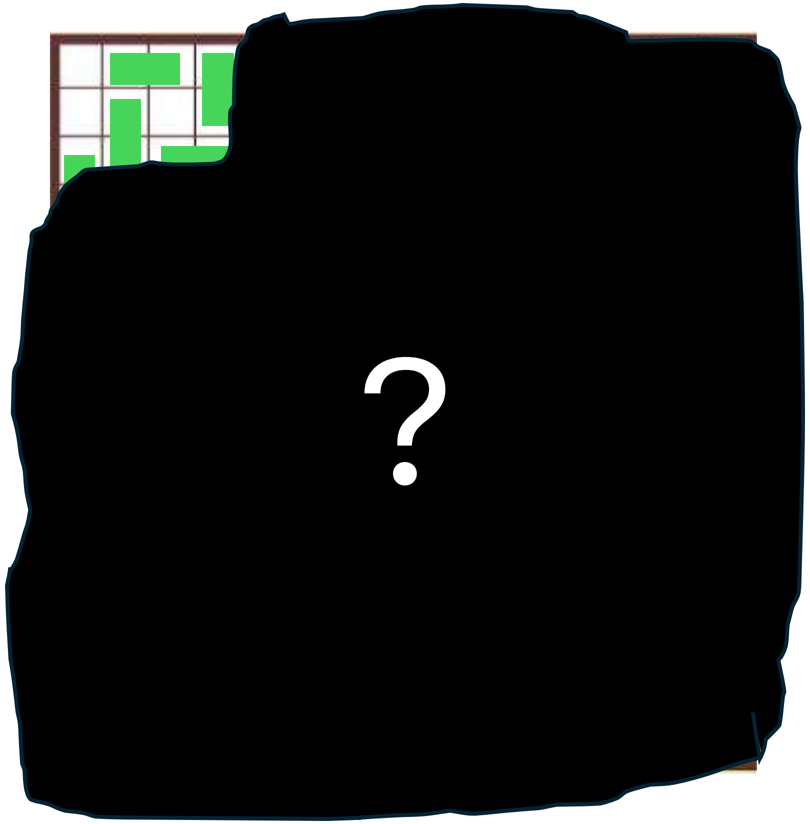
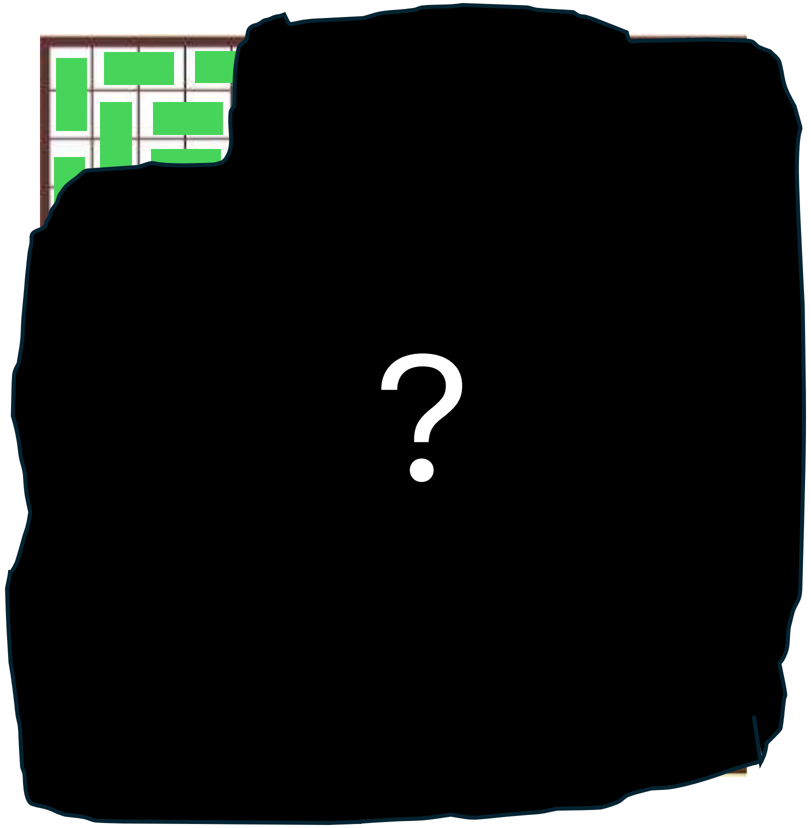
If a player makes a move, then, in a worst-case scenario, the opponent can finish the component with their answer.





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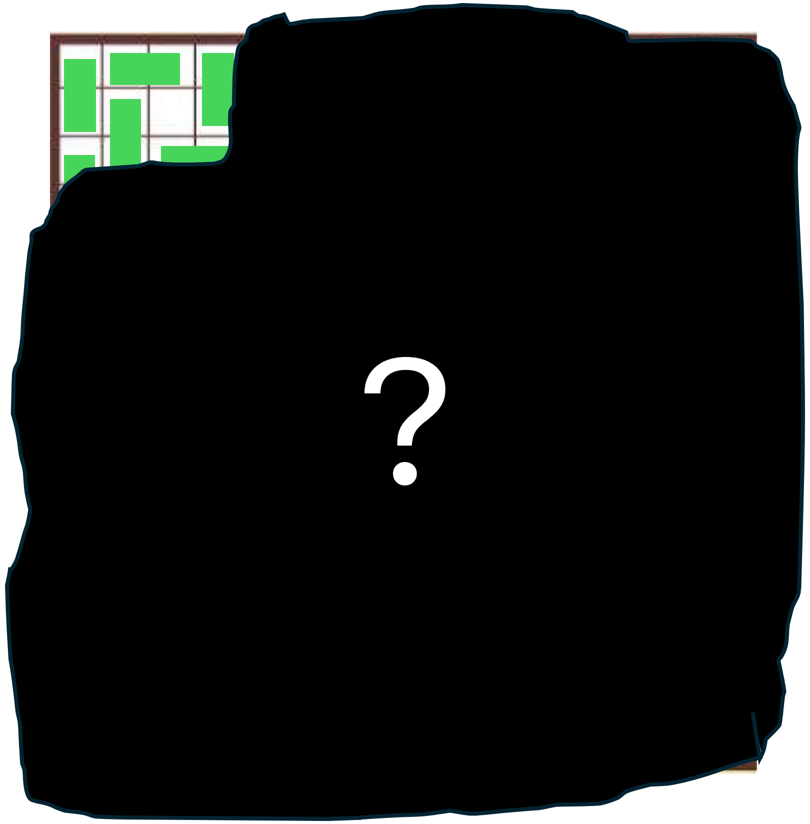

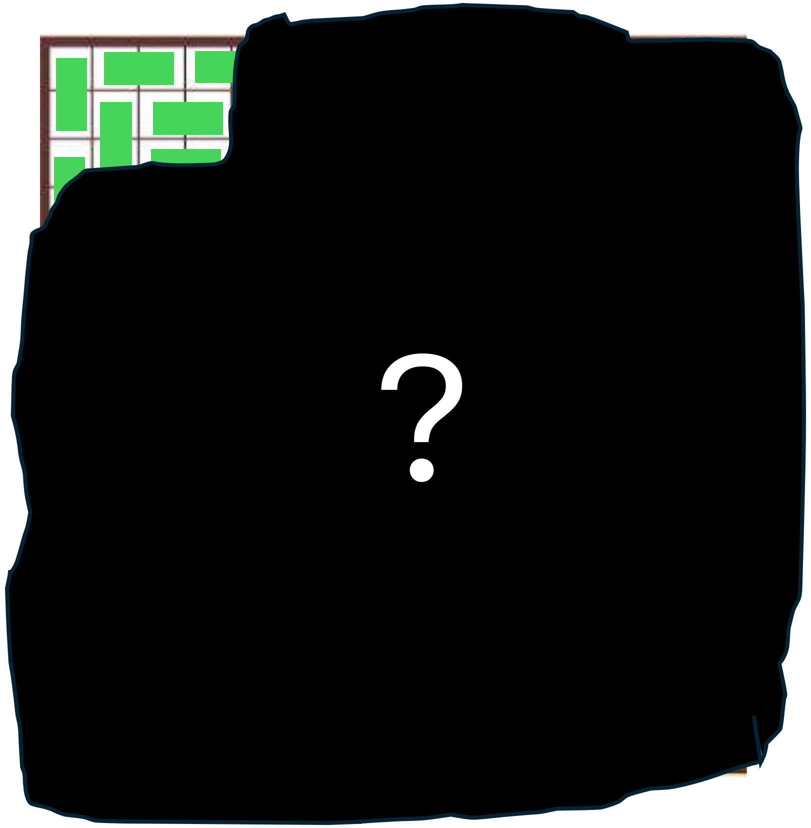

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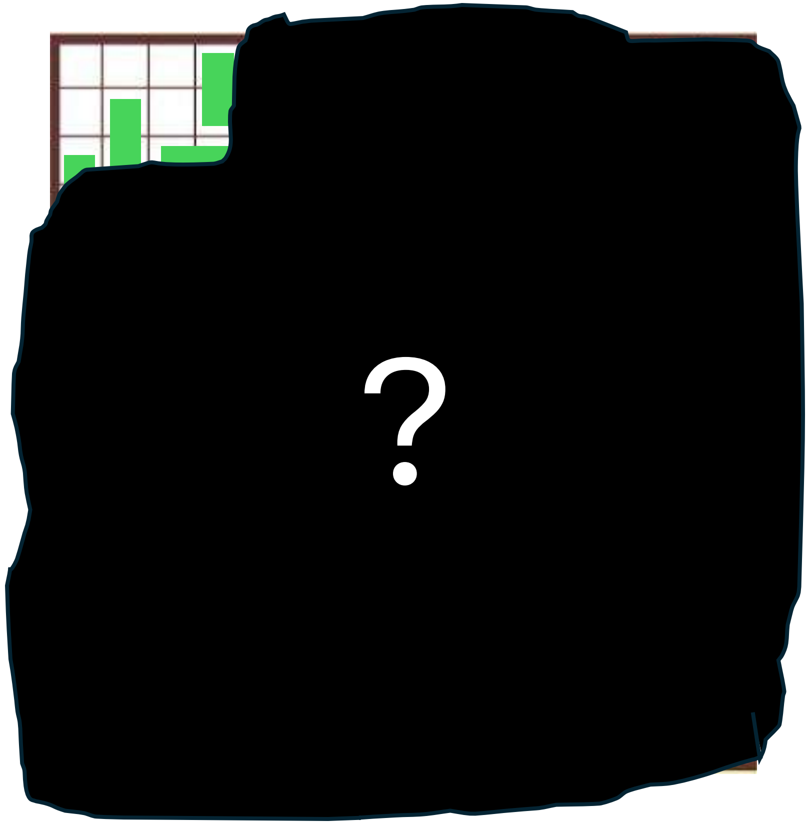

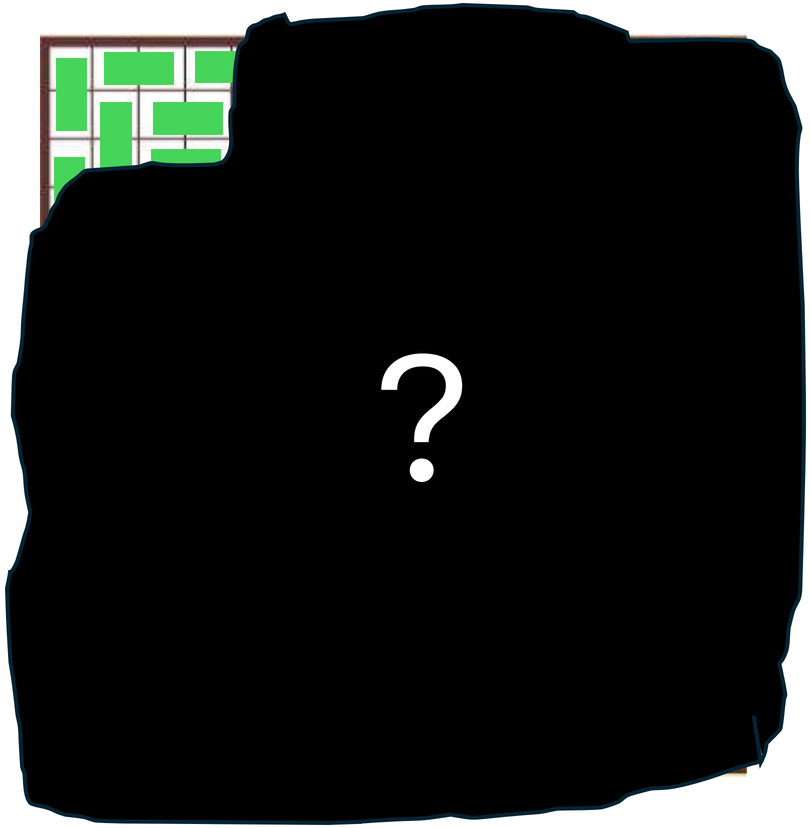

The component is irrelevant.

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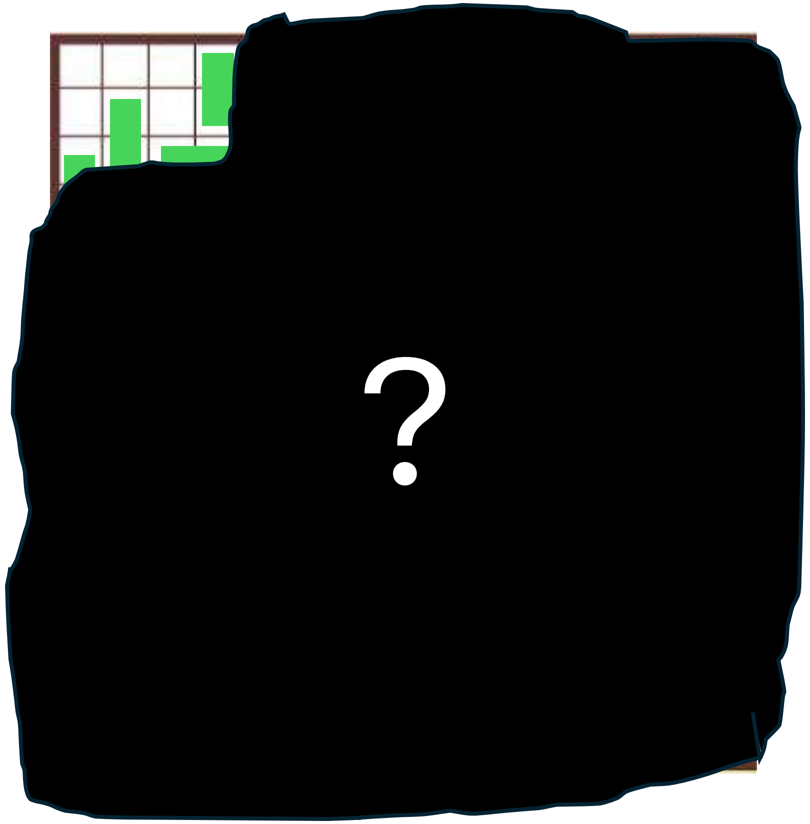

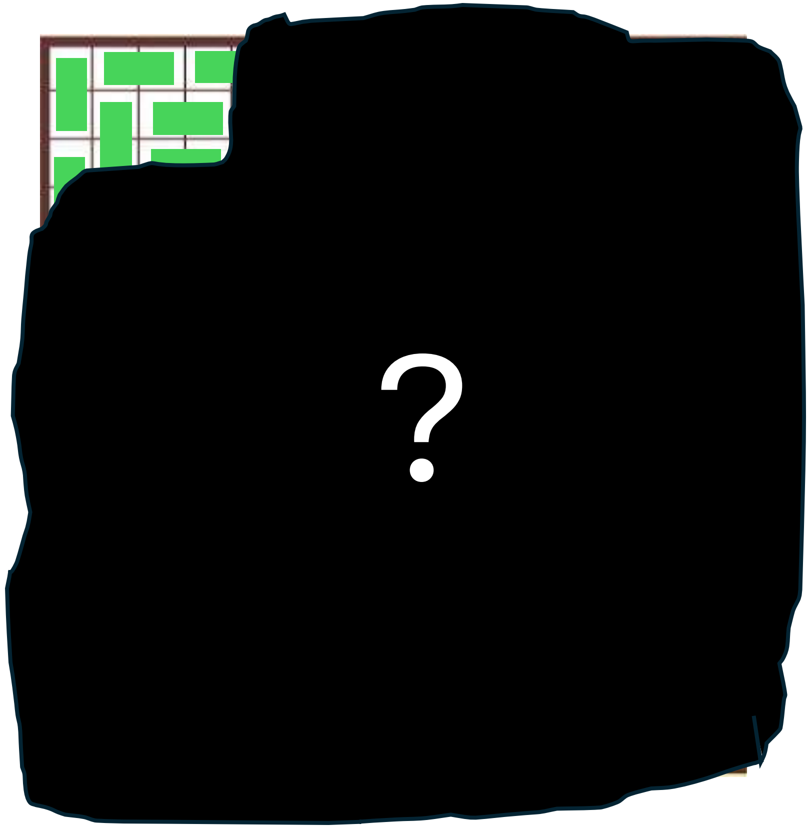

The component is irrelevant. Being there or not **is the same in terms of outcome**. The positions on the left and on the right have **the same outcome**, **regardless of what the other components may be**.

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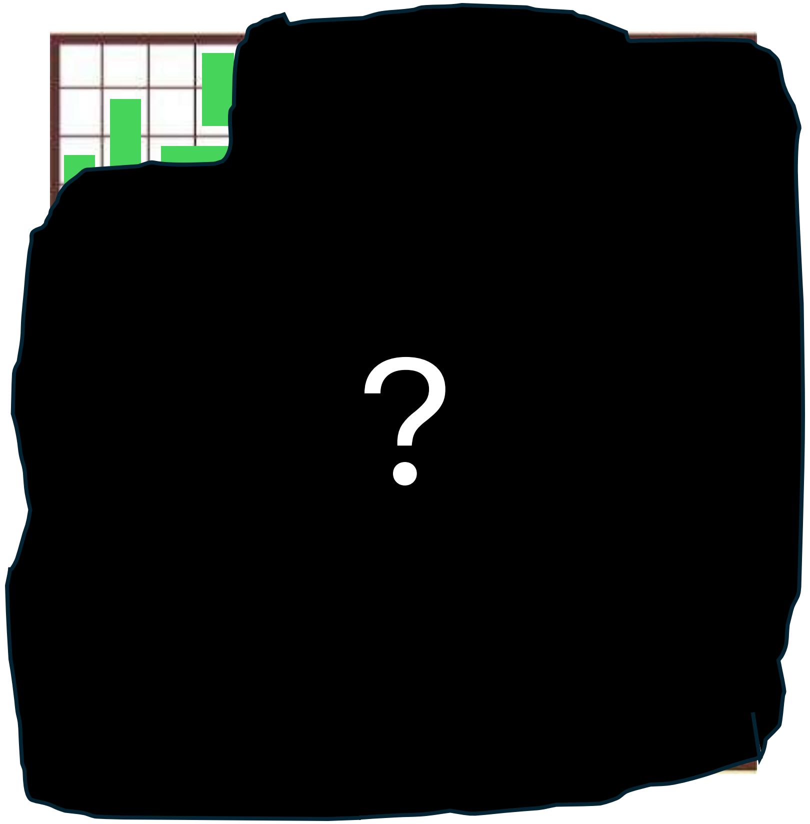
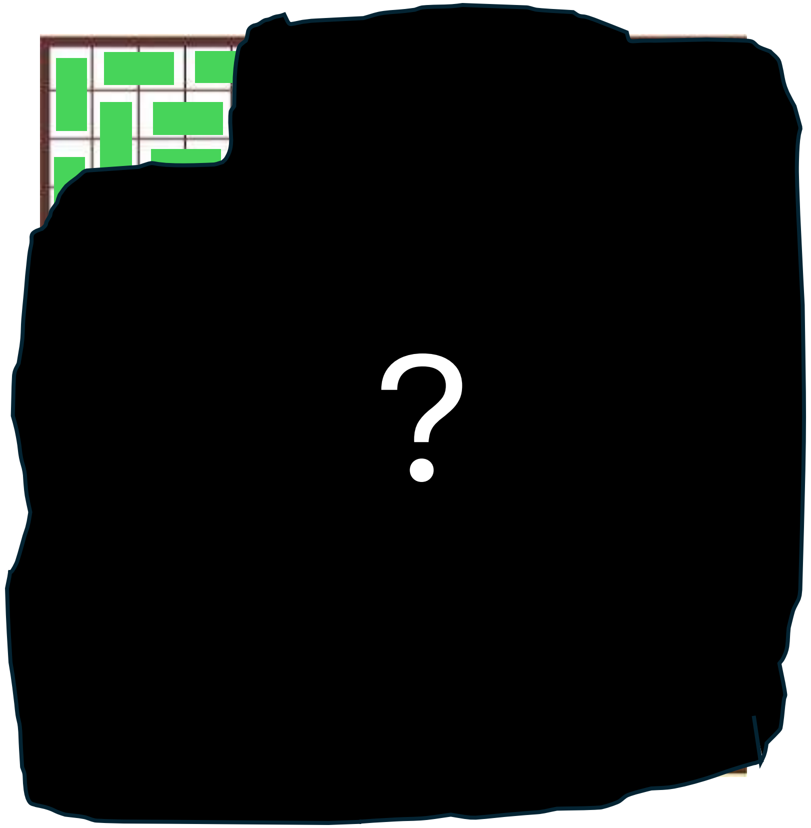
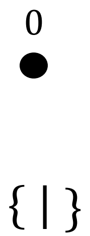
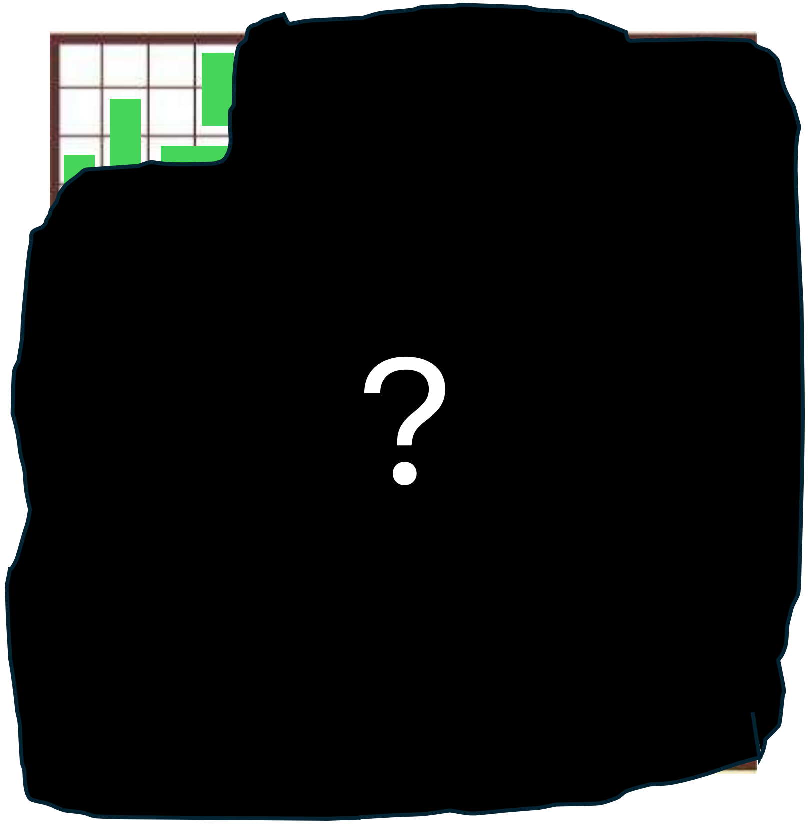




The component is irrelevant. Being there or not **is the same in terms of outcome**. The positions on the left and on the right have **the same outcome**, **regardless of what the other components may be**. In terms of outcome, **they cannot be distinguished**. In all situations, **one can be replaced by the other without changing the outcome**.

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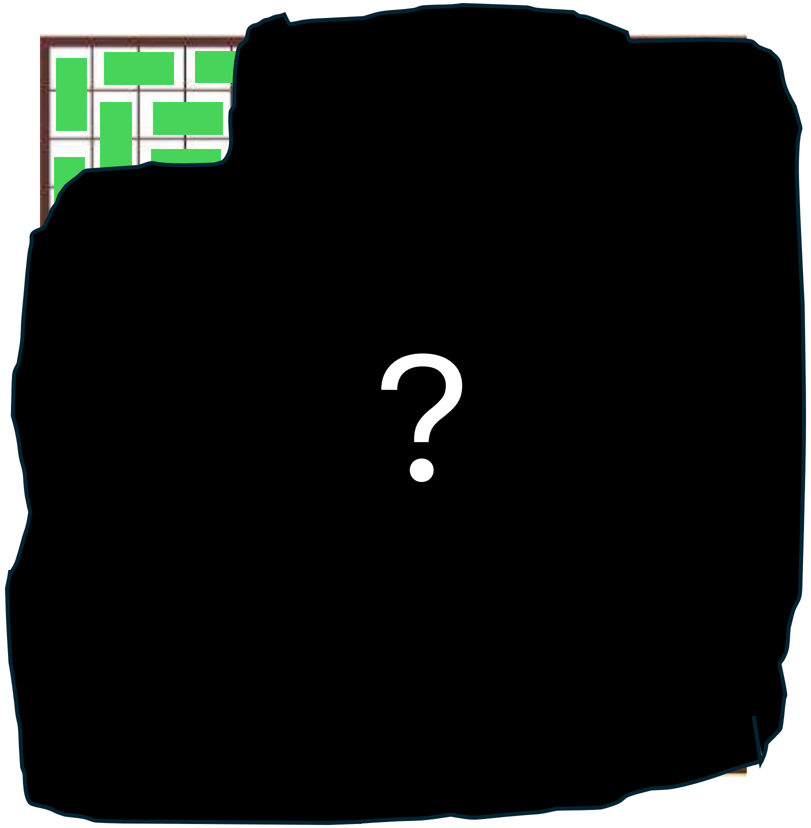
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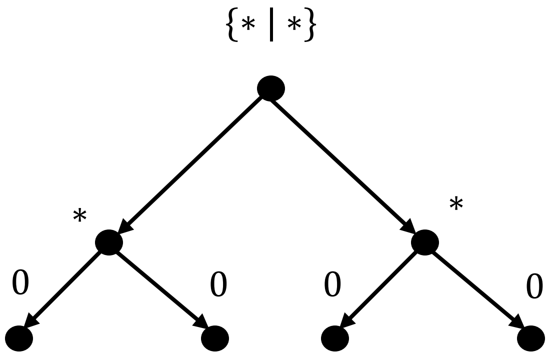
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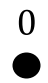
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$$G = H$$

$$G = H$$

iff

$$o(G + X) = o(H + X), \text{ for all } X$$

Sprague-Grundy Theory

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- *Omnipresence of nimbers*: Given an impartial form G, there is a nonnegative integer n such that $G = *n$ (the Grundy-value of G is n, written as $\mathcal{G}(G) = n$).

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- *Determination of the Grundy-value of a disjunctive sum, knowing the Grundy-values of the components*: If G and H are impartial, then $\mathcal{G}(G + H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$.

- *Relation between the Grundy-value of G and its outcome*: Given an impartial form G, the outcome of G is $\mathcal{P}$ if and only if $\mathcal{G}(G) = 0$. An important consequence of this fact is that $\mathcal{G}(G) = k$ if and only if $G + *k$ is a $\mathcal{P}$-position.

NIM, A GAME WITH A COMPLETE MATHEMATICAL THEORY

BY CHARLES L. BOUTON.

THE game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory.* The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

1. Description of the Game. The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:—The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

It is the writer's purpose to prove that if one of the players, say A, can leave one of a certain set of numbers upon the table, and after that plays without mistake, the other player, B, *cannot* win. Such a set of numbers will be called a *safe combination*. In outline the proof consists in showing that if A leaves a safe combination on the table, B at his next move cannot leave a safe combination, and whatever B may draw, A at his next move can again leave a safe combination. The piles are then reduced, A always leaving a safe combination, and B never doing so, and A must eventually take the last counter (or counters).

2. Its Theory. A *safe combination* is determined as follows: Write the number of the counters in each pile in the binary scale of notation,† and

---

* The modification of the game given in §6 was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

† For example, the number 9, written in this notation, will appear as

$$1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 = 1001.$$

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By Charles L. Bouton.

The game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory.* The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

1. Description of the Game. The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:—The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

It is the writer's purpose to prove that if one of the players, say A, can leave one of a certain set of numbers upon the table, and after that plays without mistake, the other player, B, *cannot* win. Such a set of numbers will be called a *safe combination*. In outline the proof consists in showing that if A leaves a safe combination on the table, B at his next move cannot leave a safe combination, and whatever B may draw, A at his next move can again leave a safe combination. The piles are then reduced, A always leaving a safe combination, and B never doing so, and A must eventually take the last counter (or counters).

2. Its Theory. A *safe combination* is determined as follows: Write the number of the counters in each pile in the binary scale of notation,† and

* The modification of the game given in §6 was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

† For example, the number 9, written in this notation, will appear as
$$1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 = 1001.$$

- *Omnipresence of nimbers*: Given an impartial form G, there is a nonnegative integer n such that $G = *n$ (the Grundy-value of G is n, written as $\mathcal{G}(G) = n$).

NIM

4 →

2 →

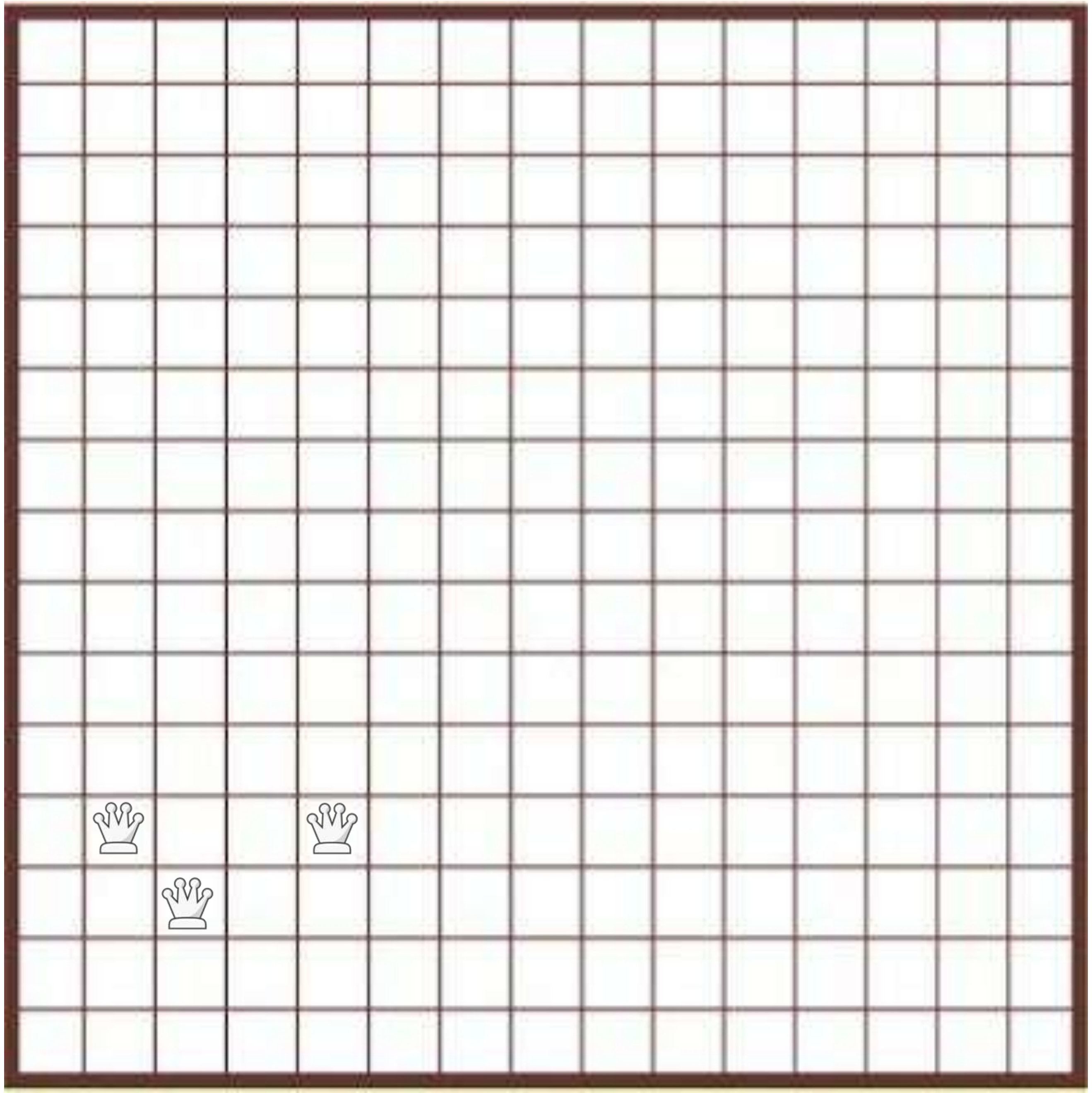
1 →

	4	2	1	
	1	0	0	
		1	0	
			1	
⊕	1	1	1	7

Part I: Impartial Games

I.1: Some famous games
I.2: Contribution of Charles Bouton (1902)
I.3: Patrick Grundy and Roland Sprague: the birth of a theory (1935, 1939)
I.4: How can you apply the theory?

WYTHOFF’S QUEENS



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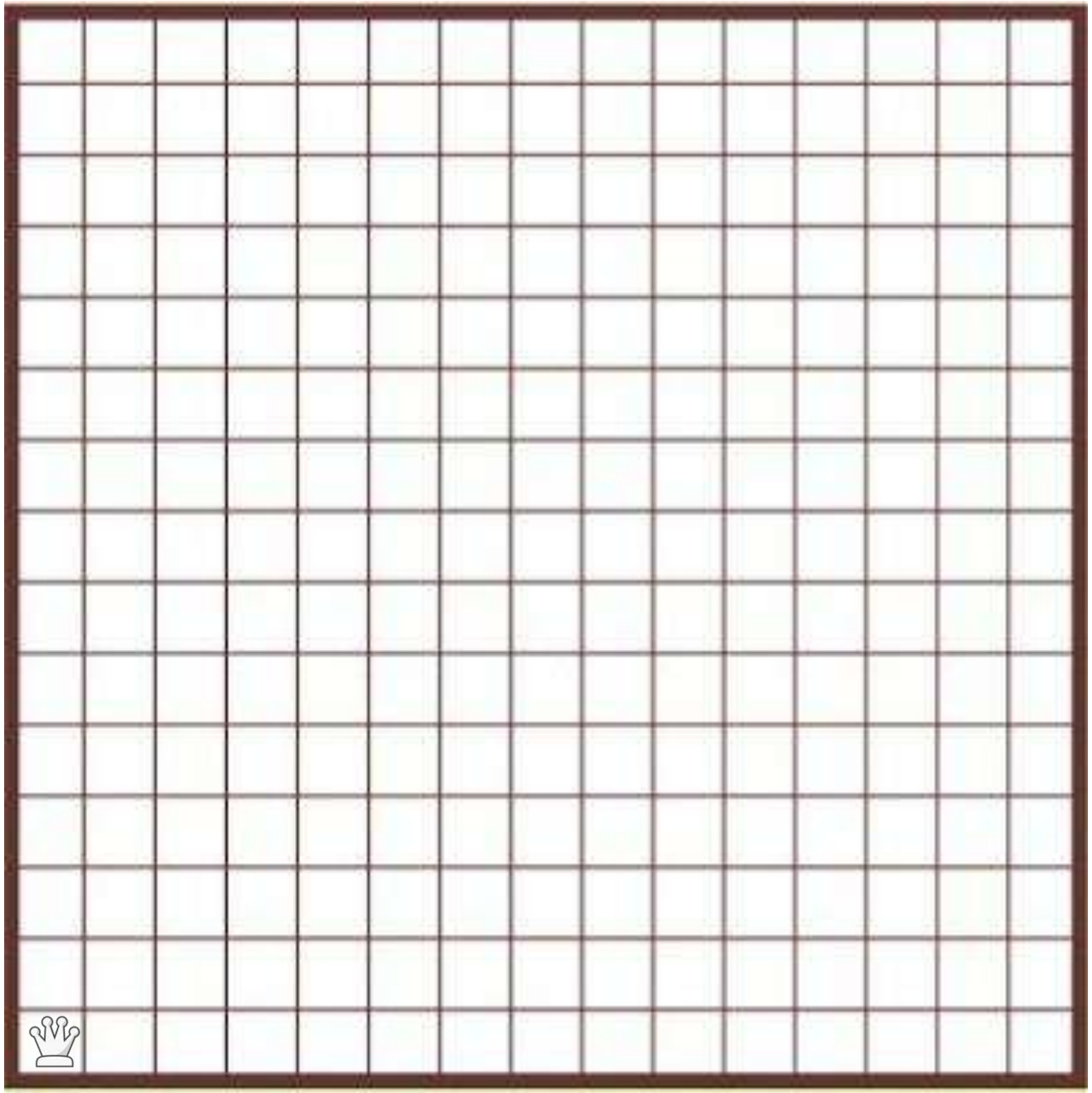

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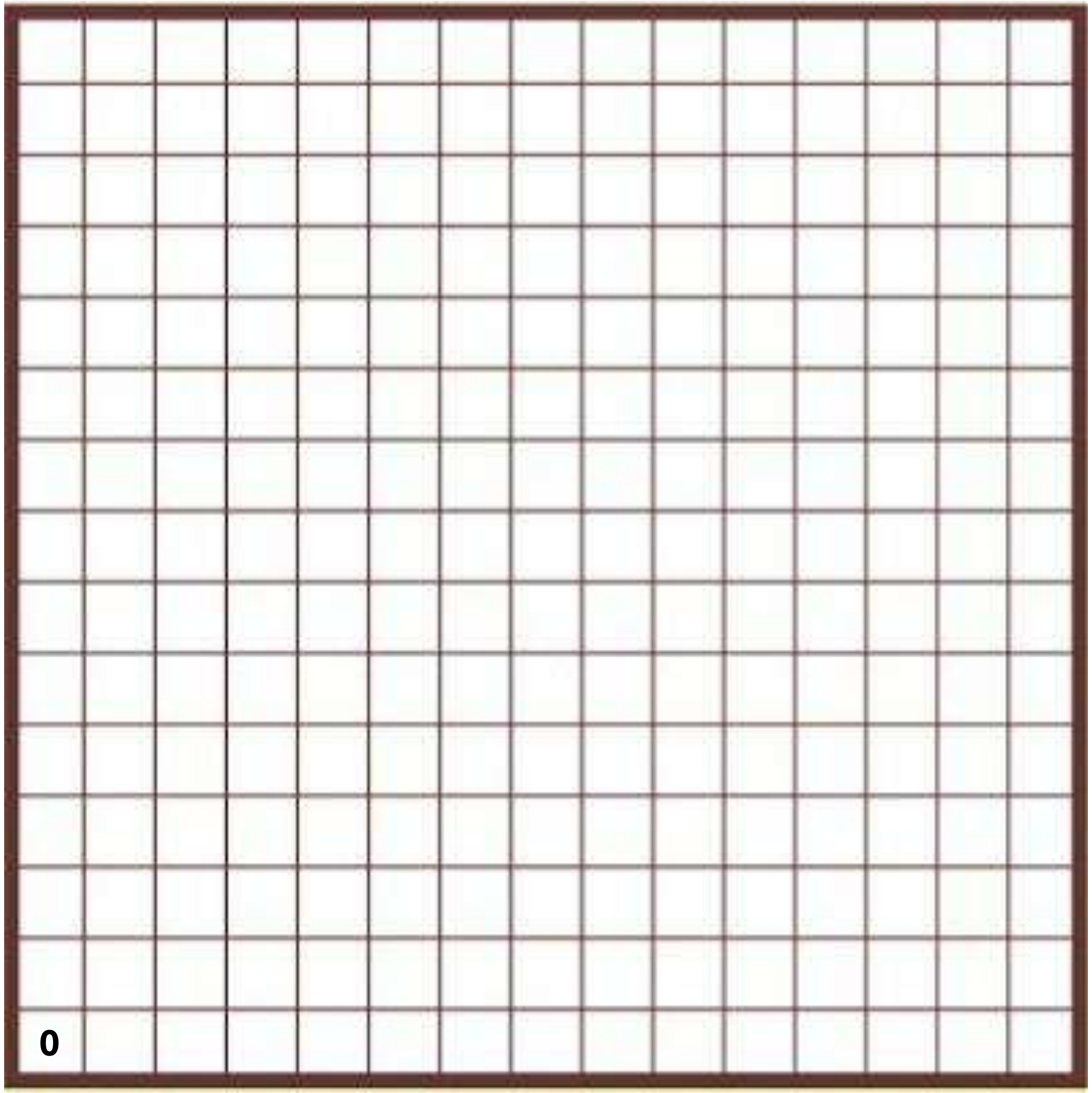

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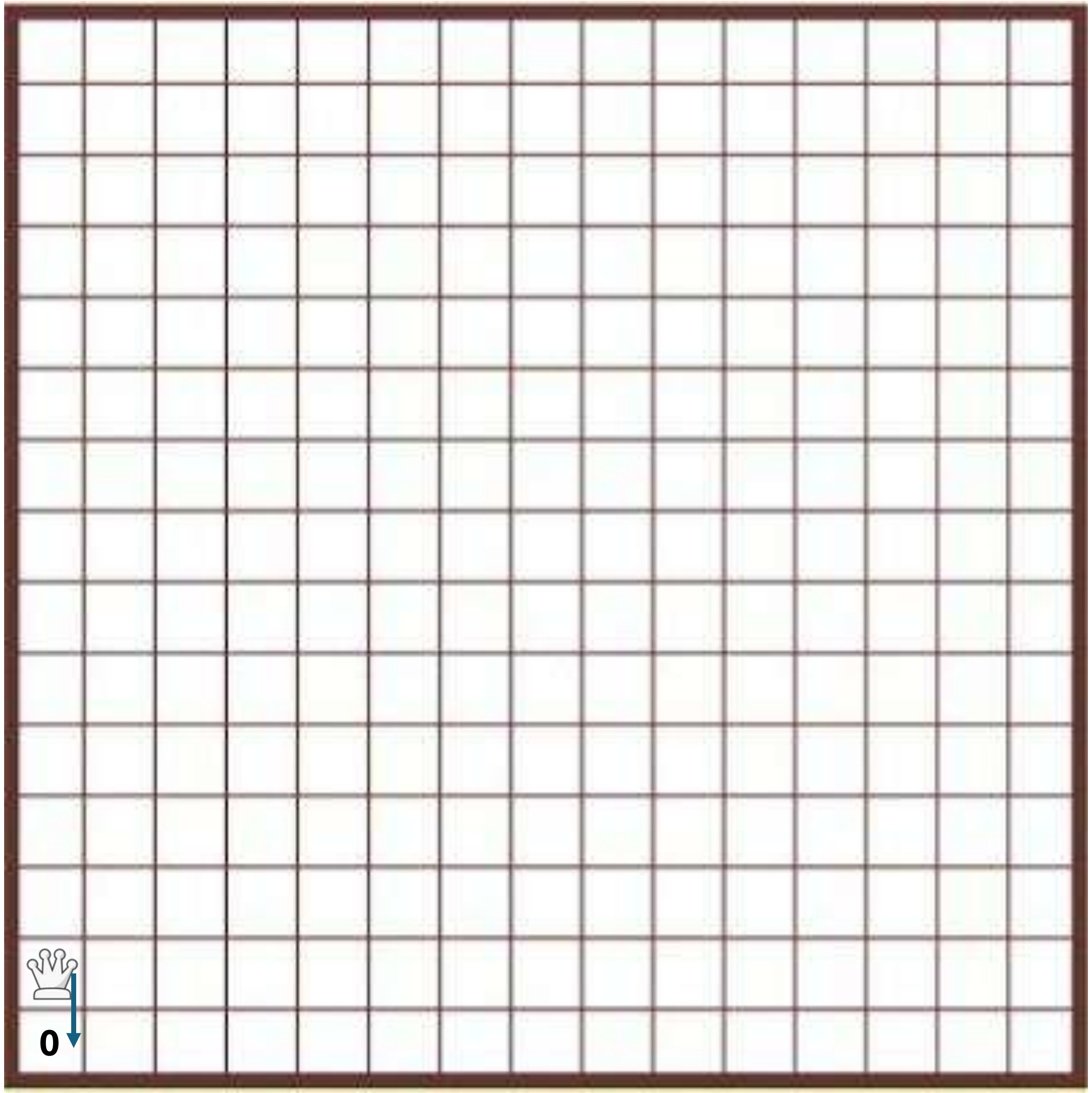

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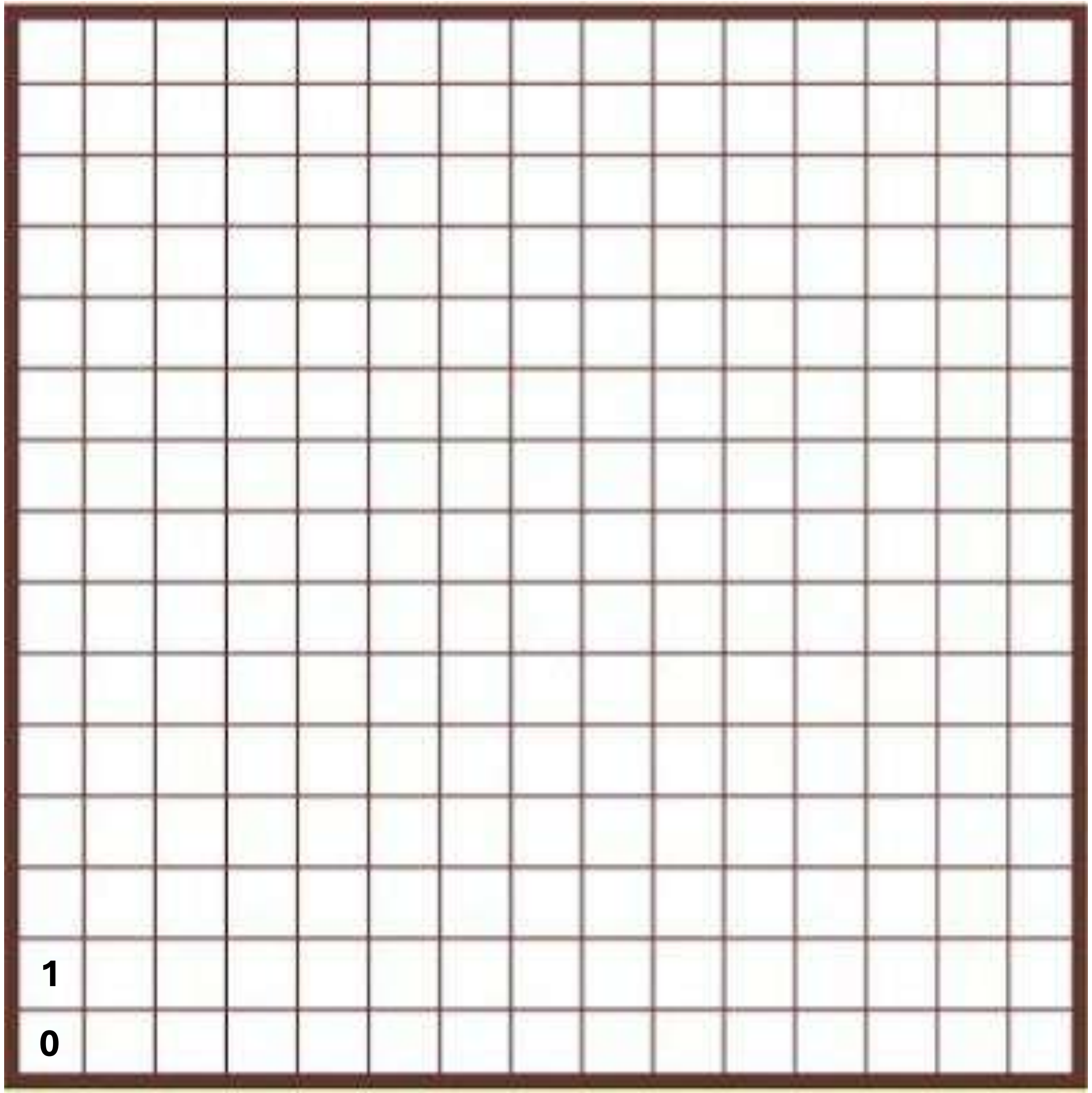

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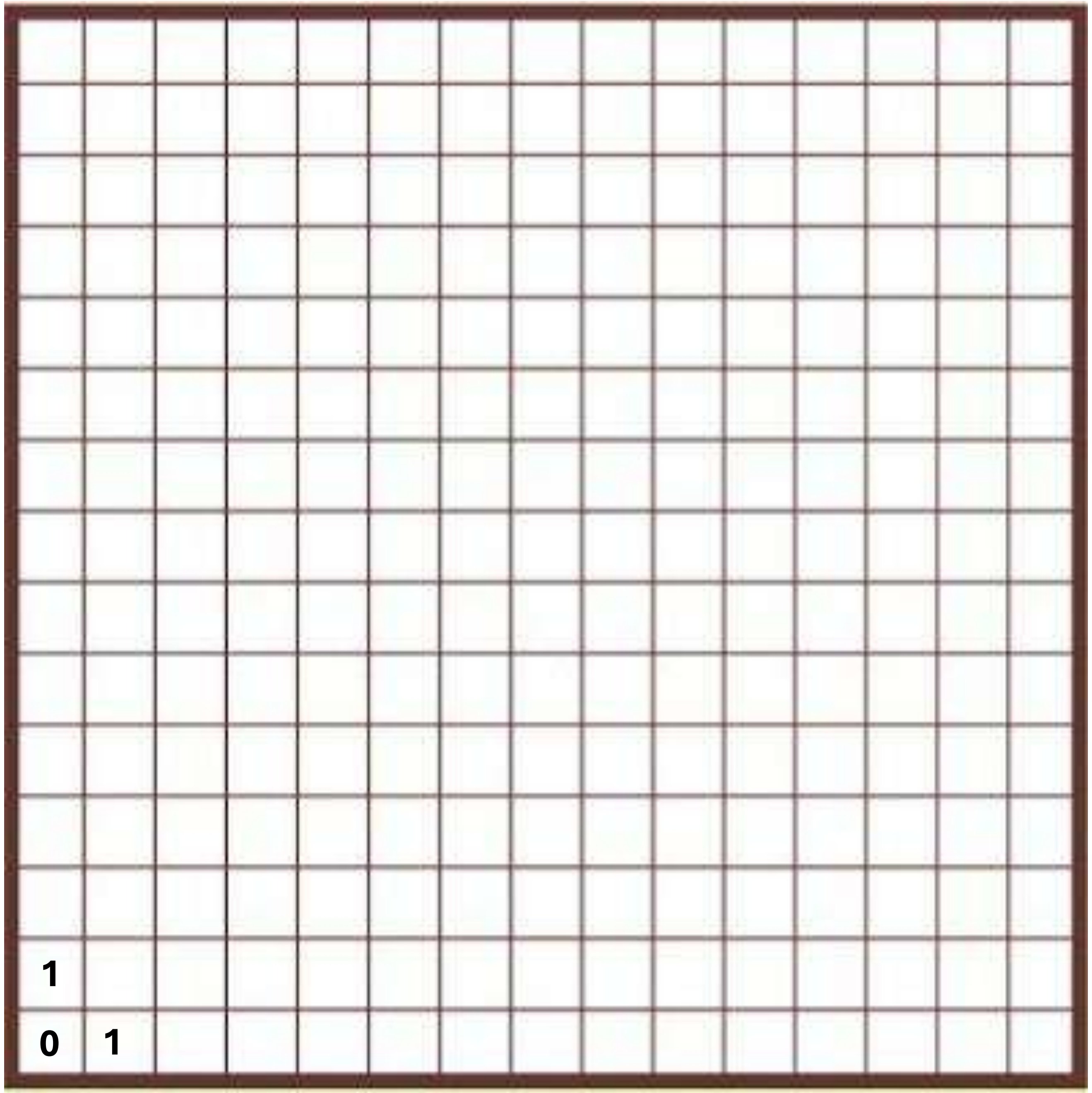

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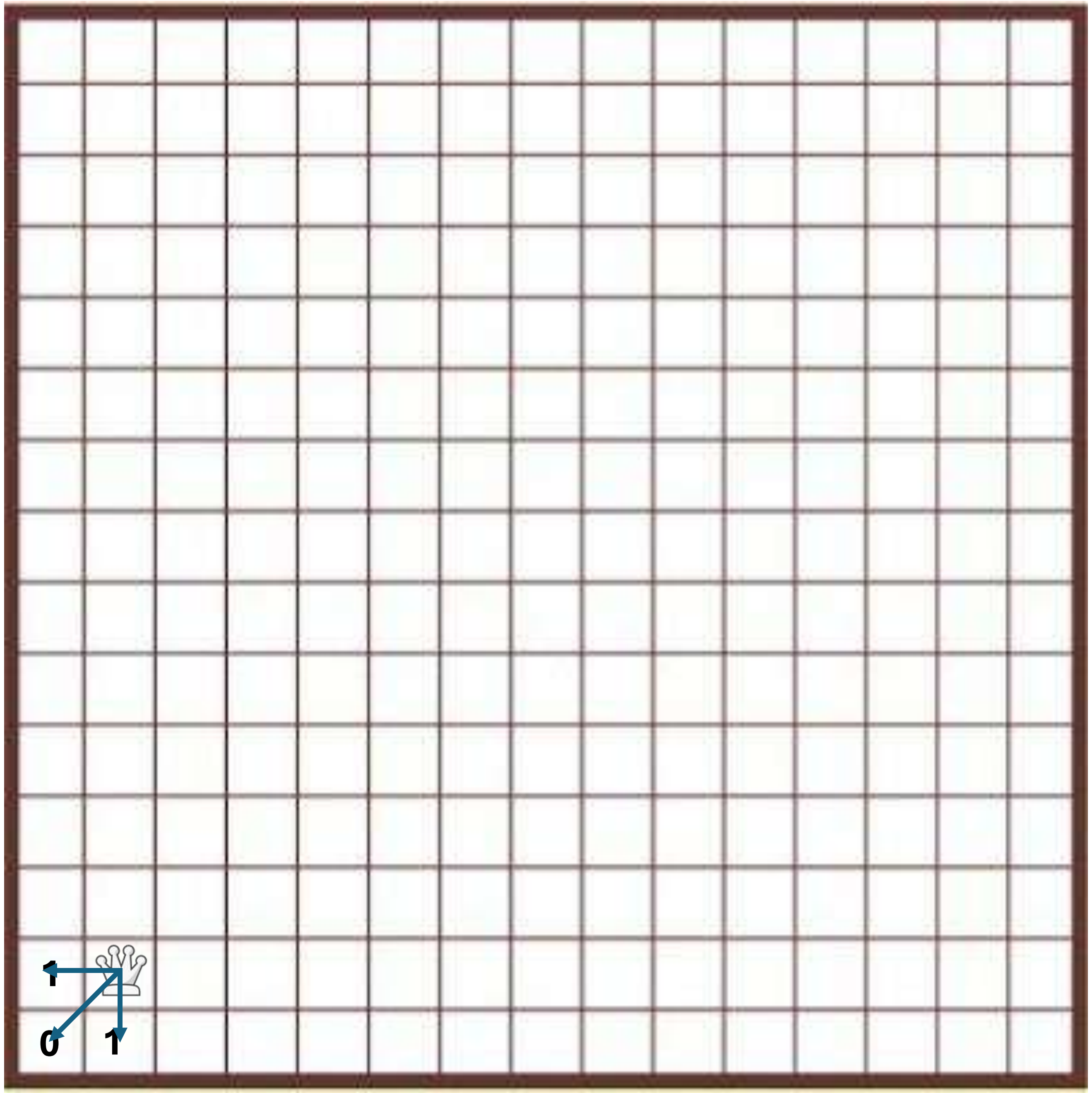

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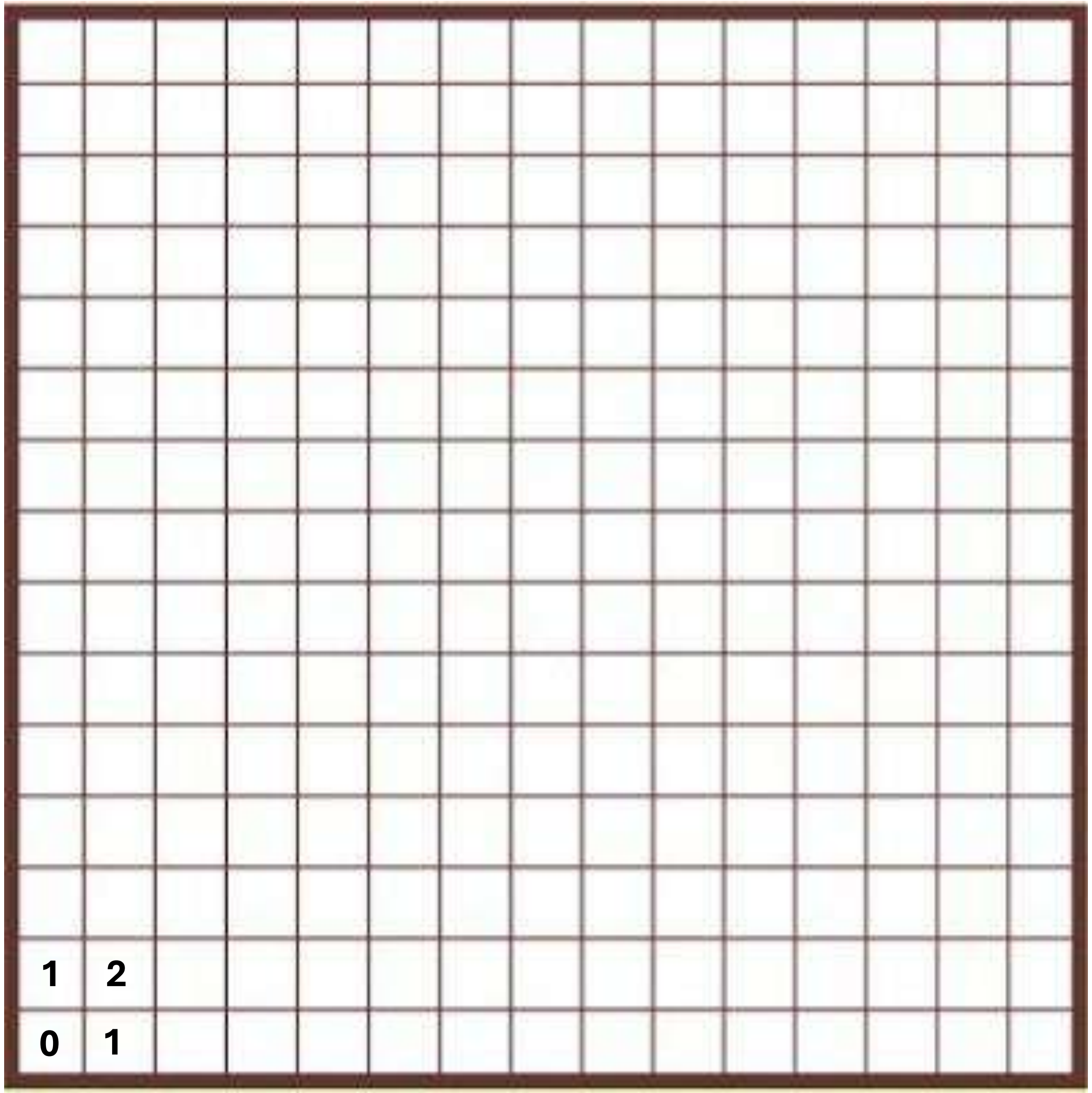

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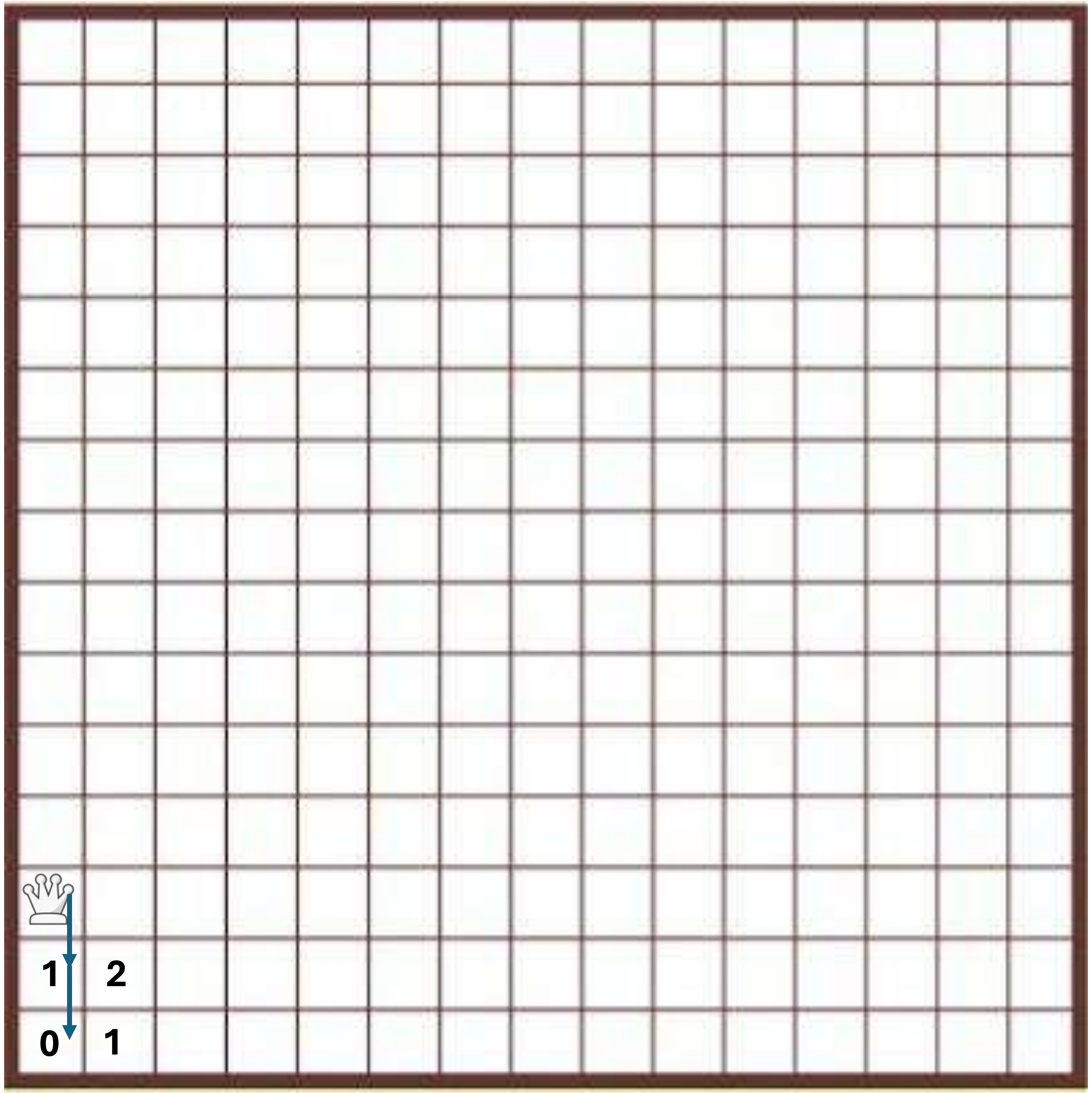

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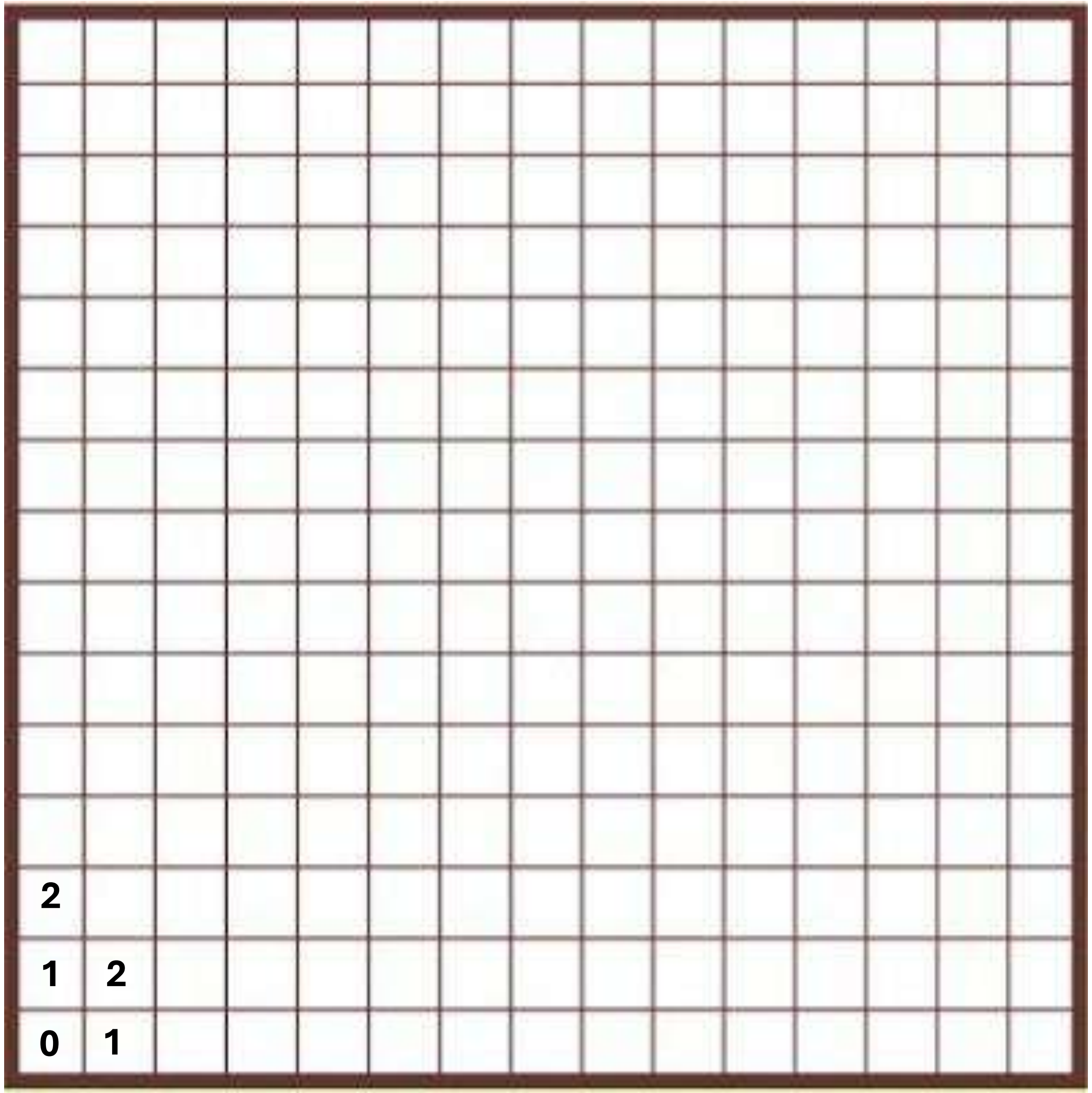

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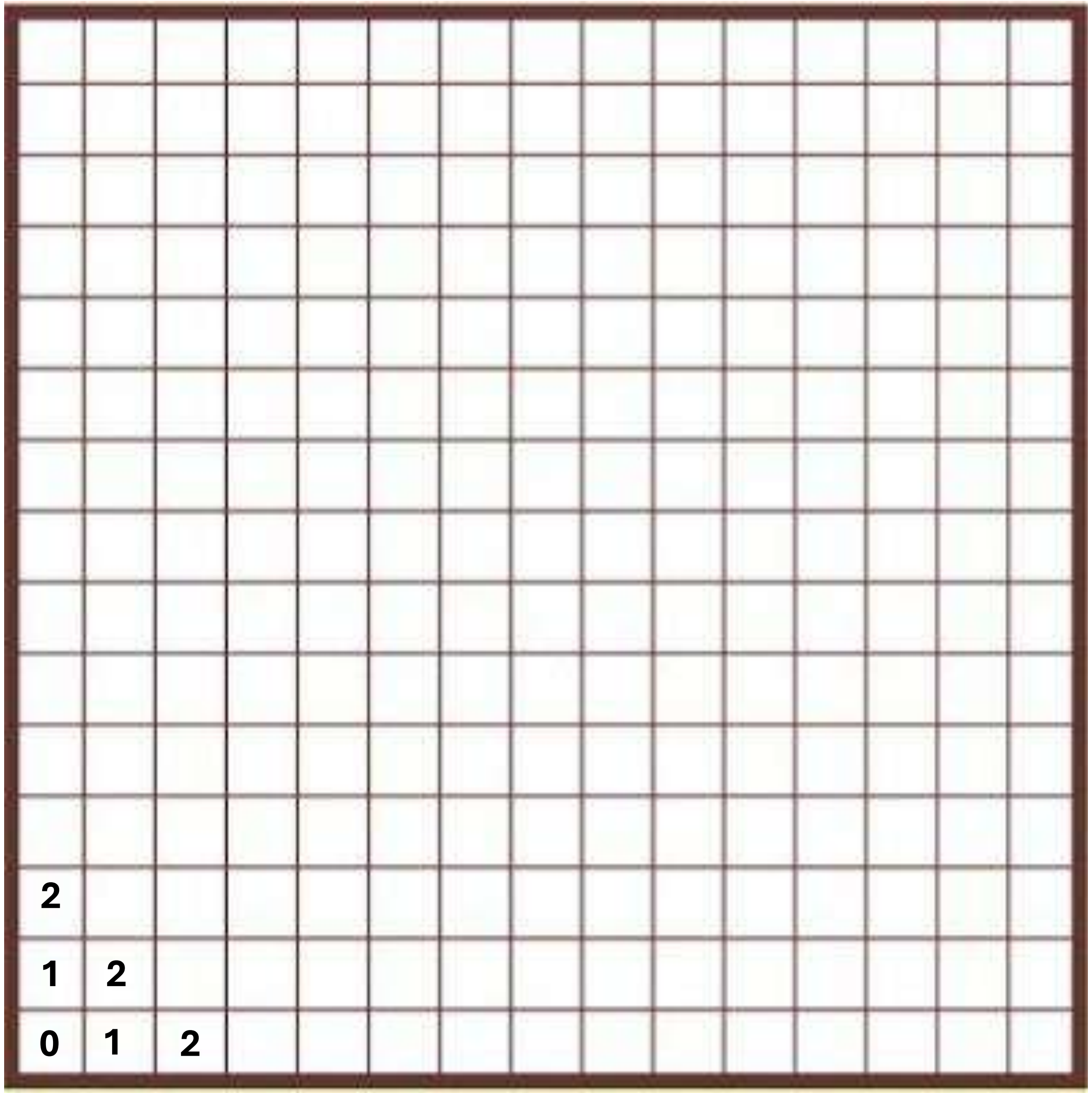

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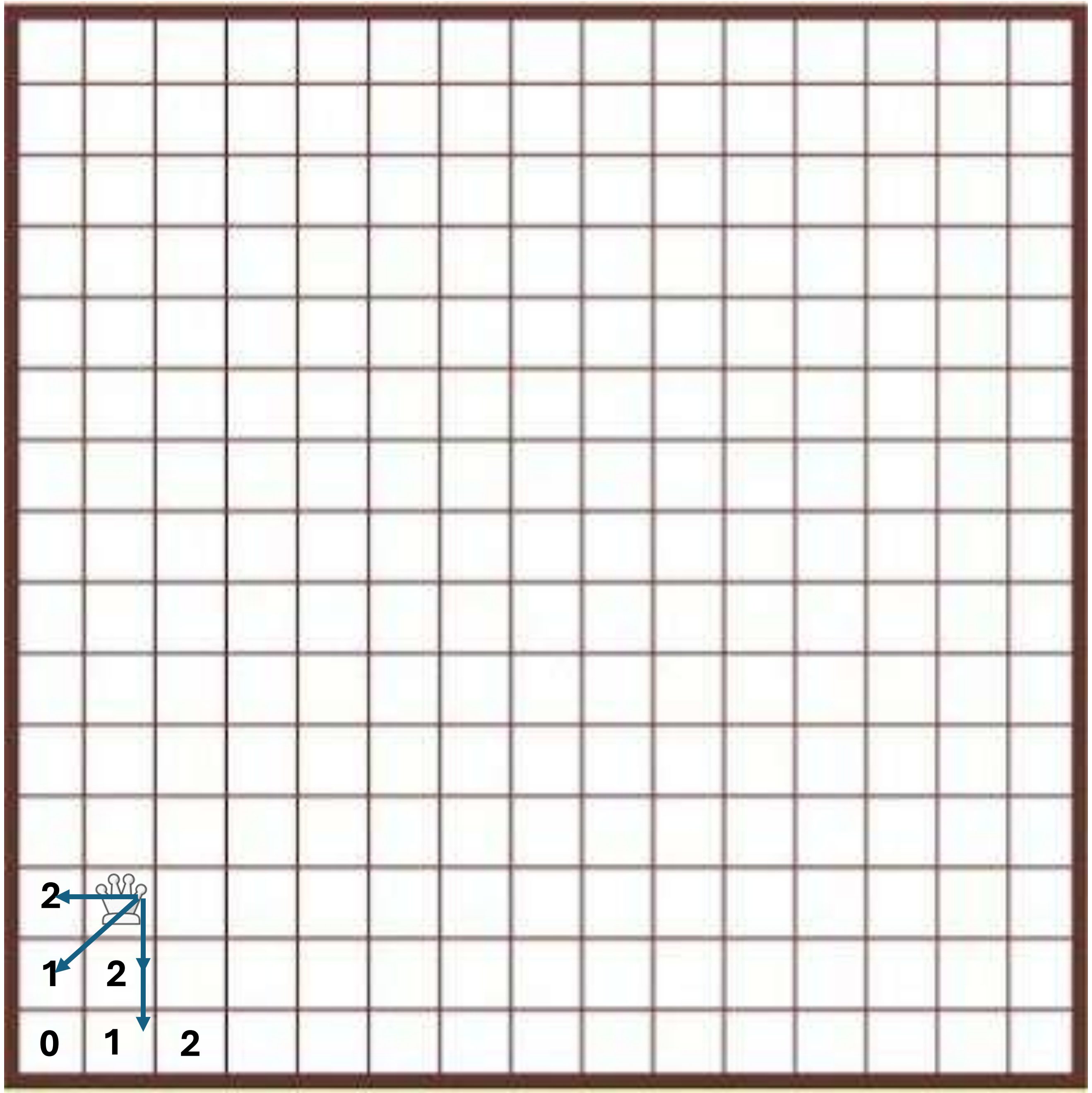

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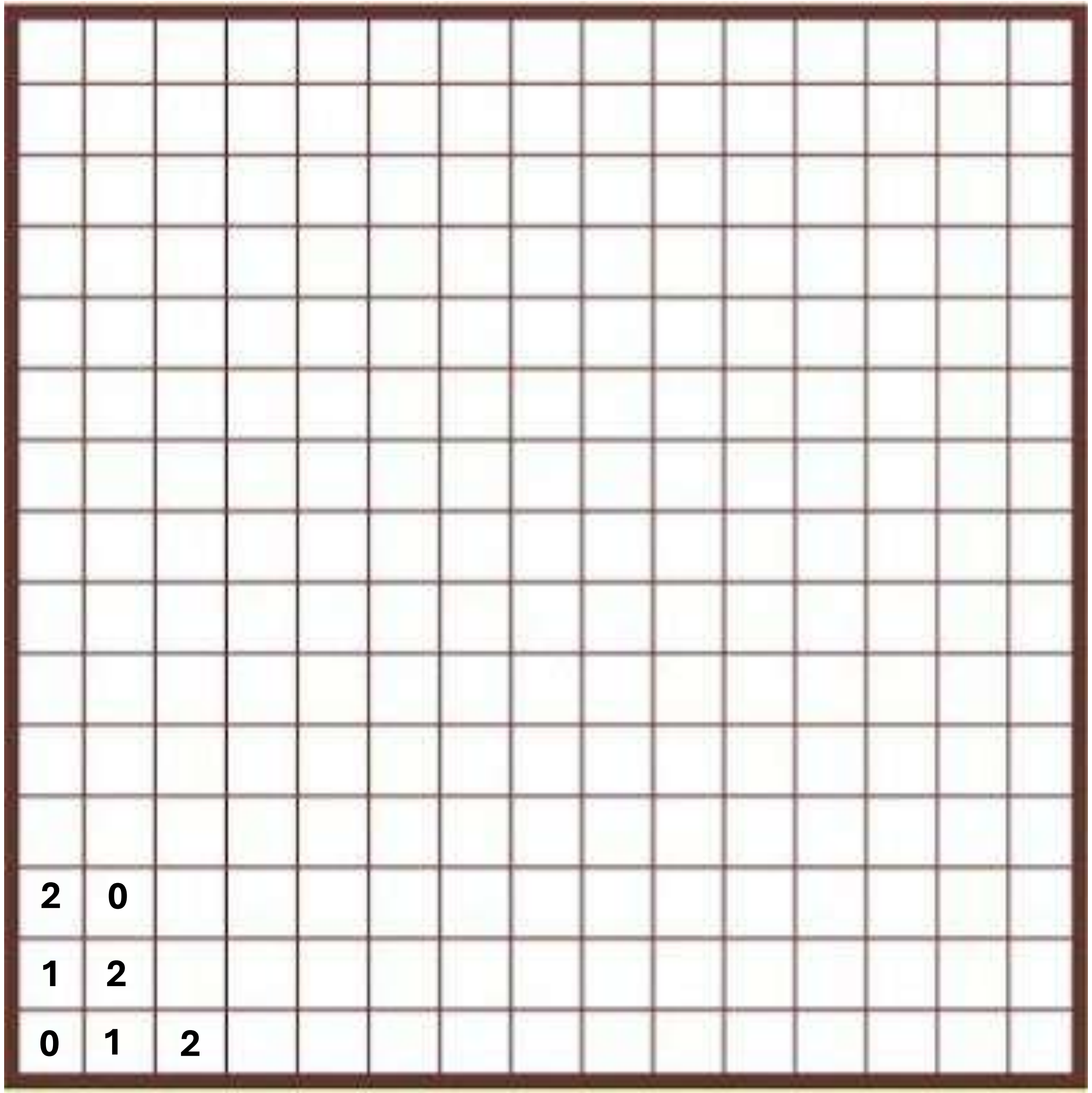

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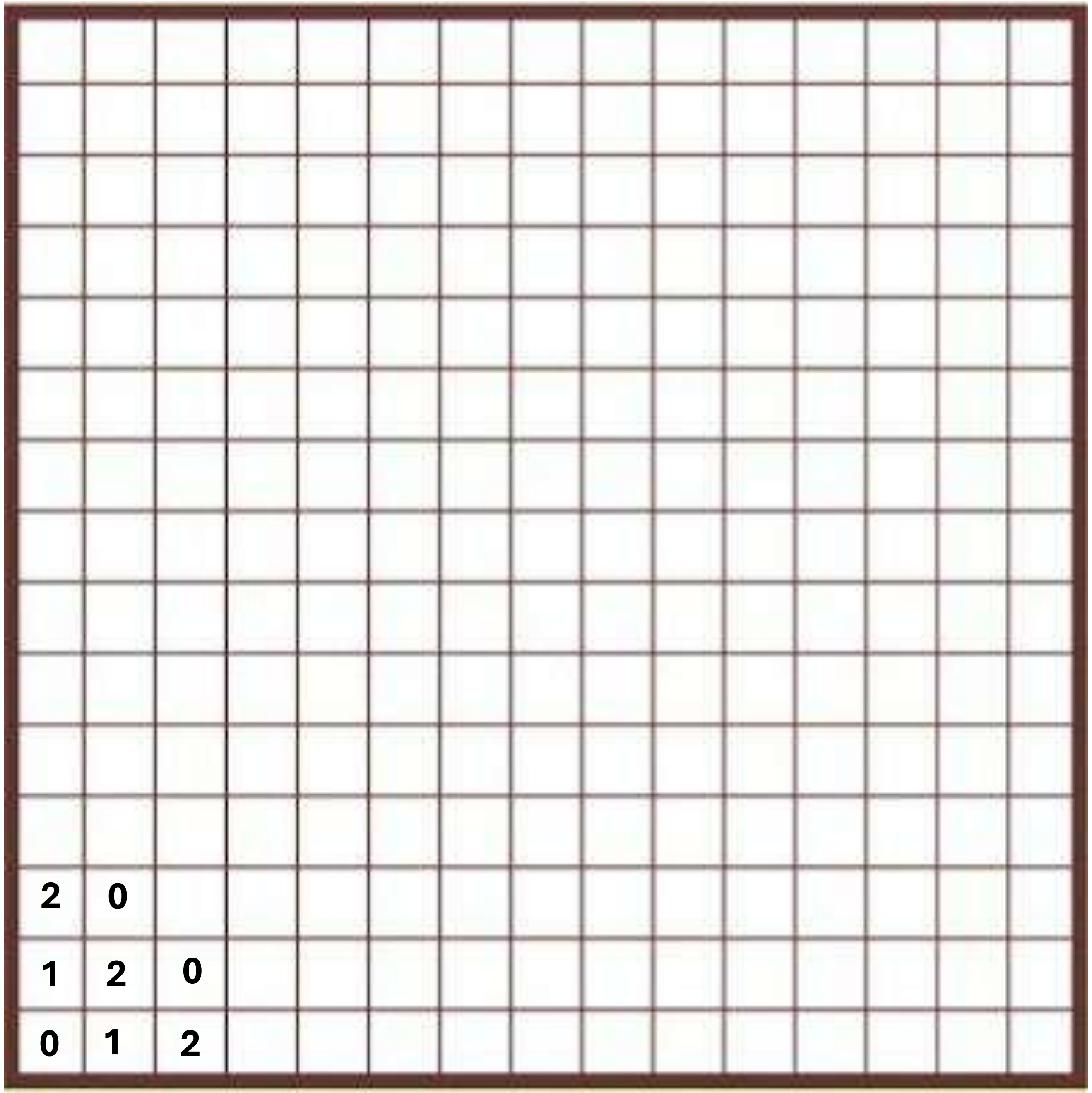

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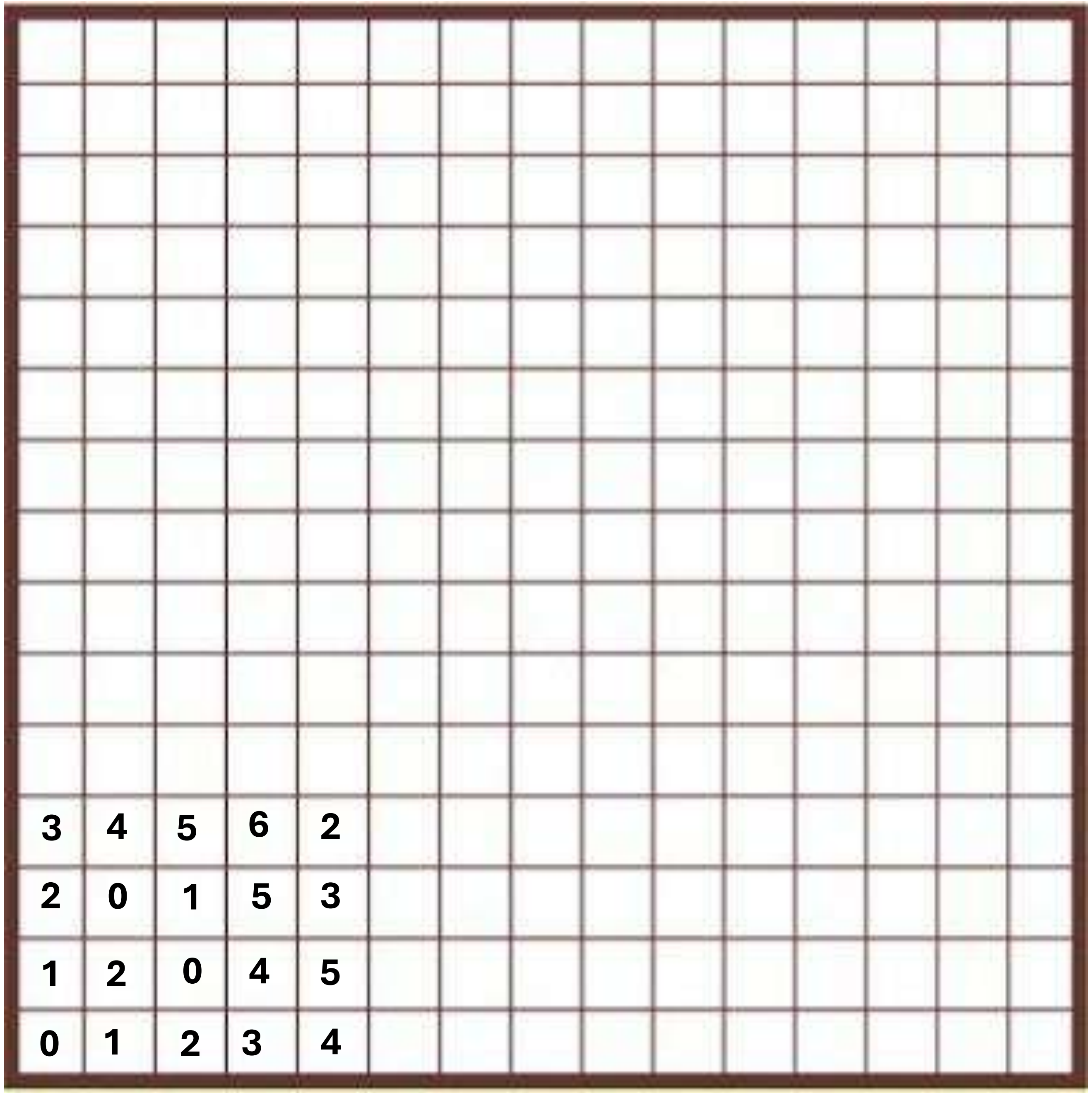

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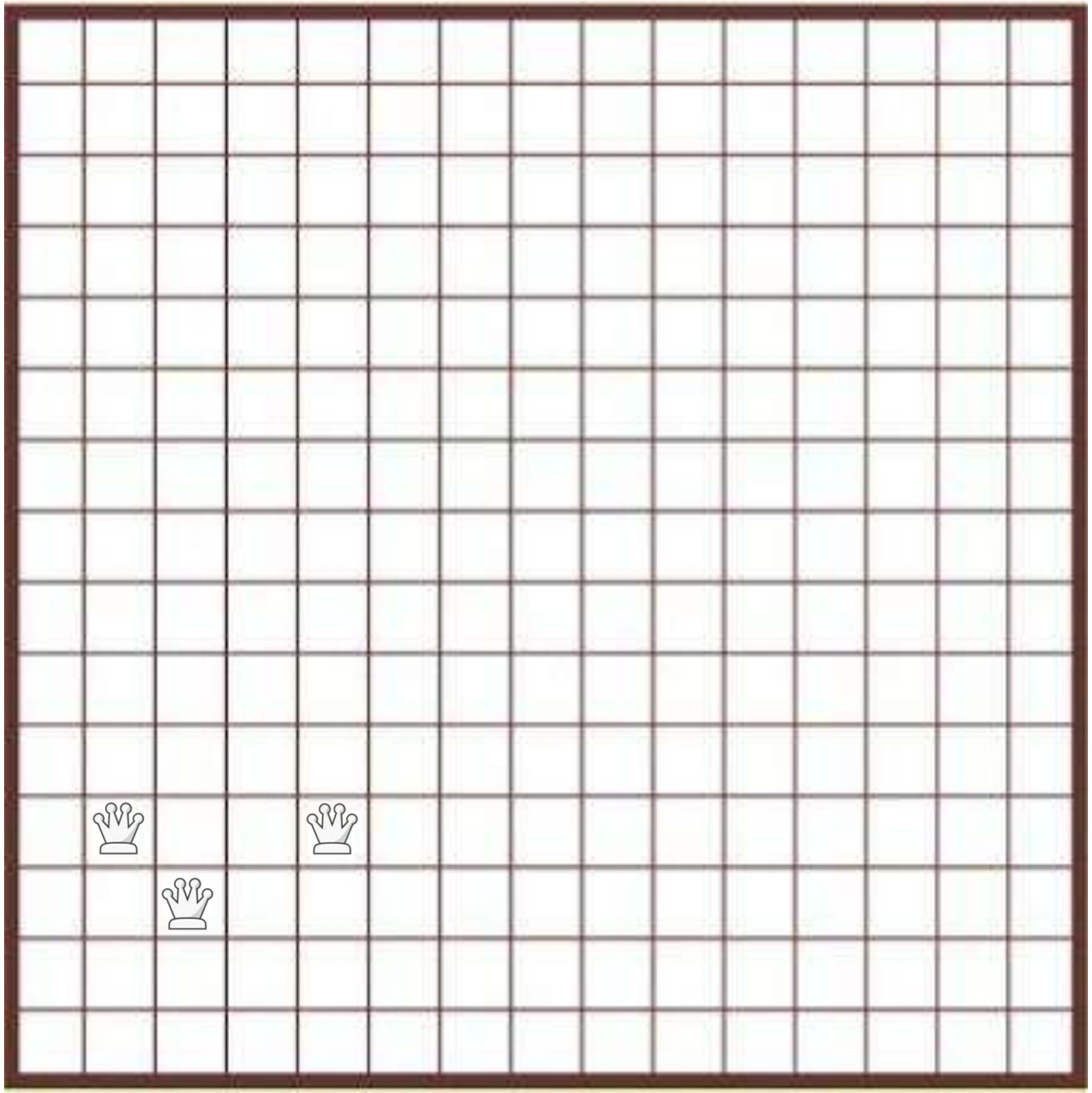

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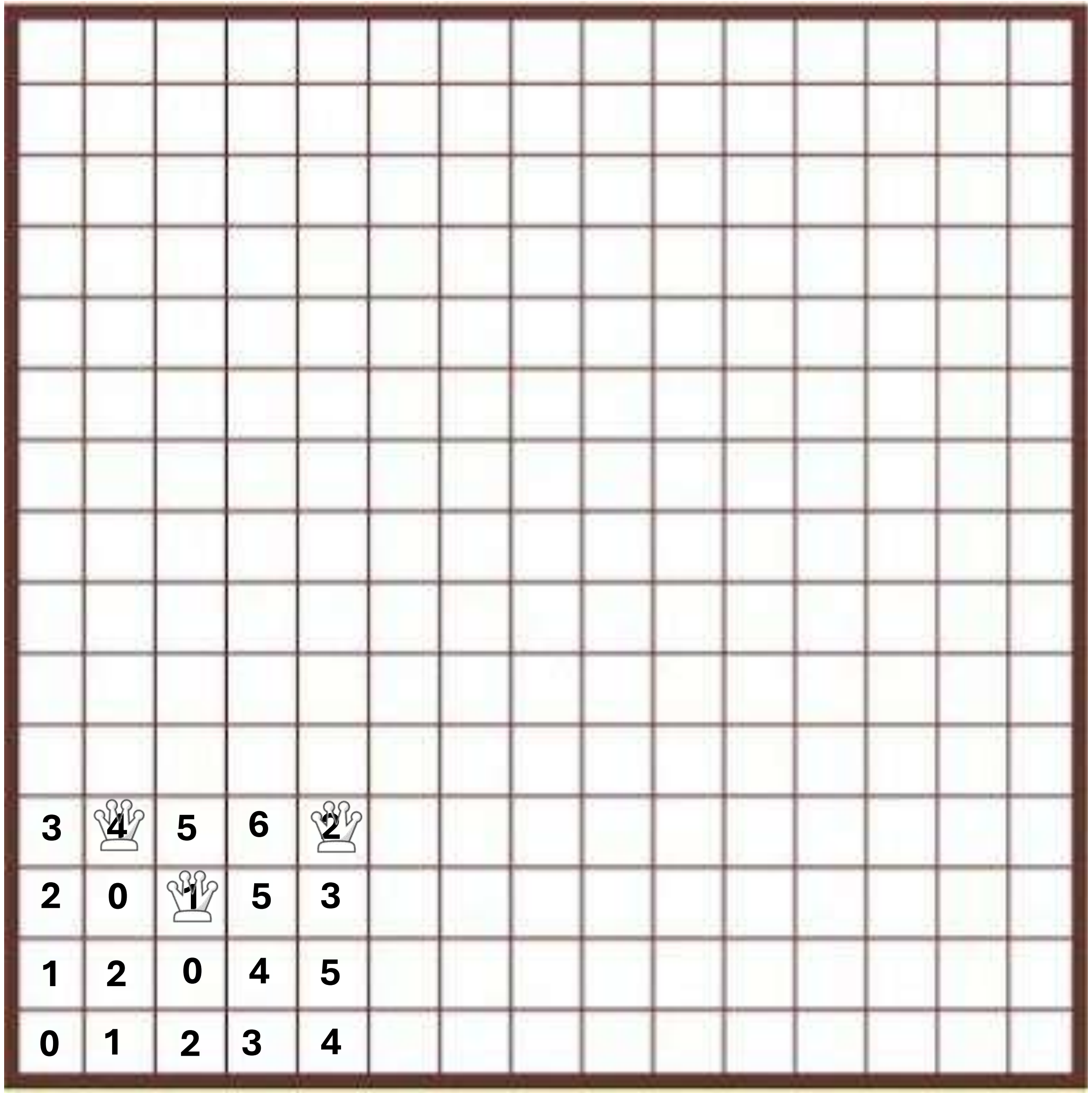

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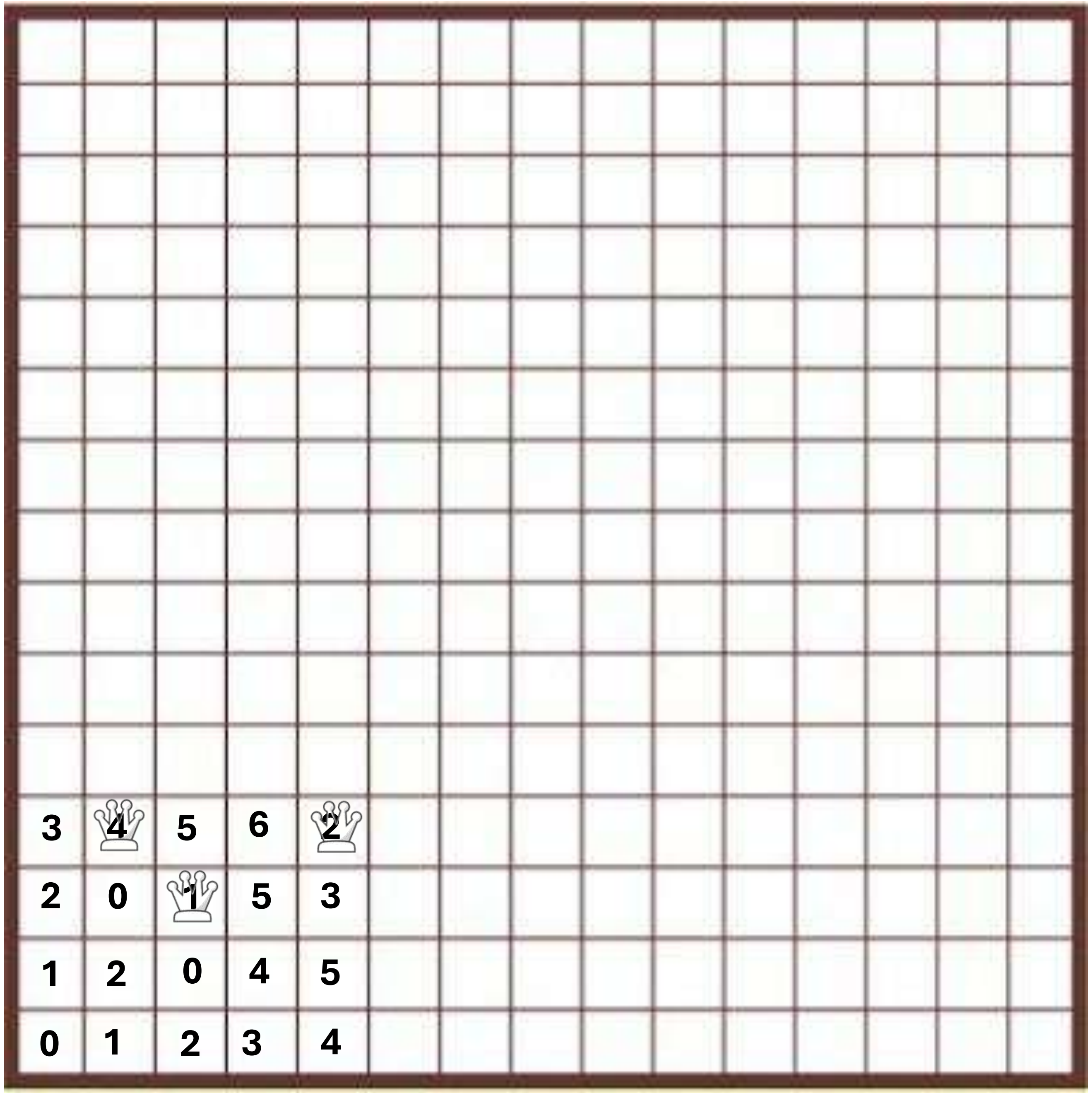
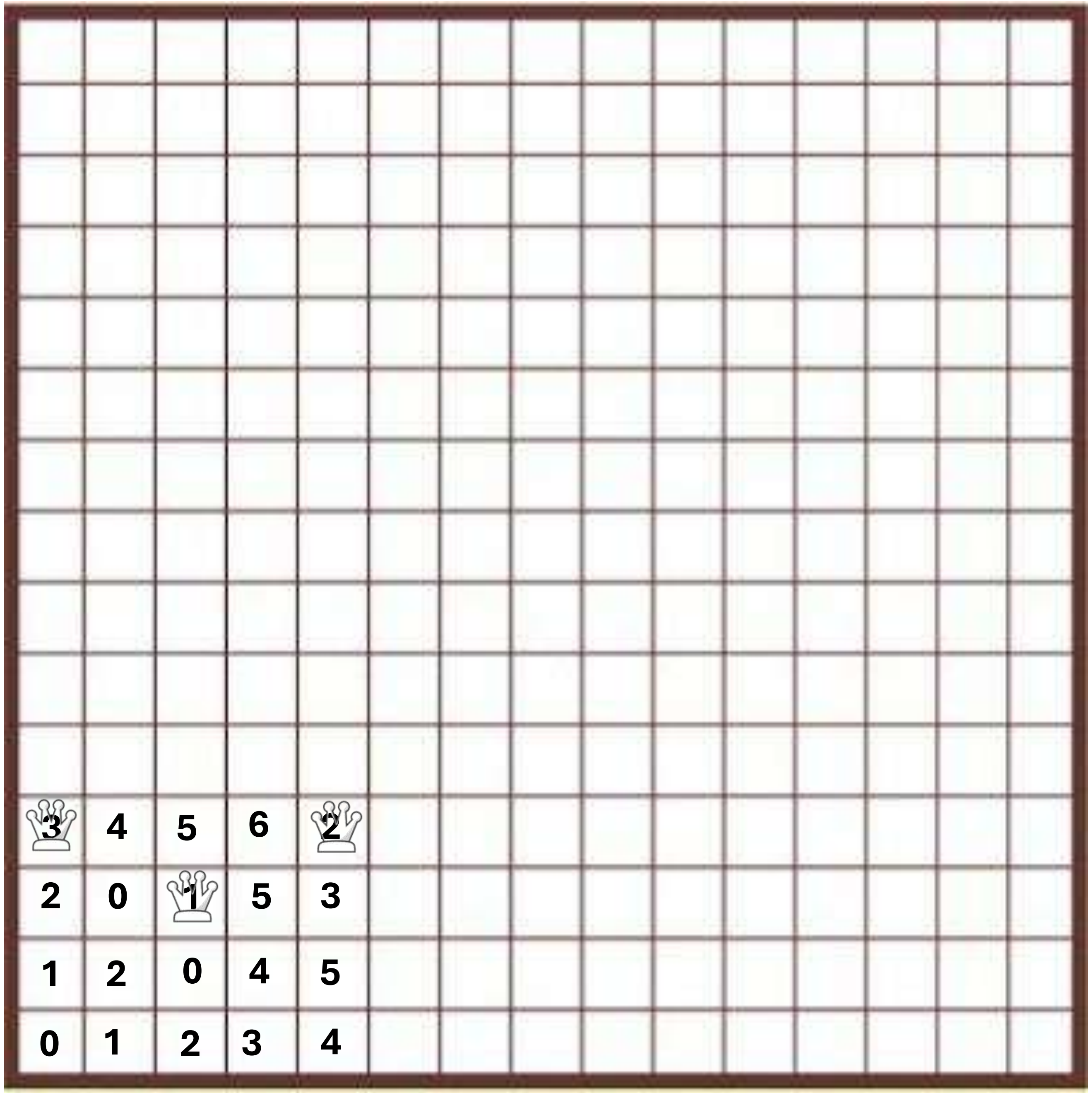


$$
\begin{array}{r}
1\ 0\ 0 \\
1\ 0 \\
1 \\
\hline
\oplus \quad 1\ 1\ 1
\end{array}
$$

WYTHOFF'S QUEENS



1 1

1 0

1

$\oplus$ 0 0

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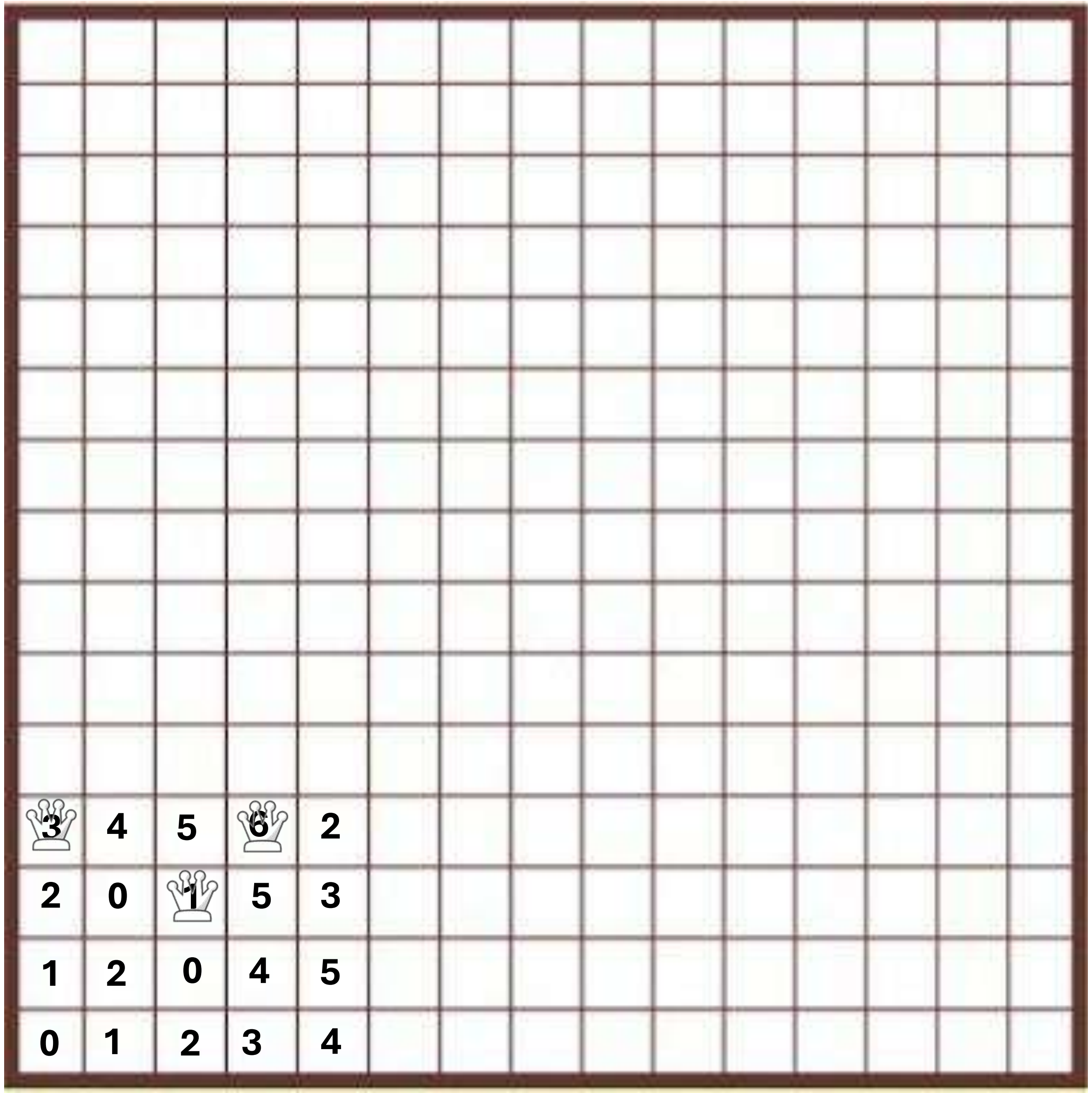

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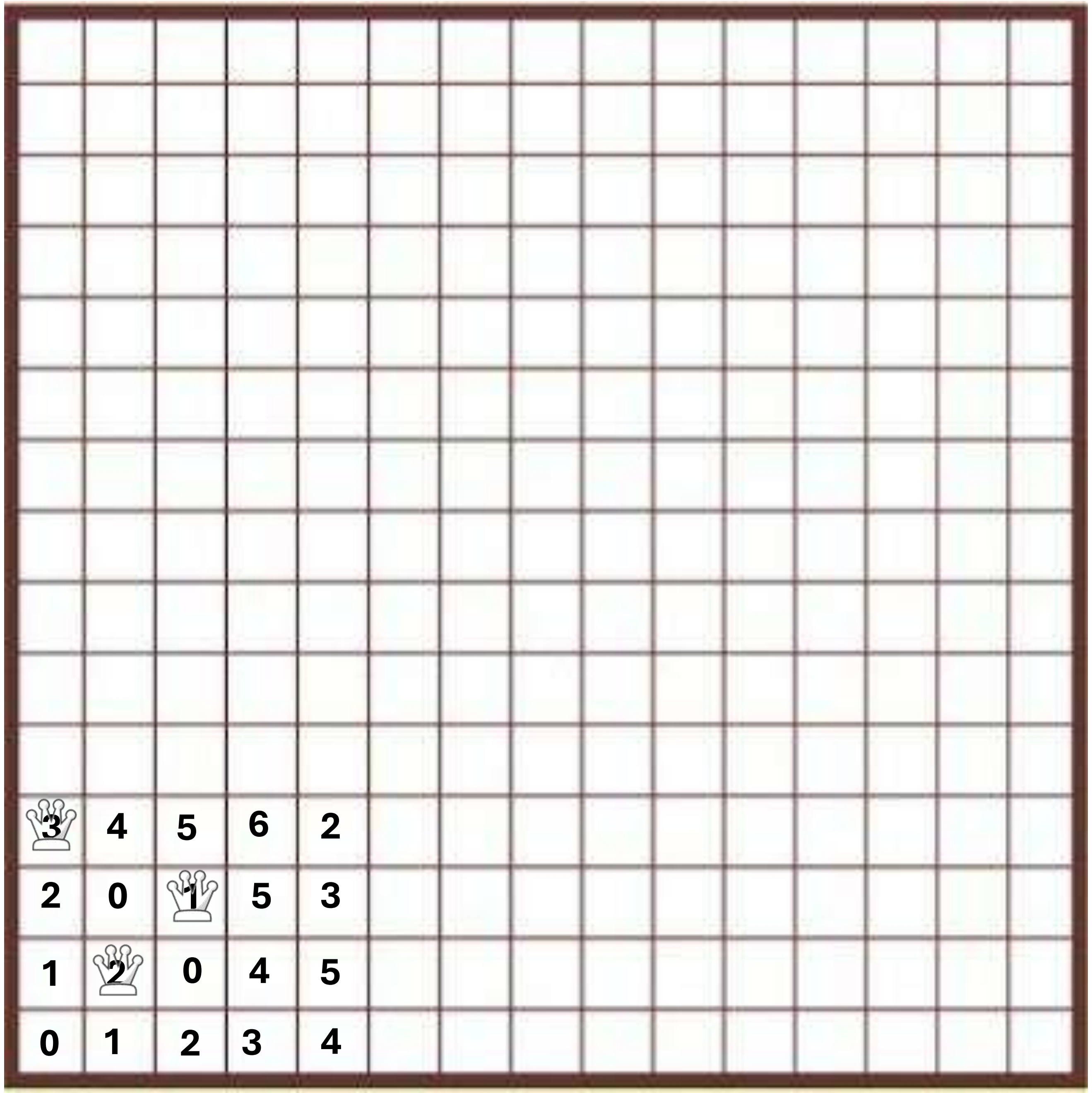
\begin{array}{r}
11 \\
110 \\
1 \\
\hline
\oplus\quad 100
\end{array}
$$

WYTHOFF’S QUEENS



	1	1
	1	0
		1
⊕	0	0

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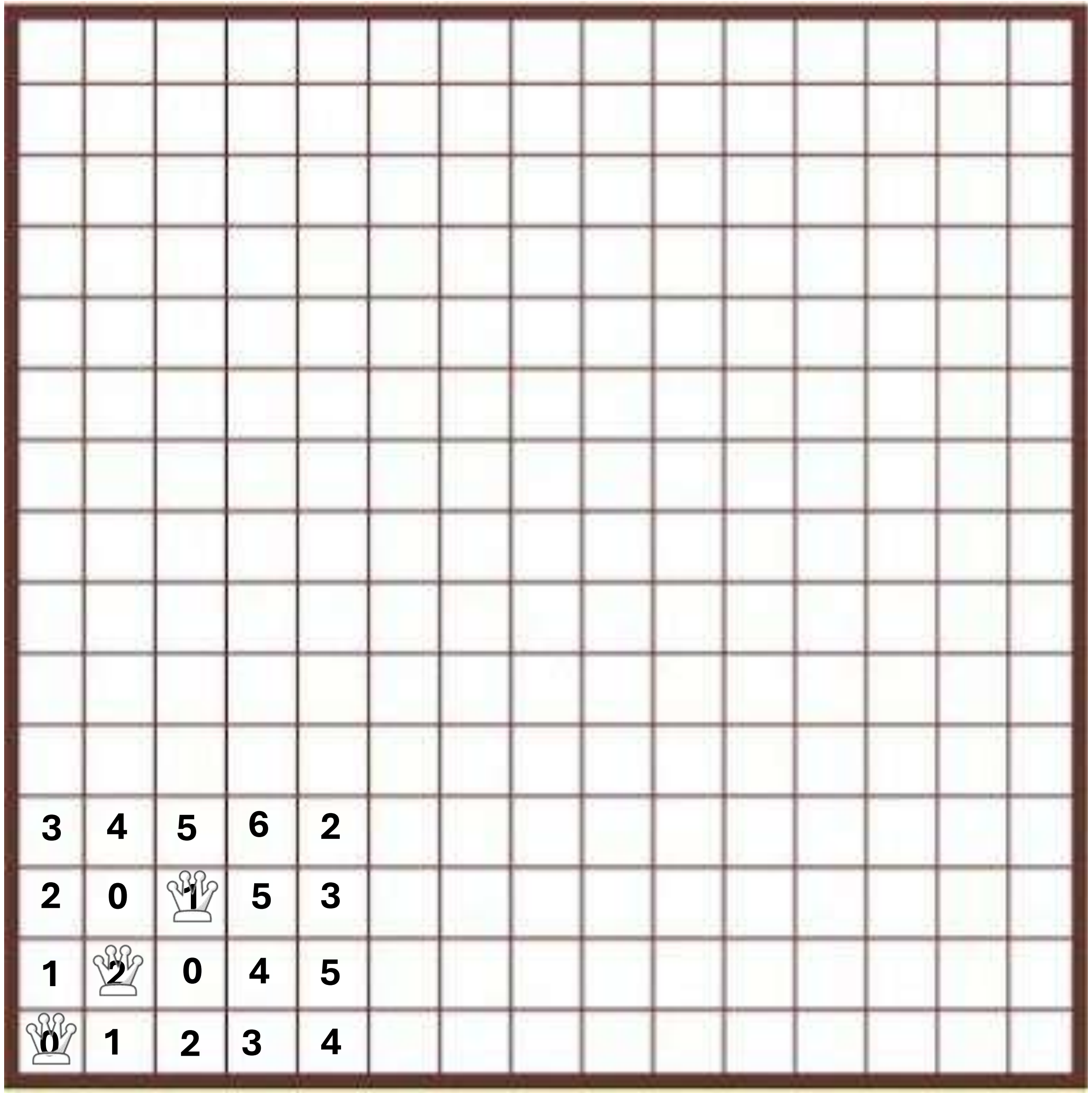

	1	0
		1
⊕	1	1

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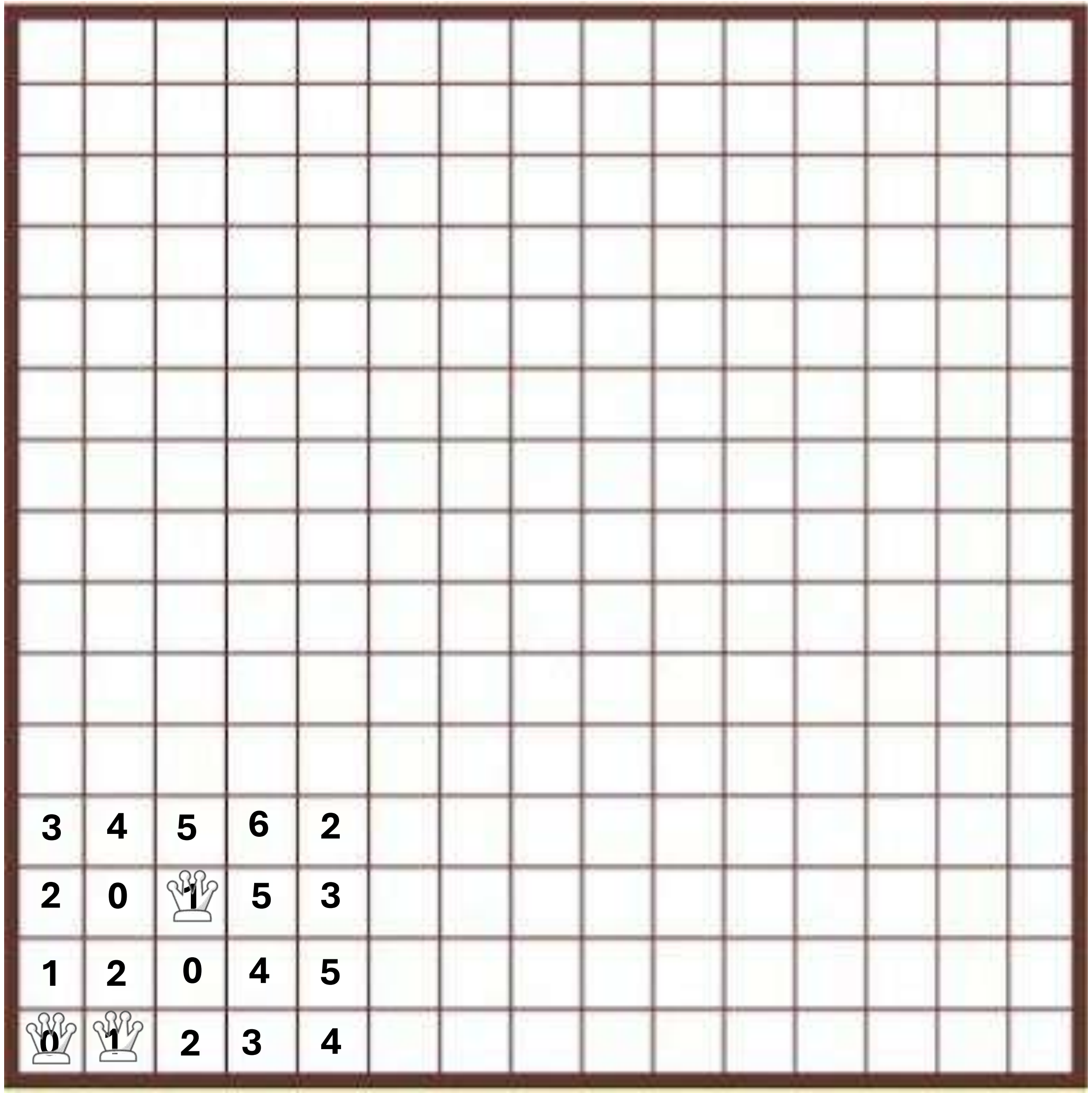

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⊕ 0 0

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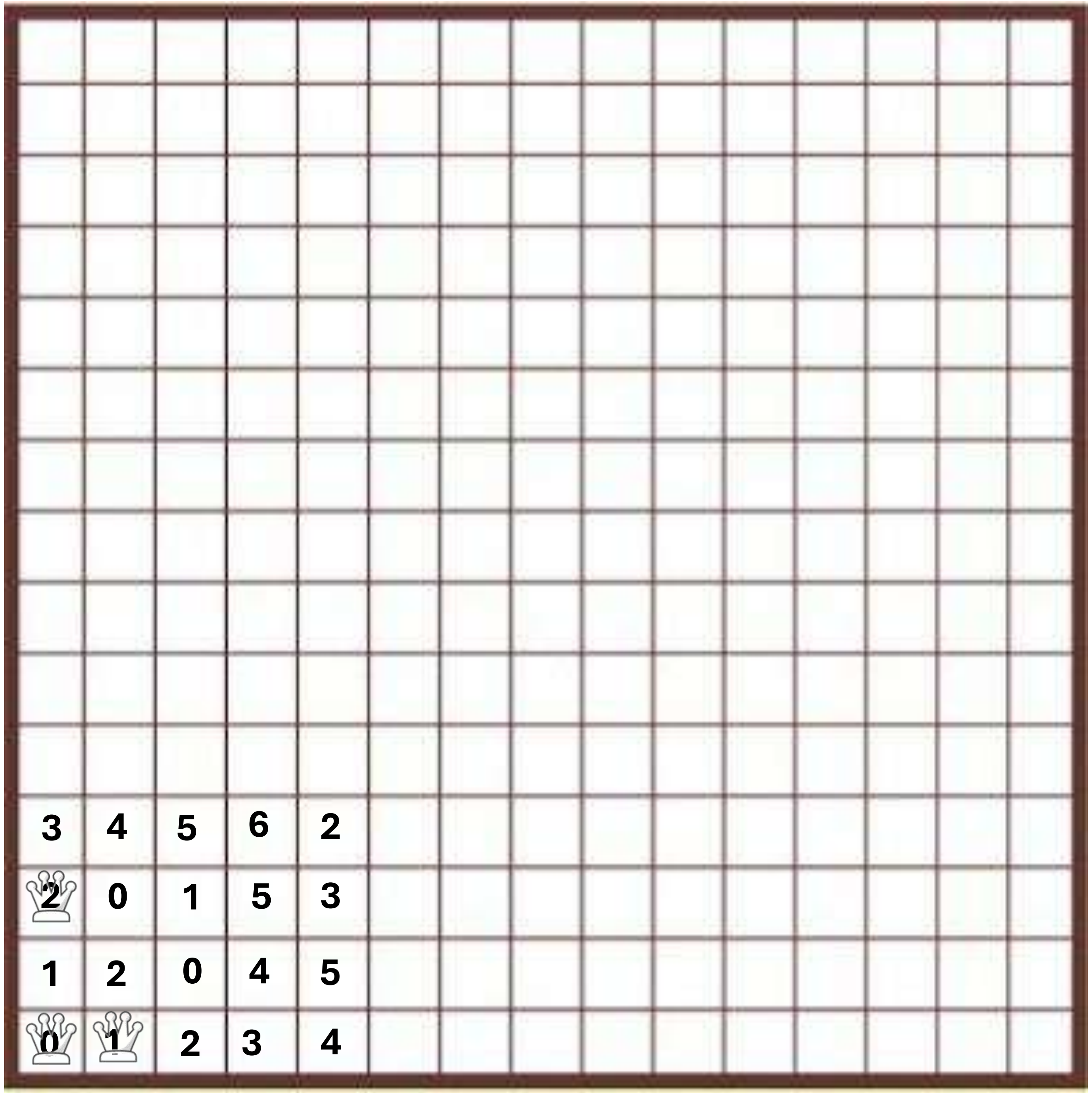
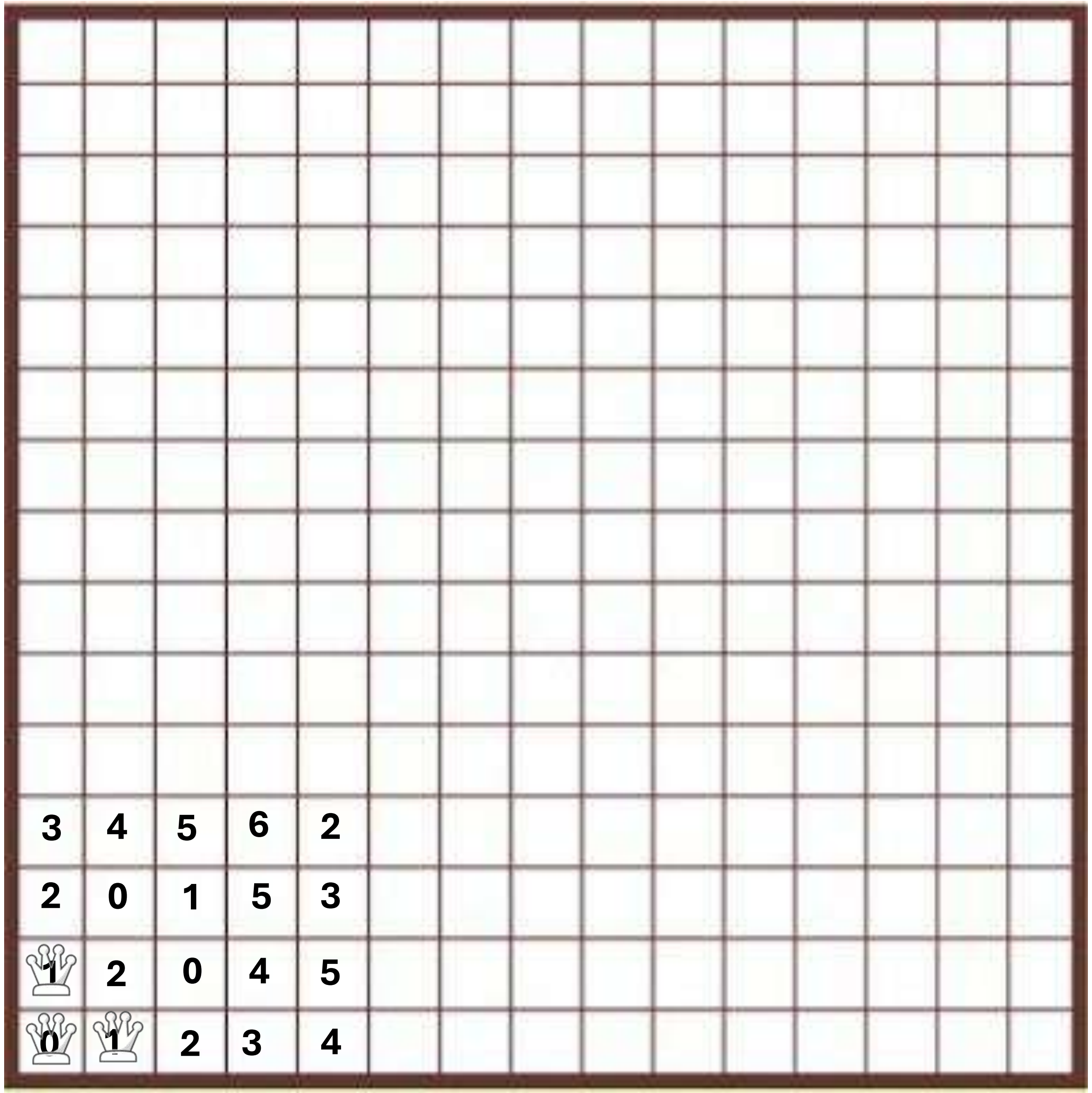


$$
\begin{array}{rr}
 & 1\;0 \\
 & 1 \\
\hline
\oplus & 1\;1
\end{array}
$$

WYTHOFF'S QUEENS

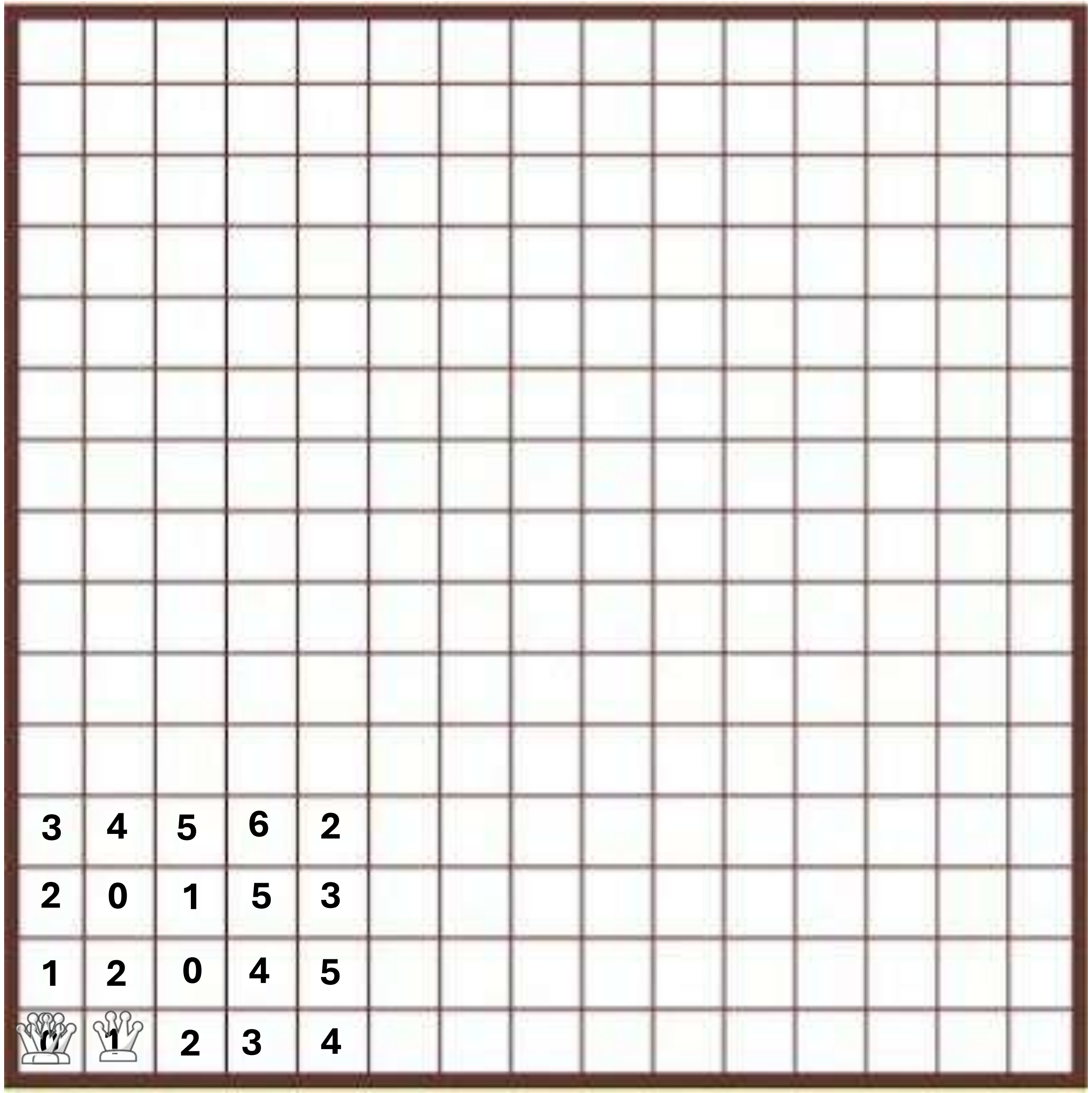


1

1

$\oplus$ 0 0

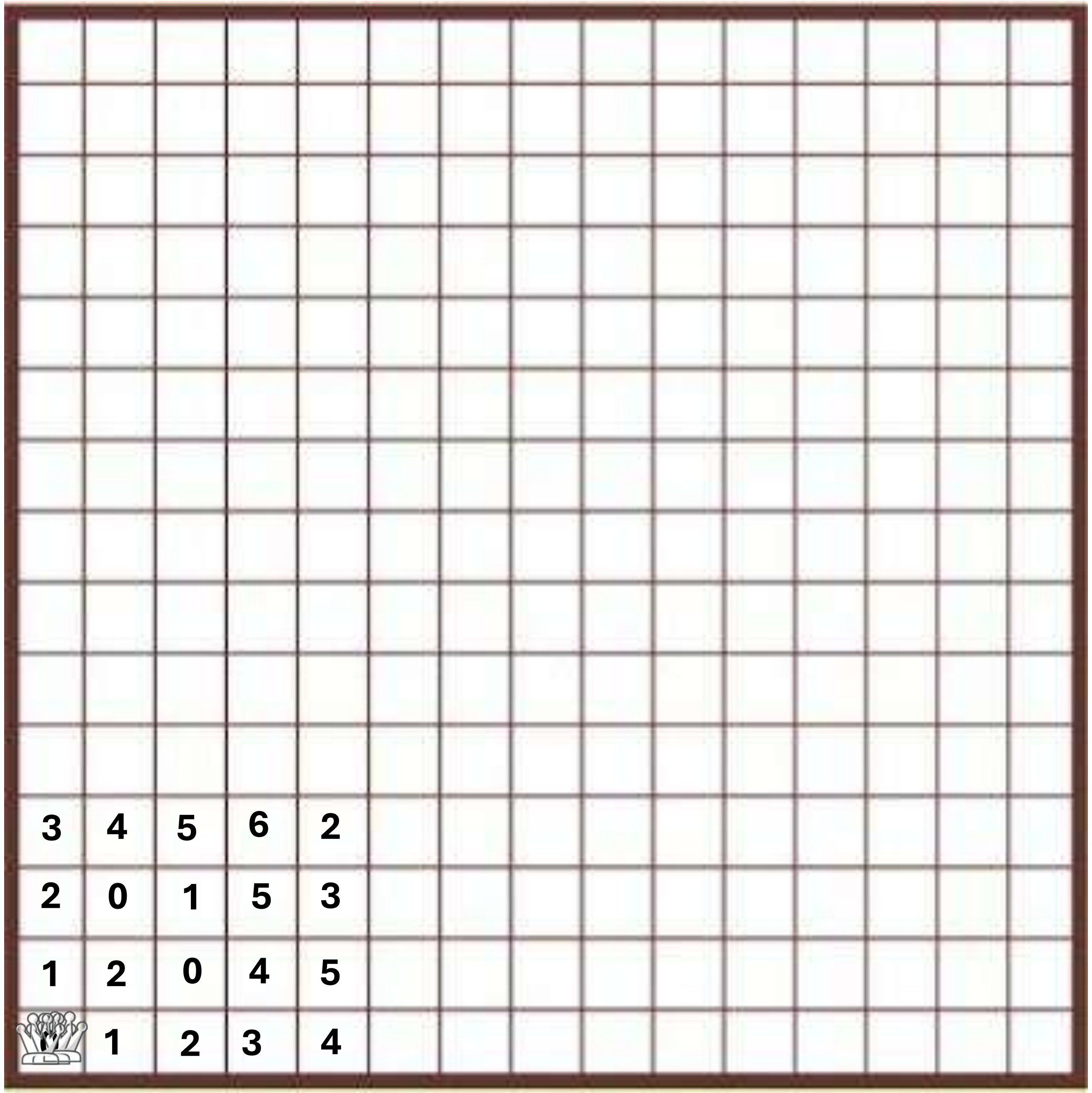
WYTHOFF'S QUEENS



  1

$\oplus$ 1

WYTHOFF’S QUEENS



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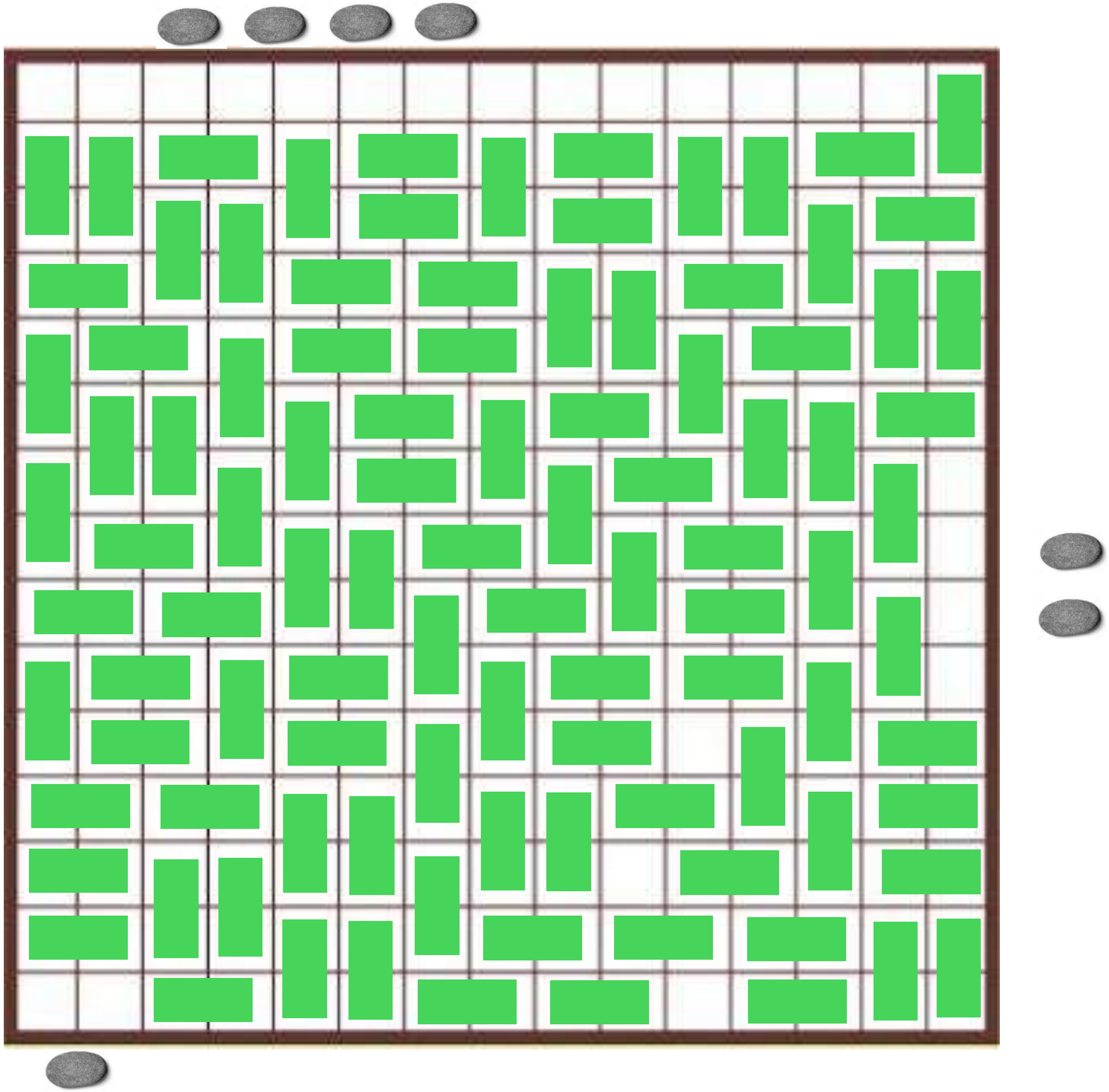

Part II: Partizan Games

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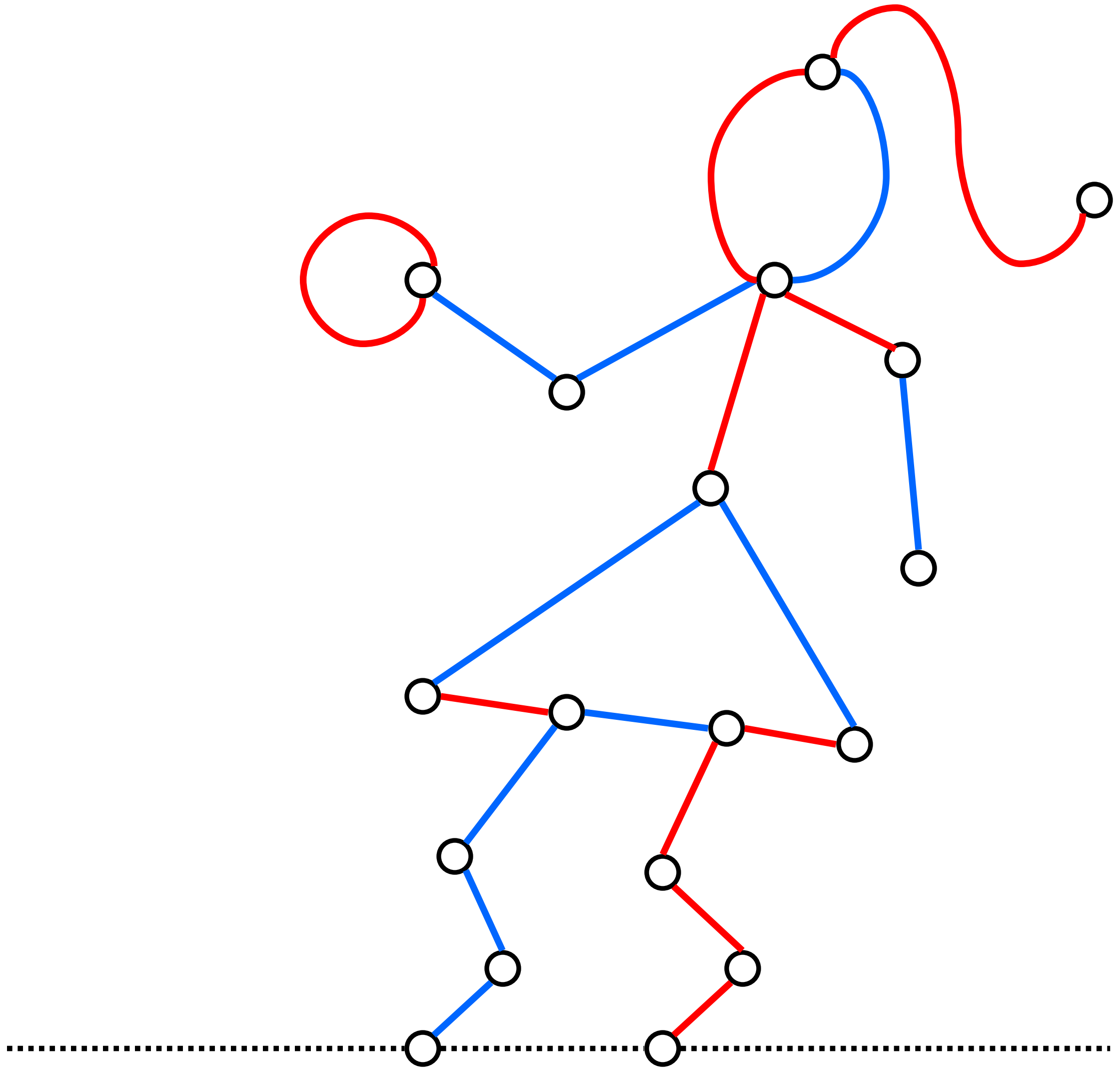
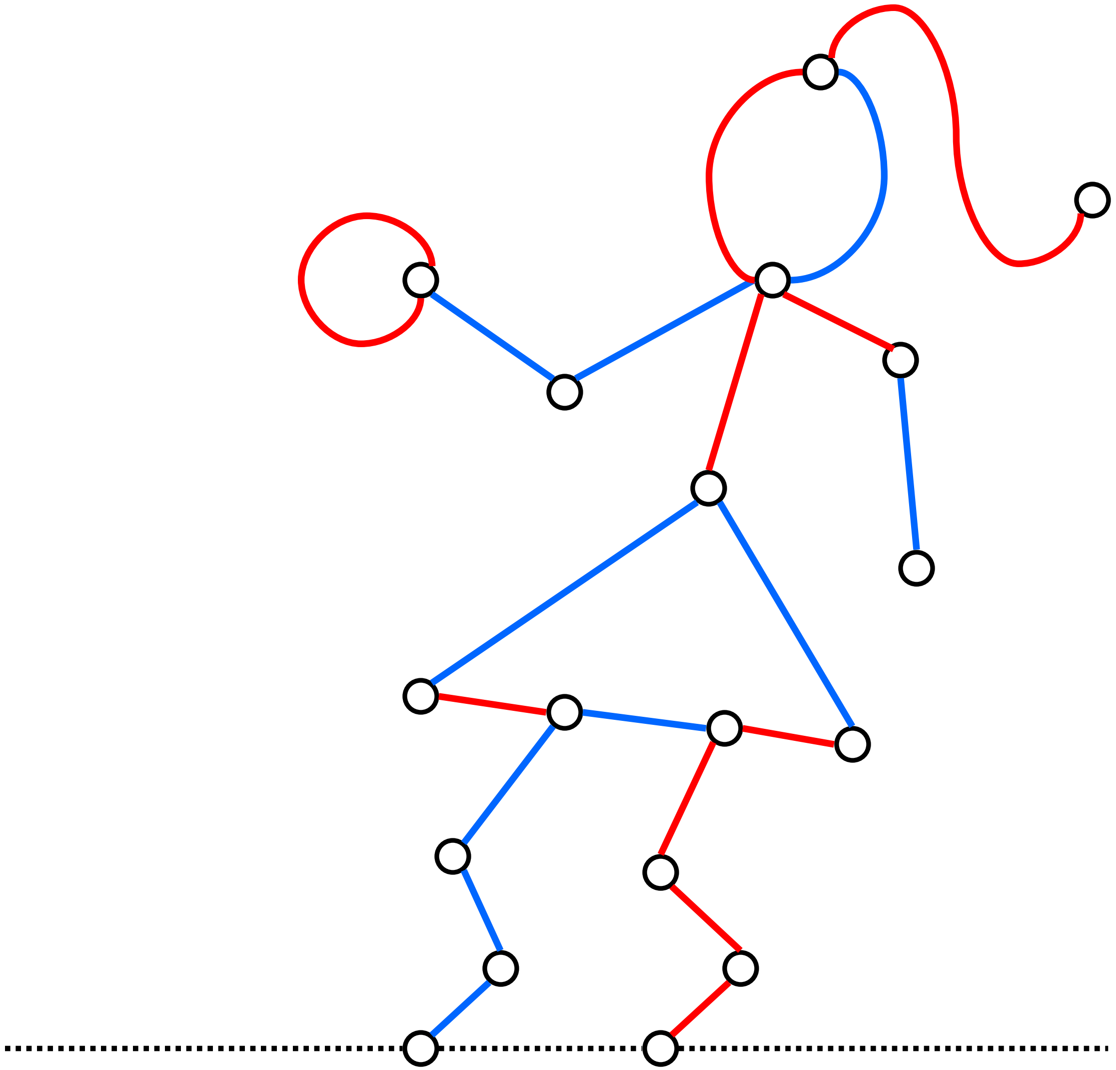
I.1: Some famous games

BLUE-RED-HACKENBUSH

BLUE-RED-HACKENBUSH

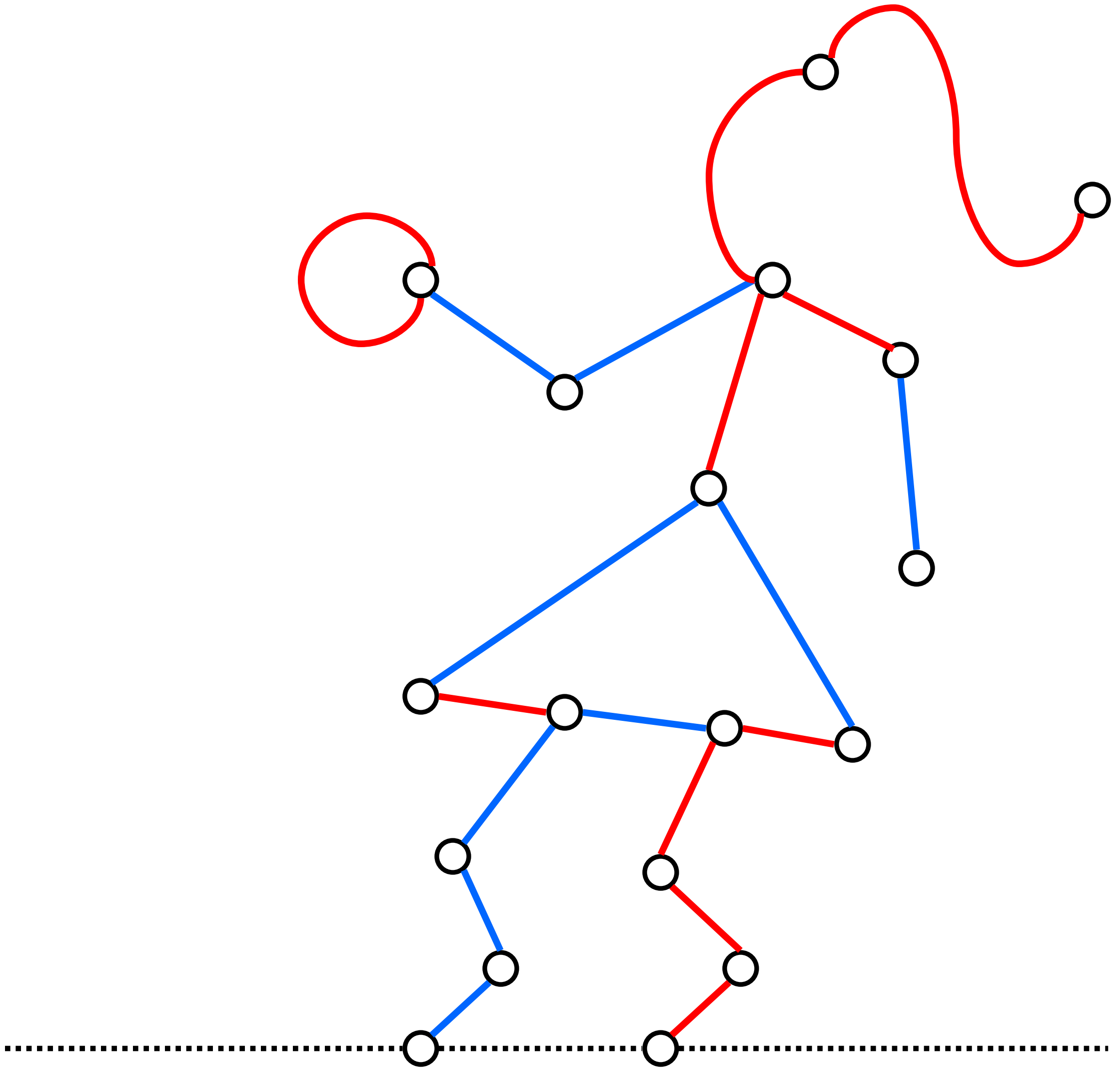


BLUE-RED-HACKENBUSH



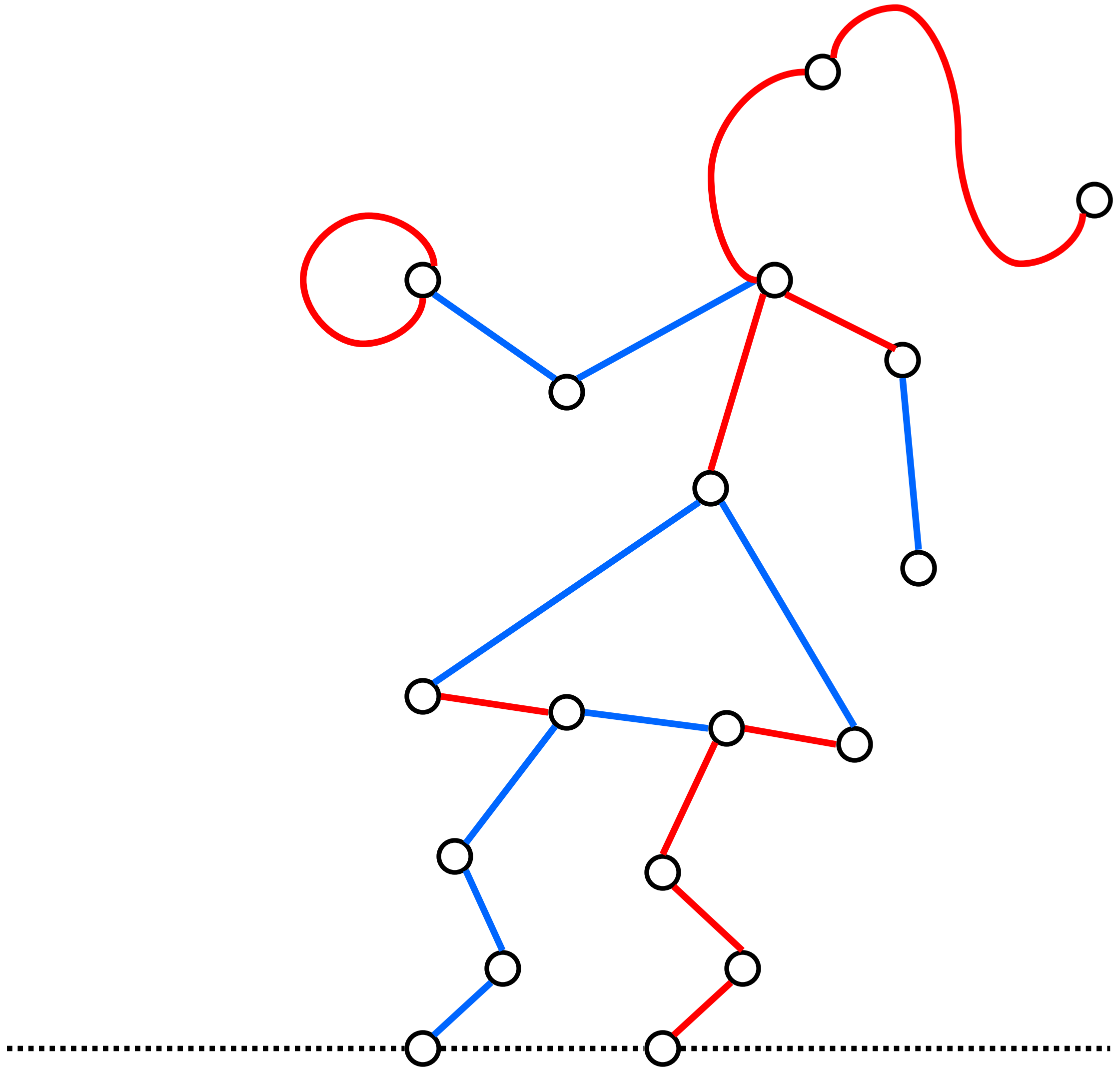
My turn

BLUE-RED-HACKENBUSH



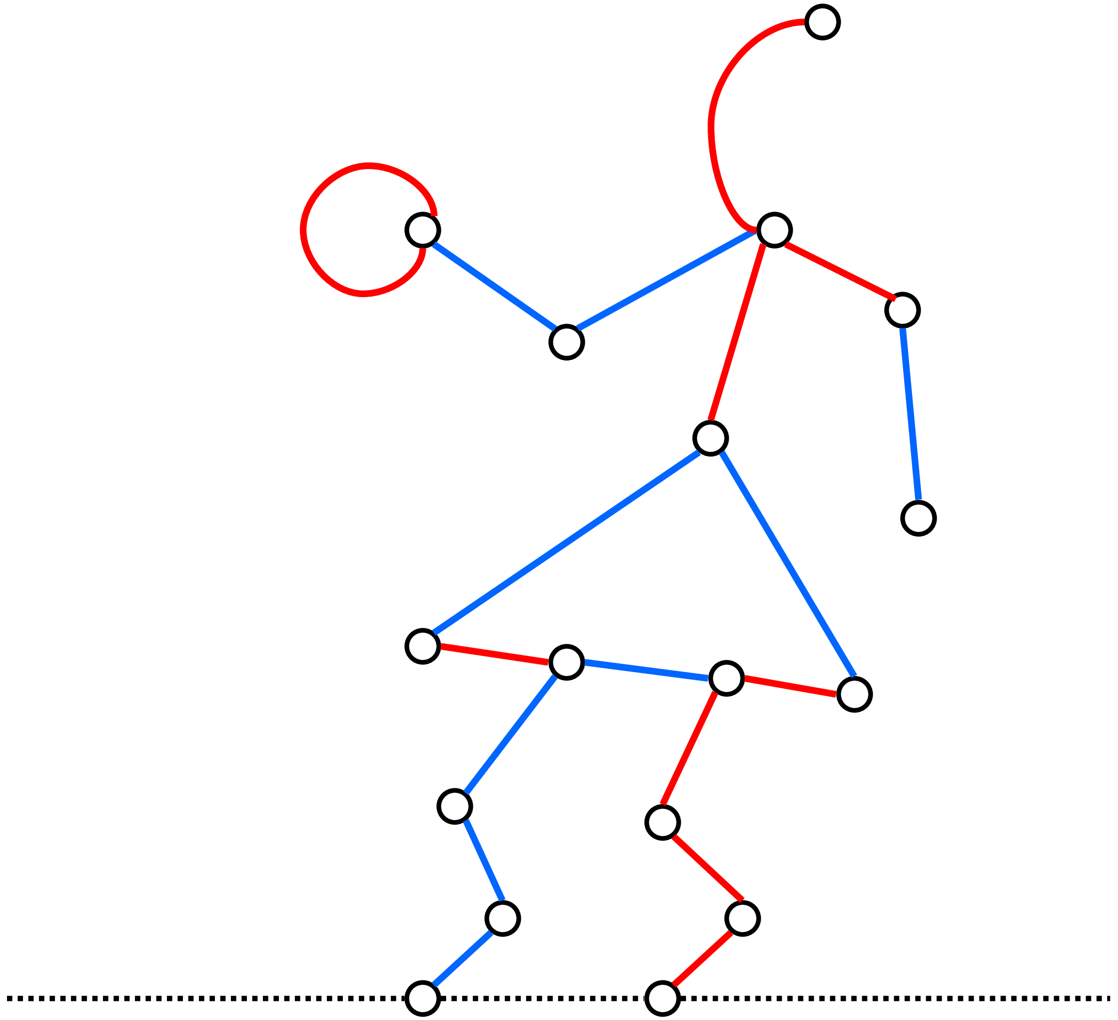
My turn

BLUE-RED-HACKENBUSH



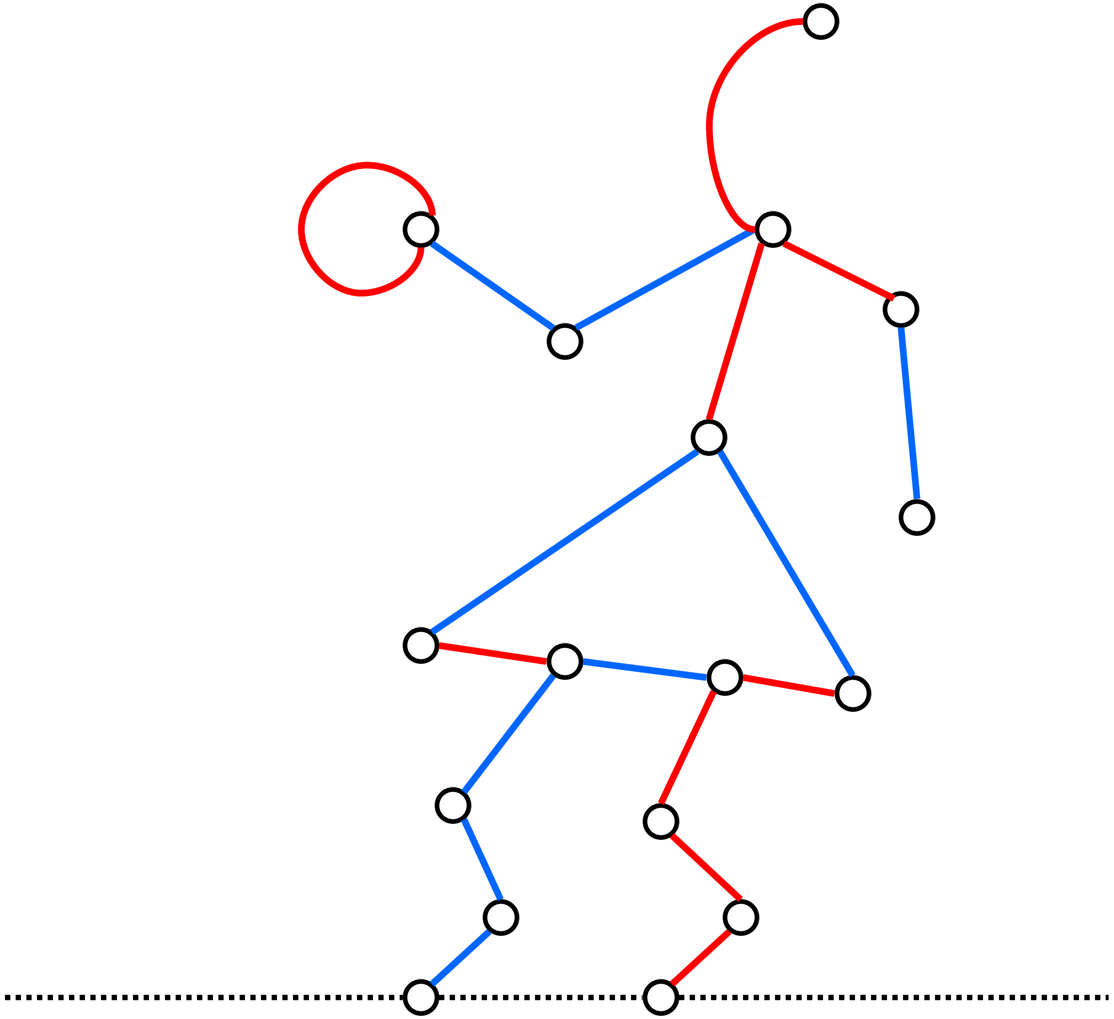
Your turn

BLUE-RED-HACKENBUSH



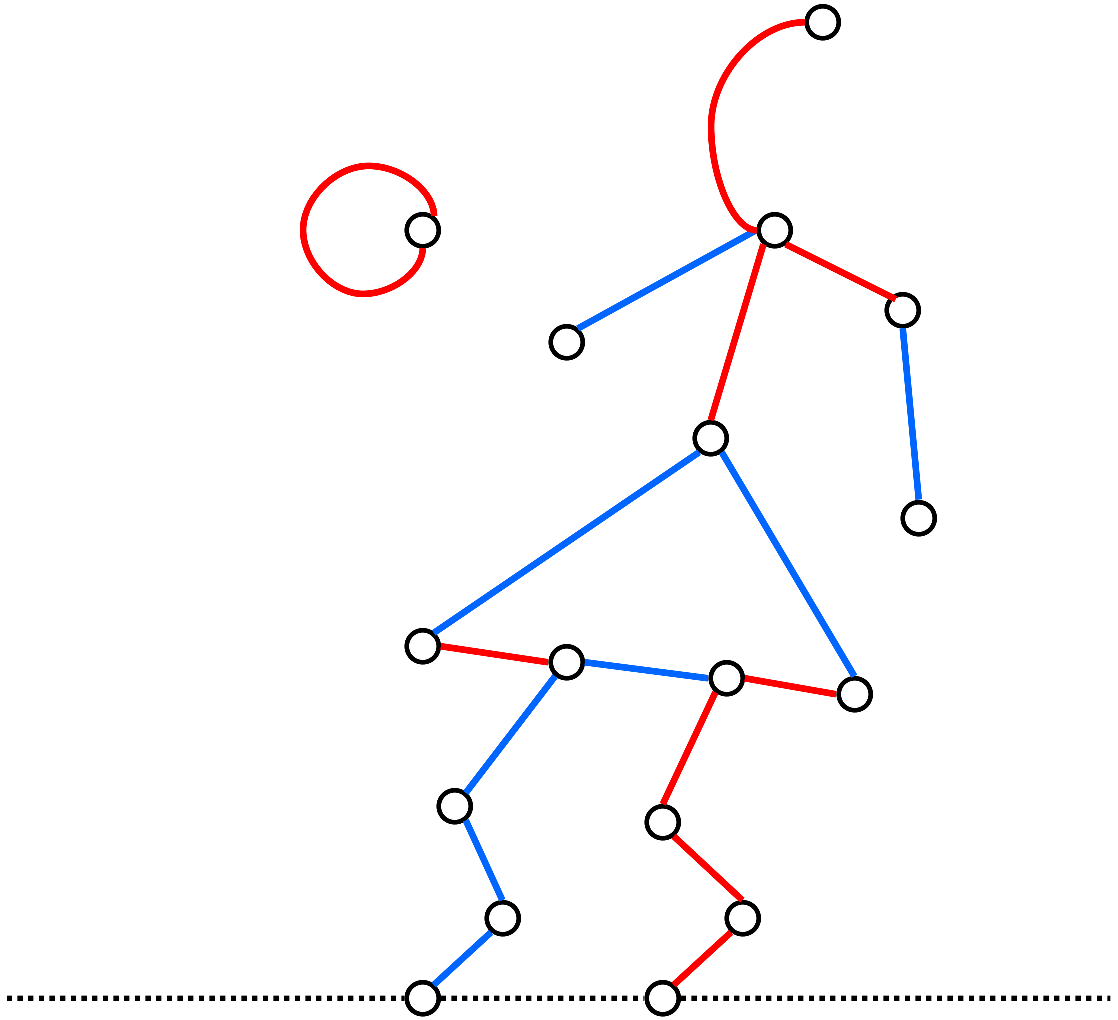
Your turn

BLUE-RED-HACKENBUSH



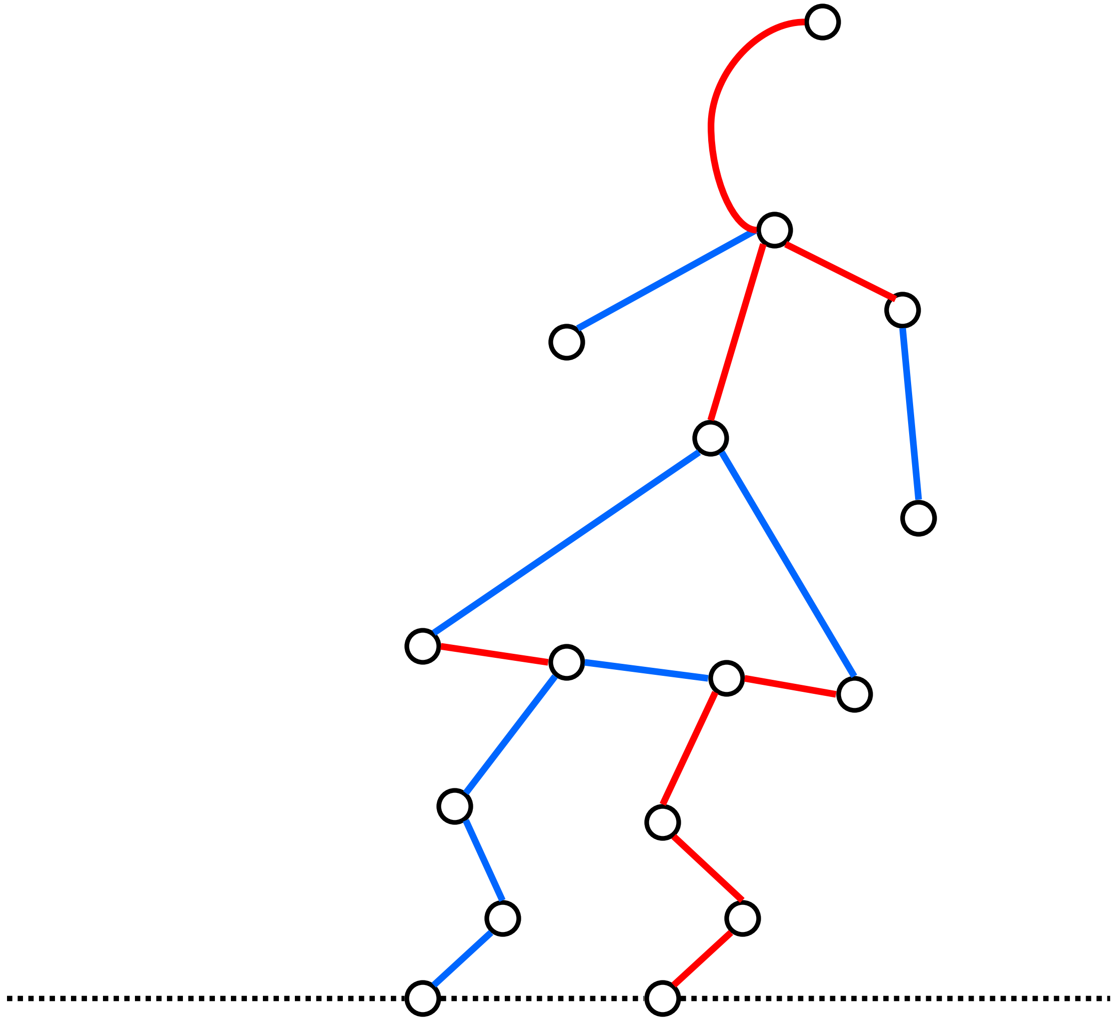
My turn

BLUE-RED-HACKENBUSH



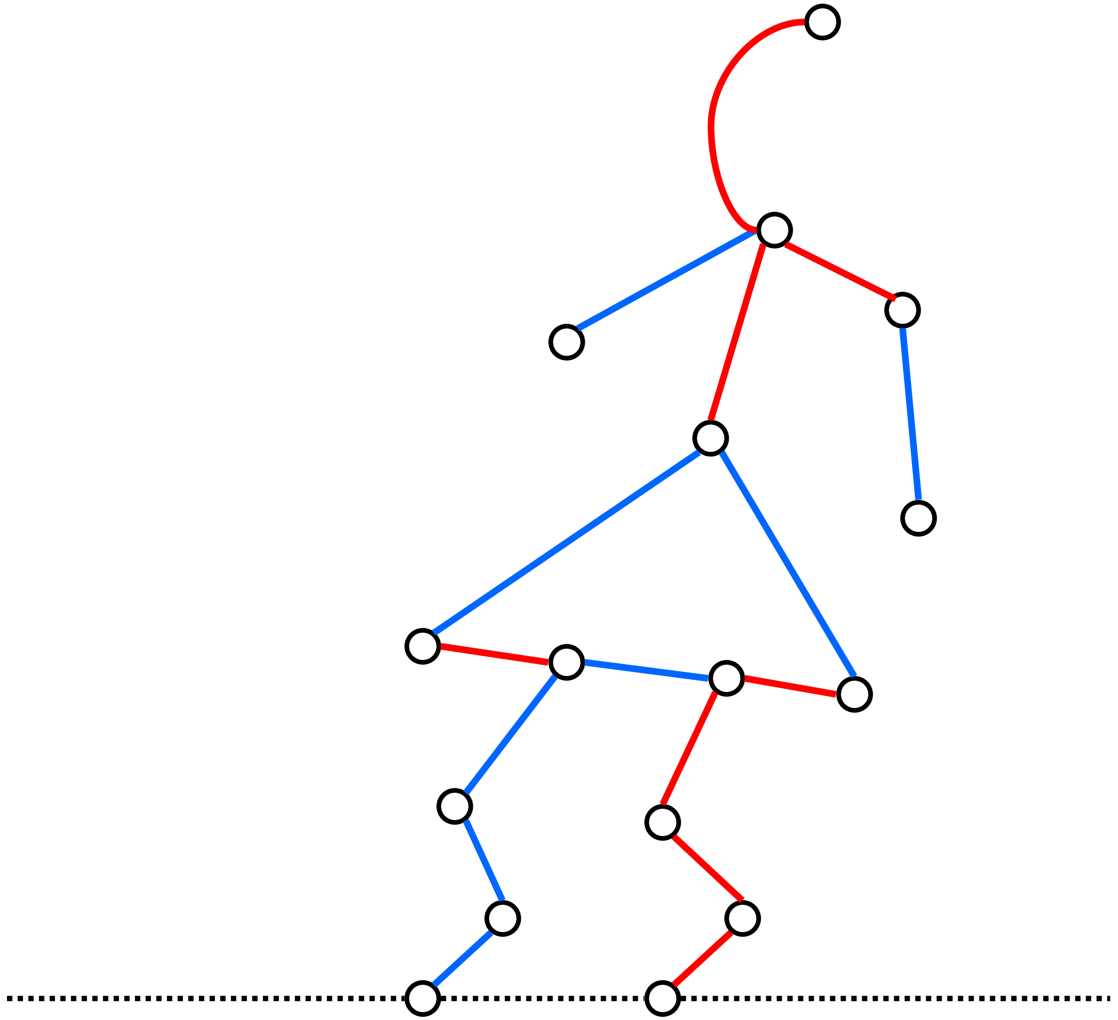
My turn

BLUE-RED-HACKENBUSH



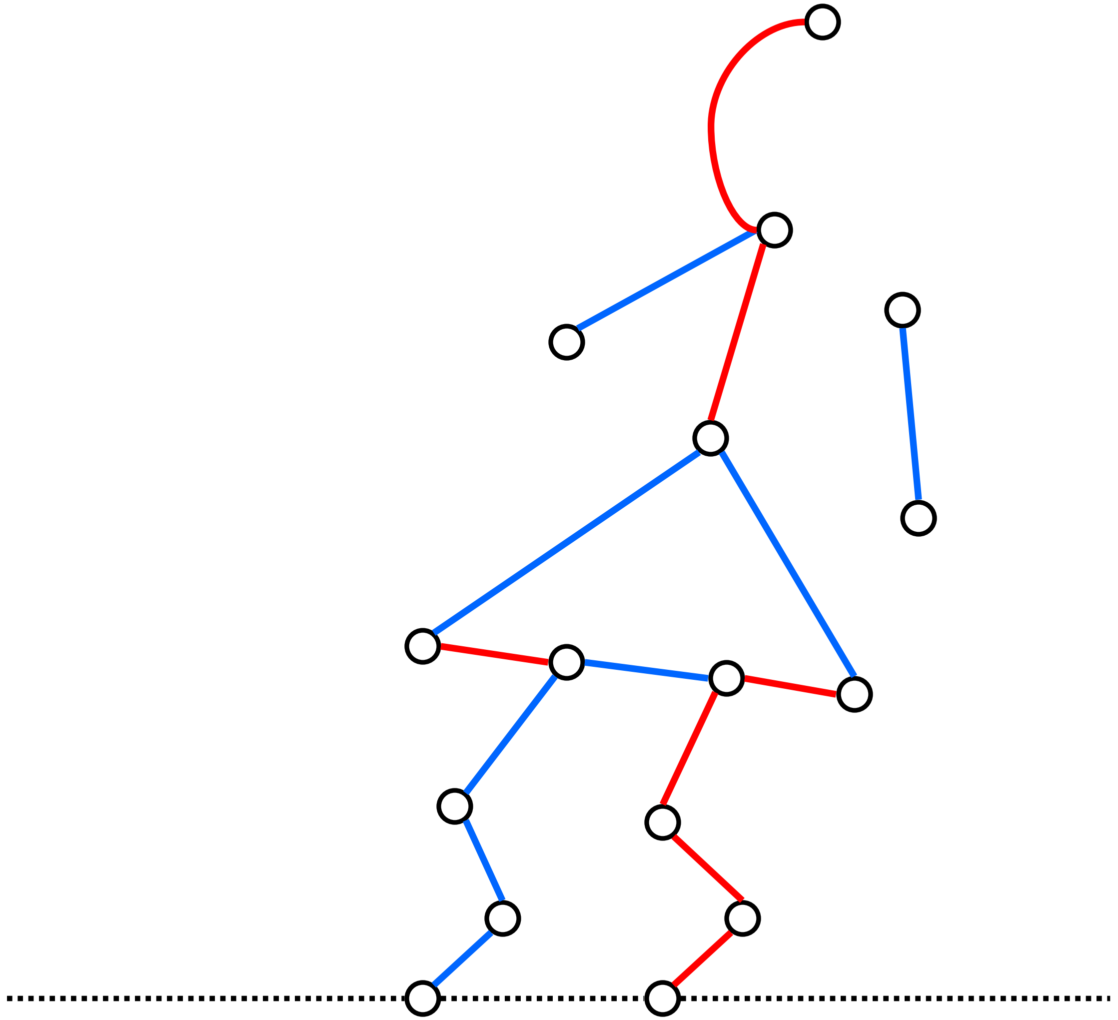
My turn

BLUE-RED-HACKENBUSH



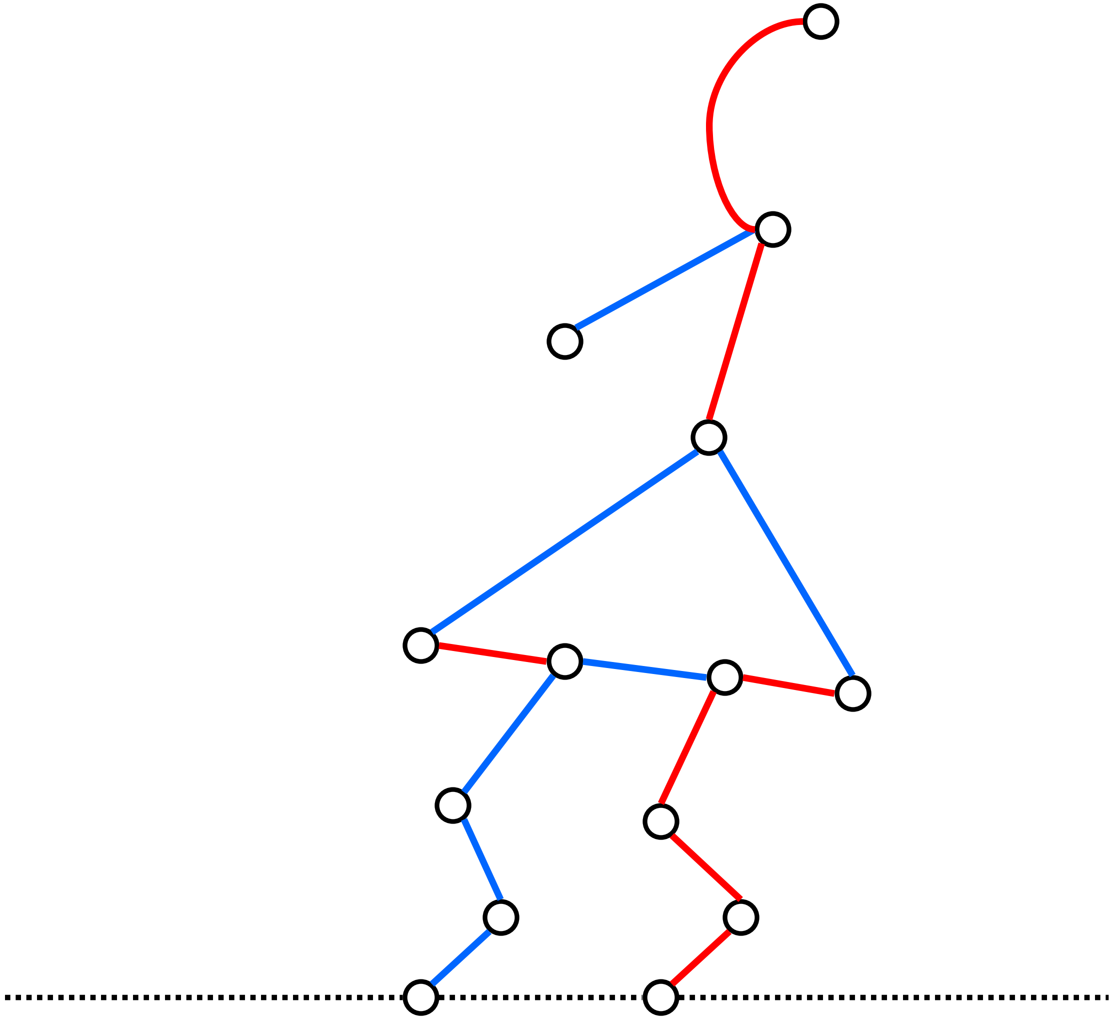
Your turn

BLUE-RED-HACKENBUSH



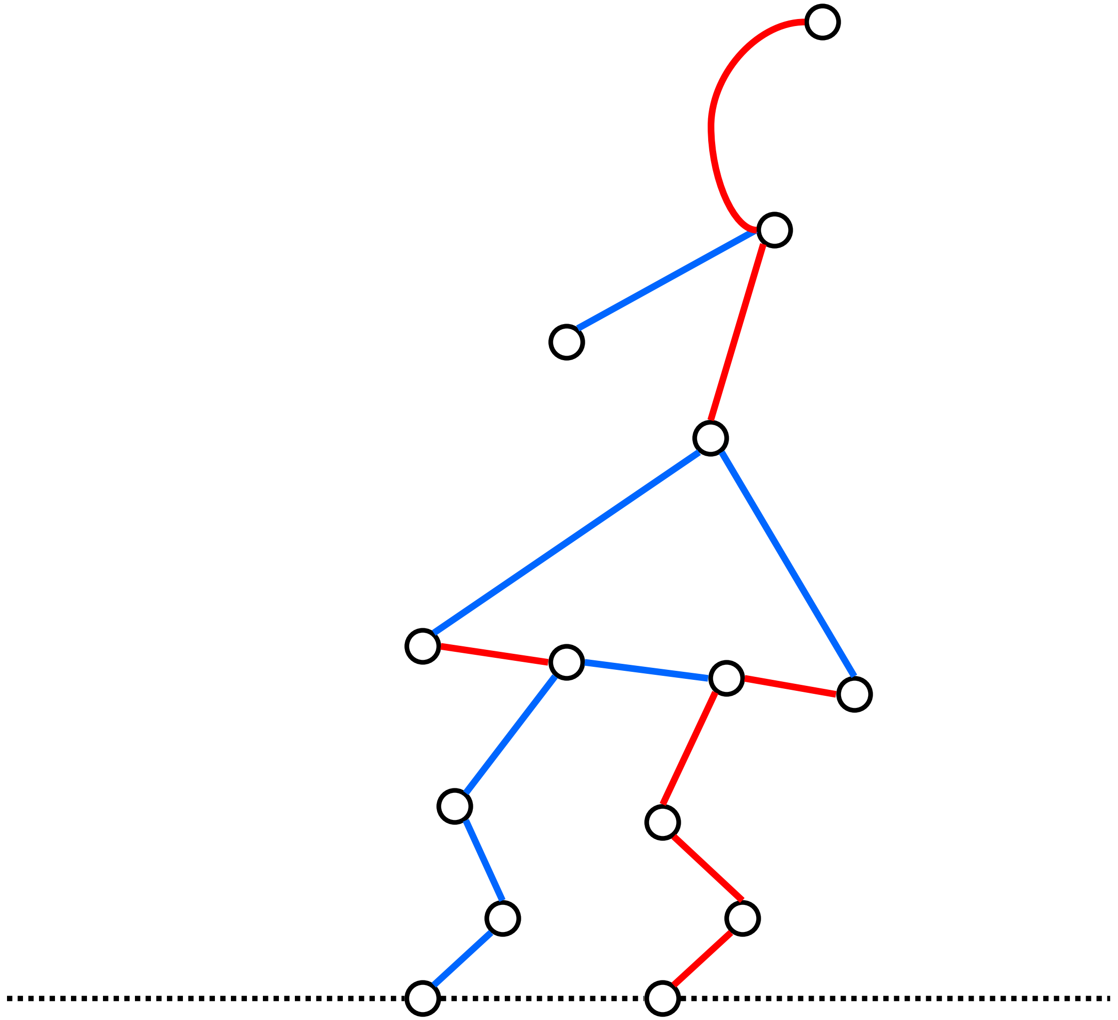
Your turn

BLUE-RED-HACKENBUSH



Your turn

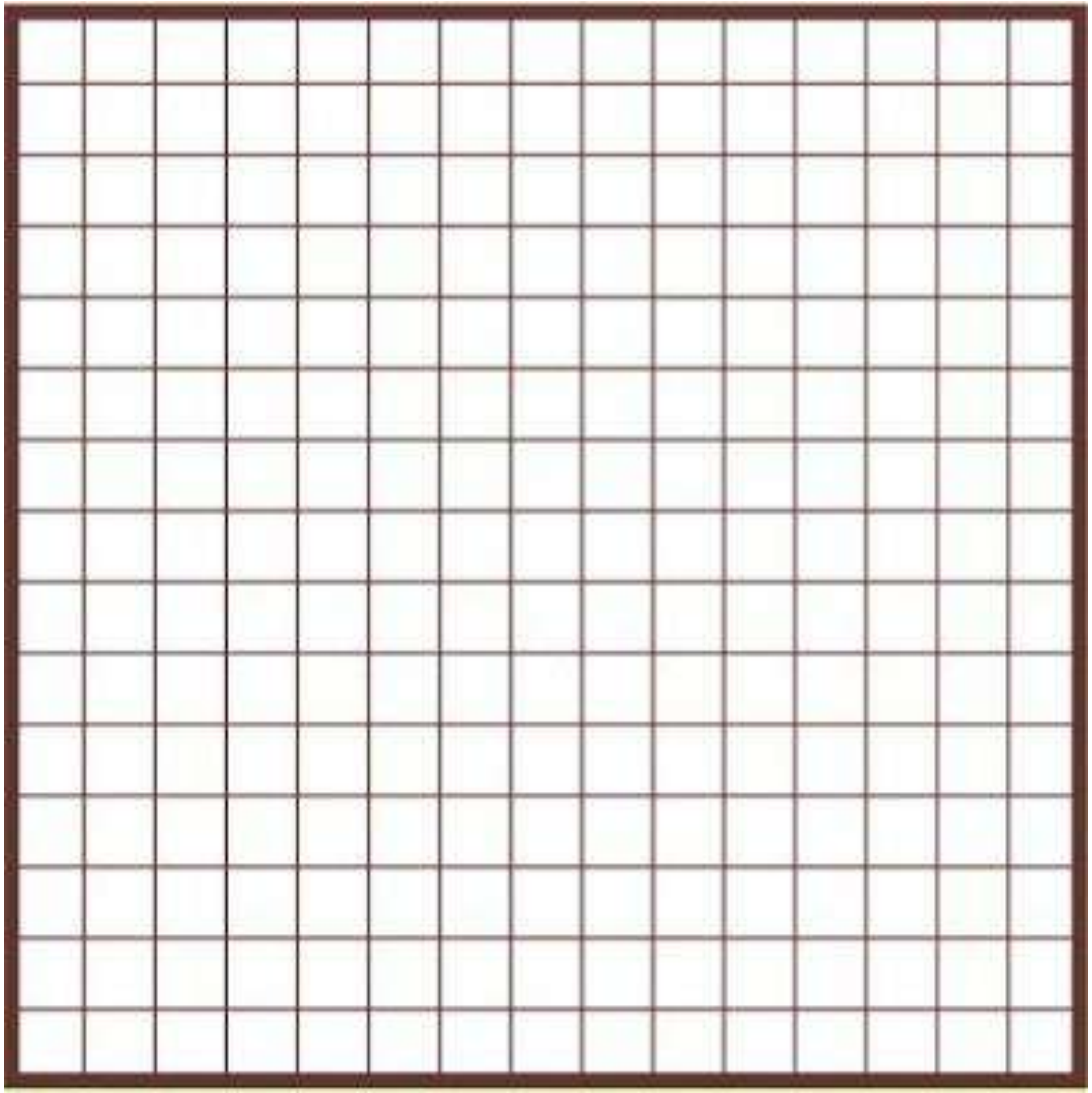
BLUE-RED-HACKENBUSH



And so on

DOMINEERING

DOMINEERING



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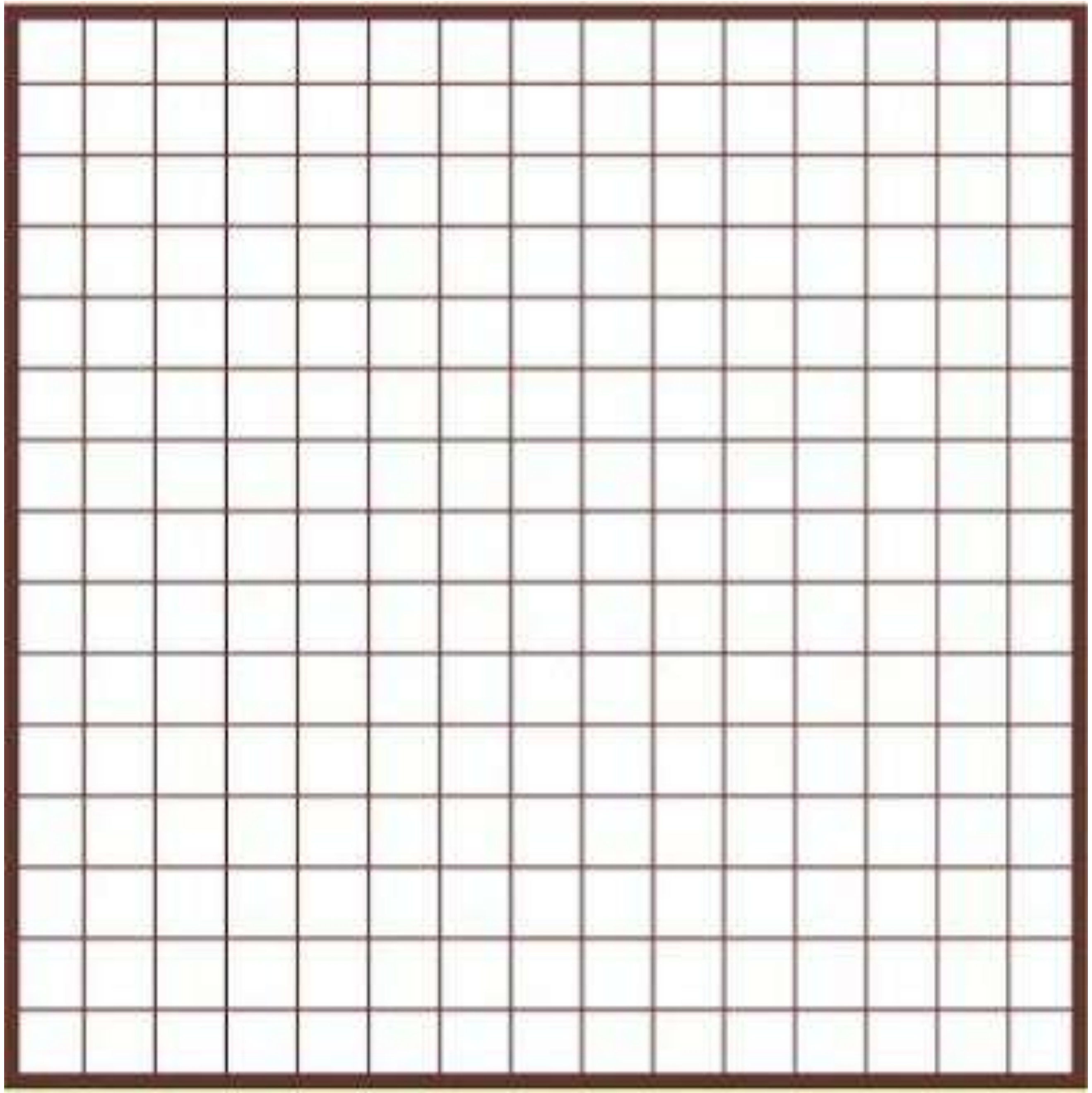

My turn

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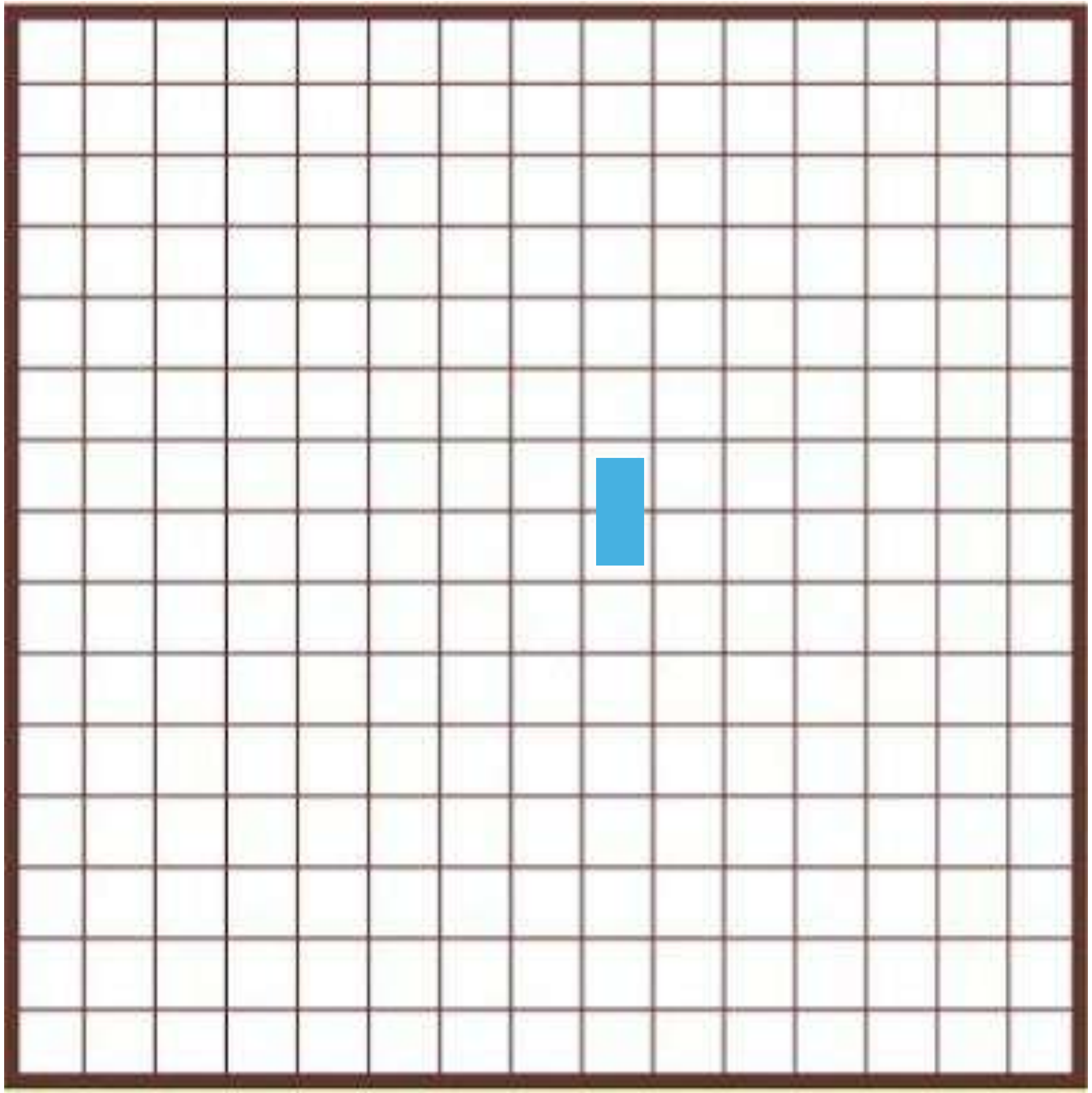

My turn

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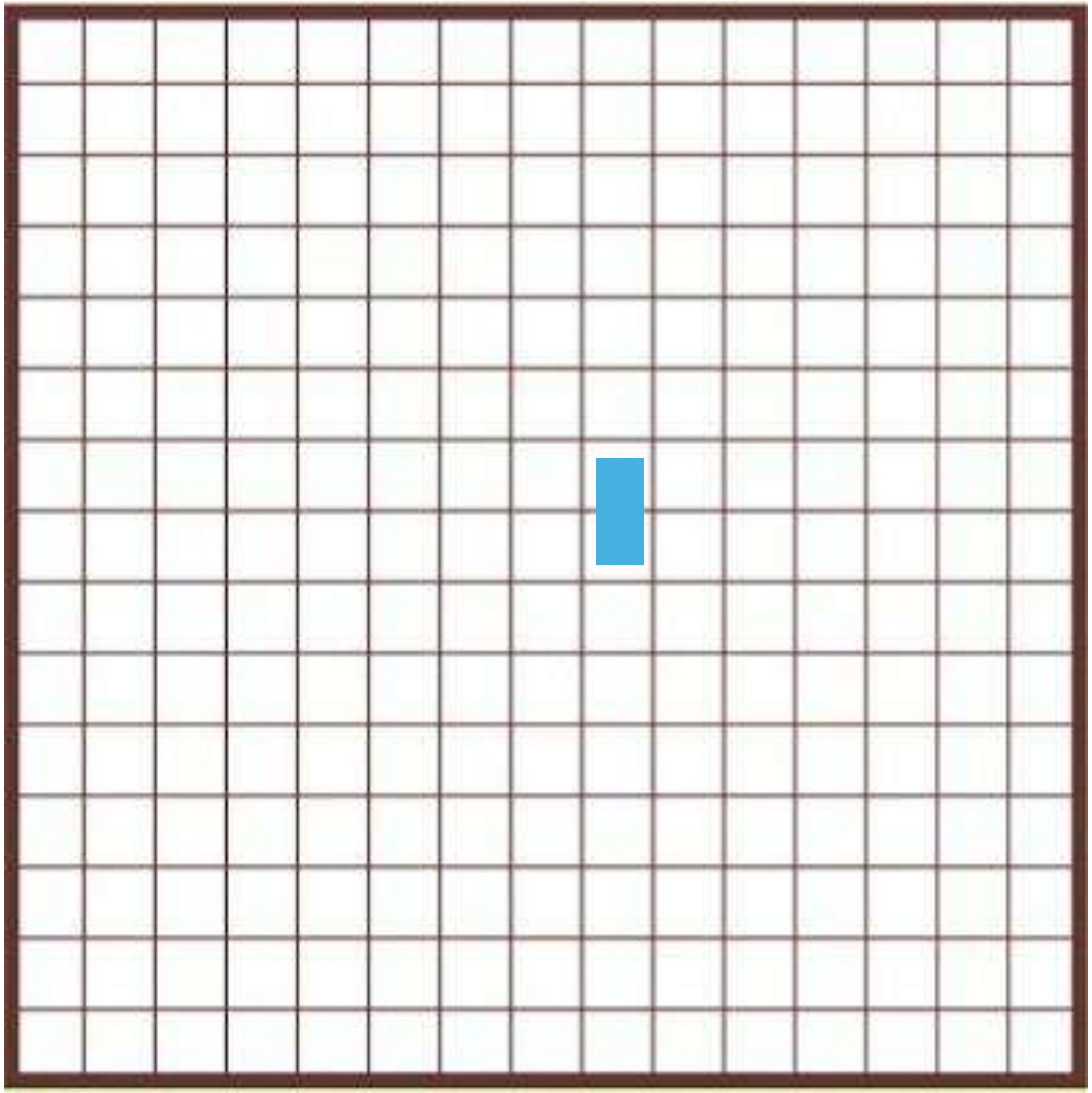

Your turn

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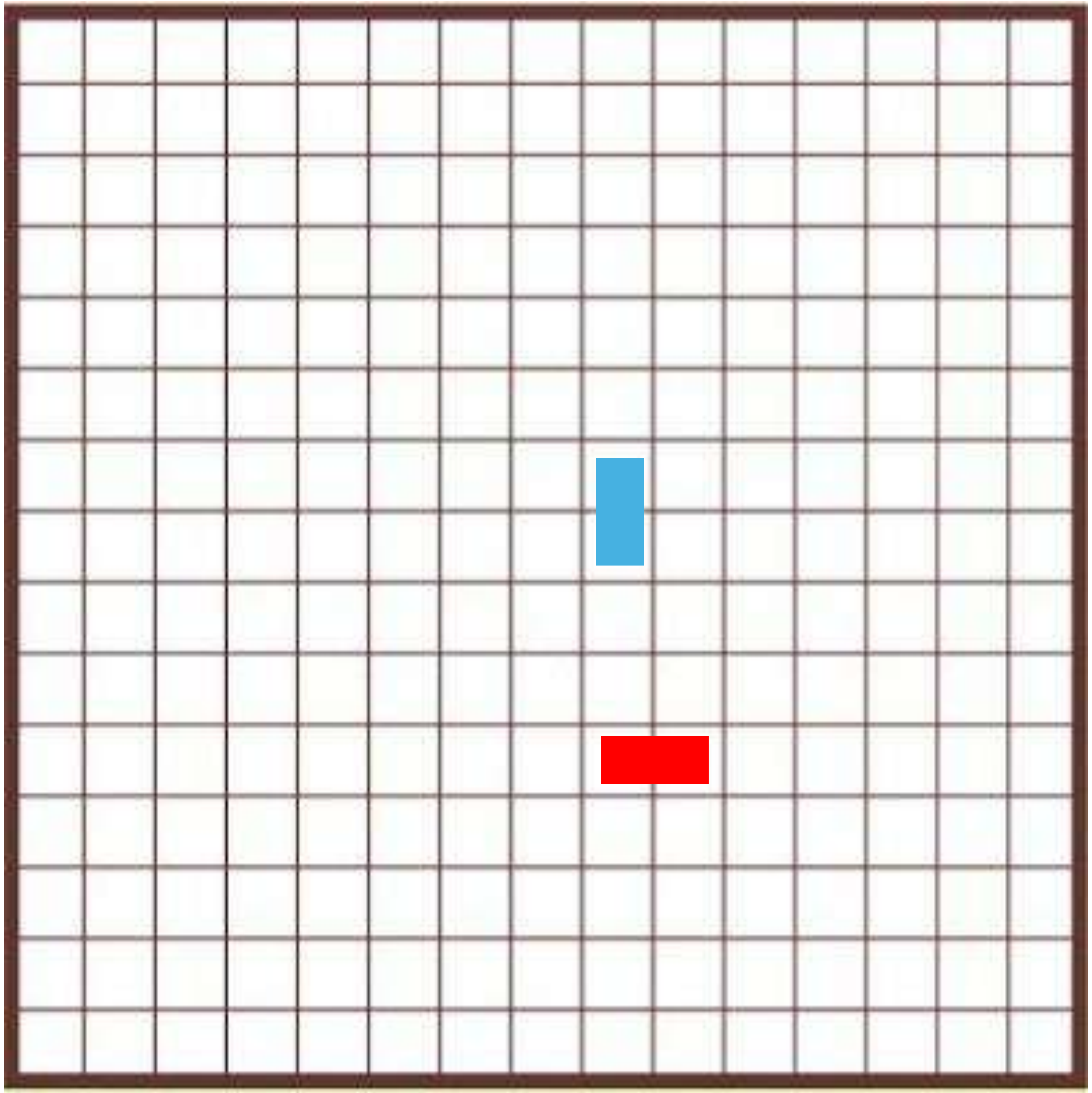

Your turn

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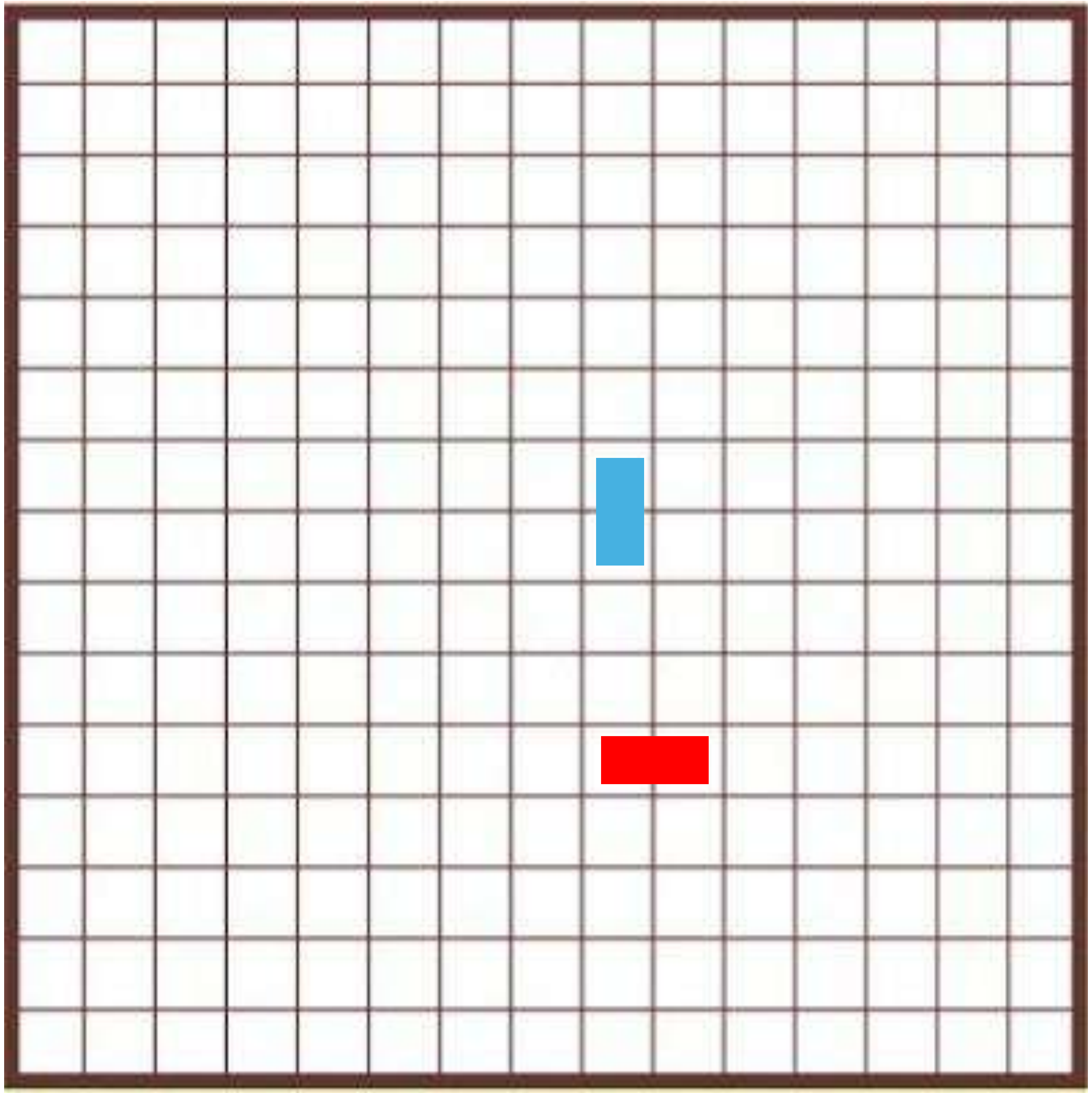

My turn

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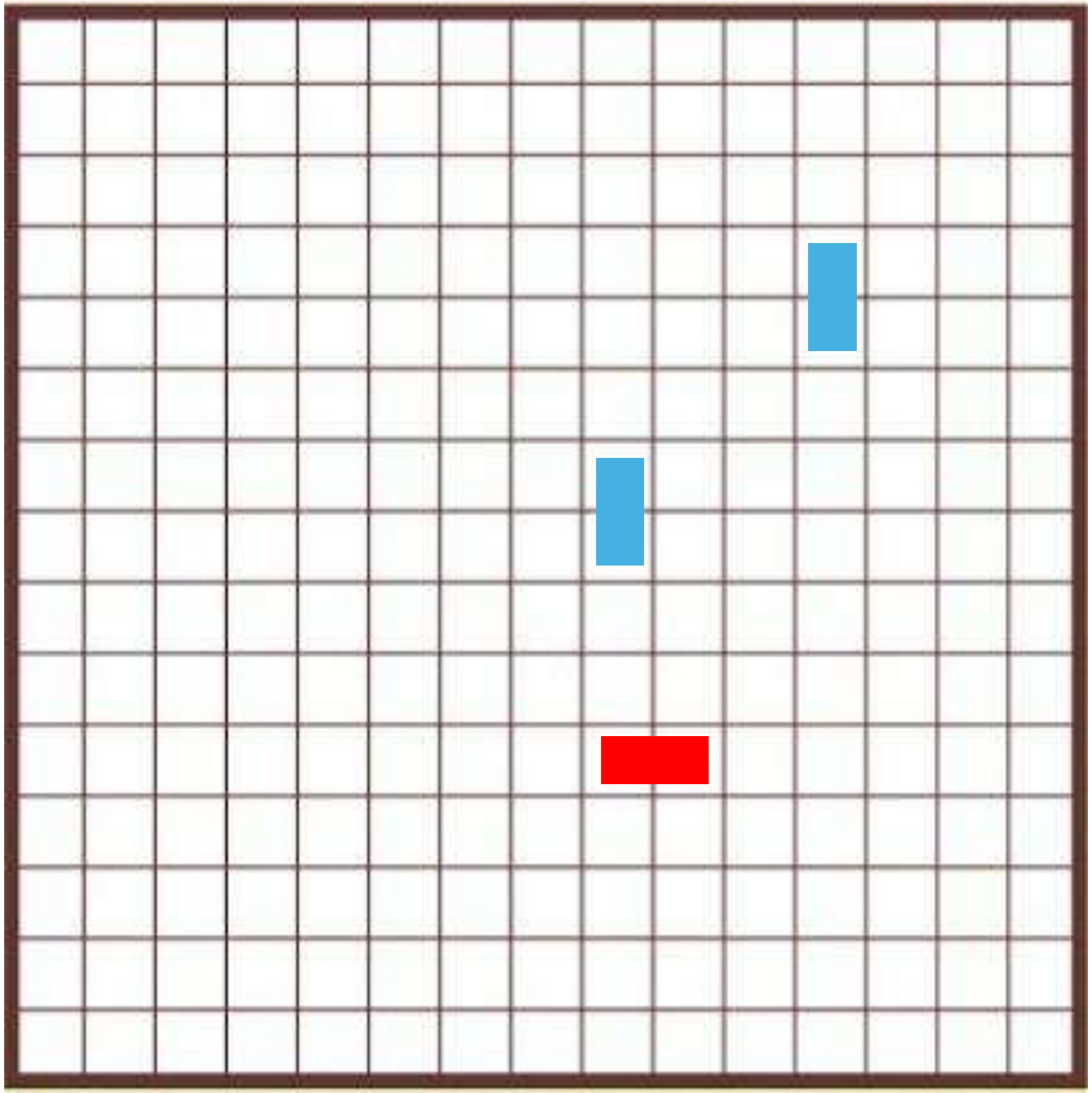

My turn

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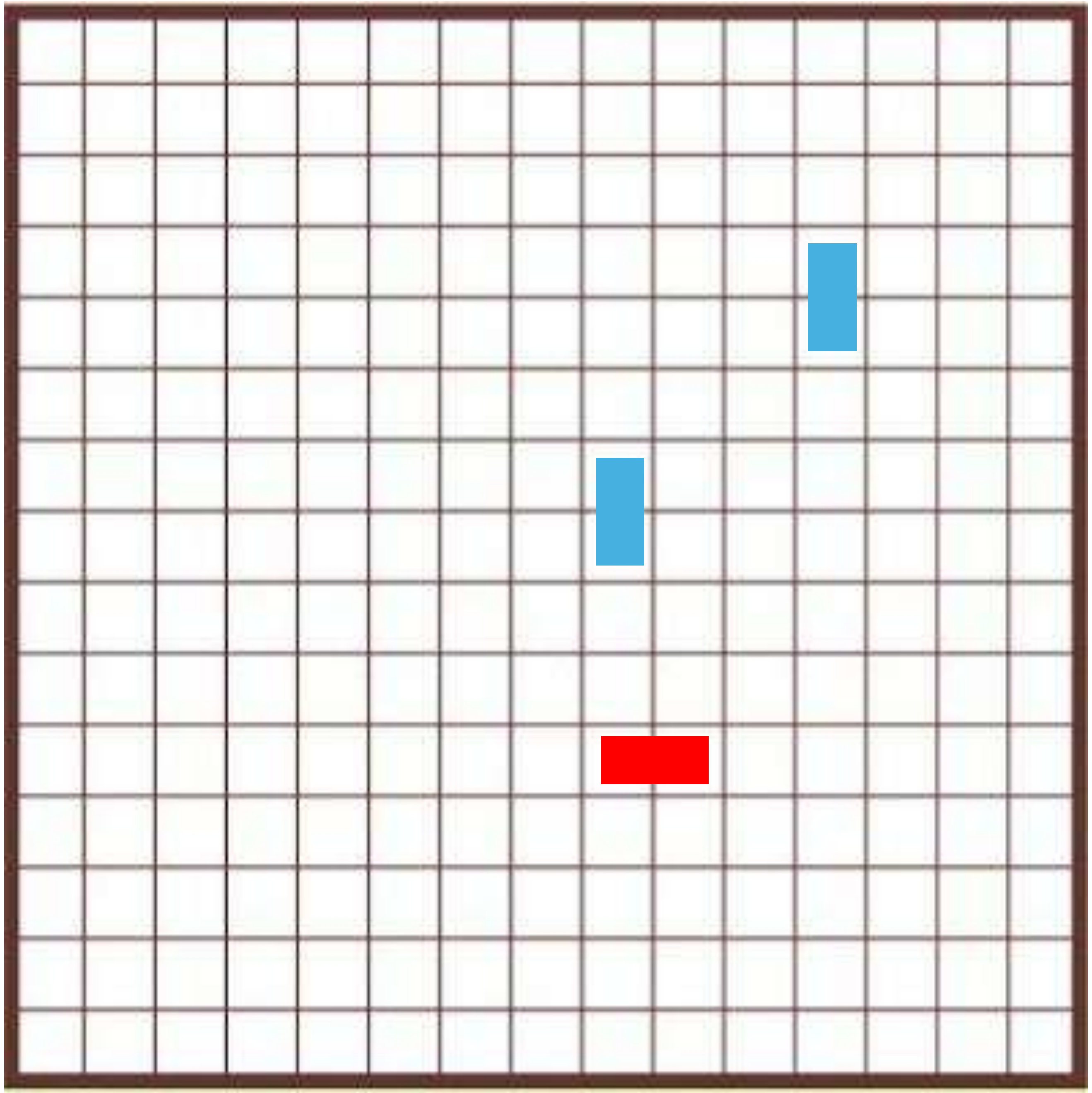

Your turn

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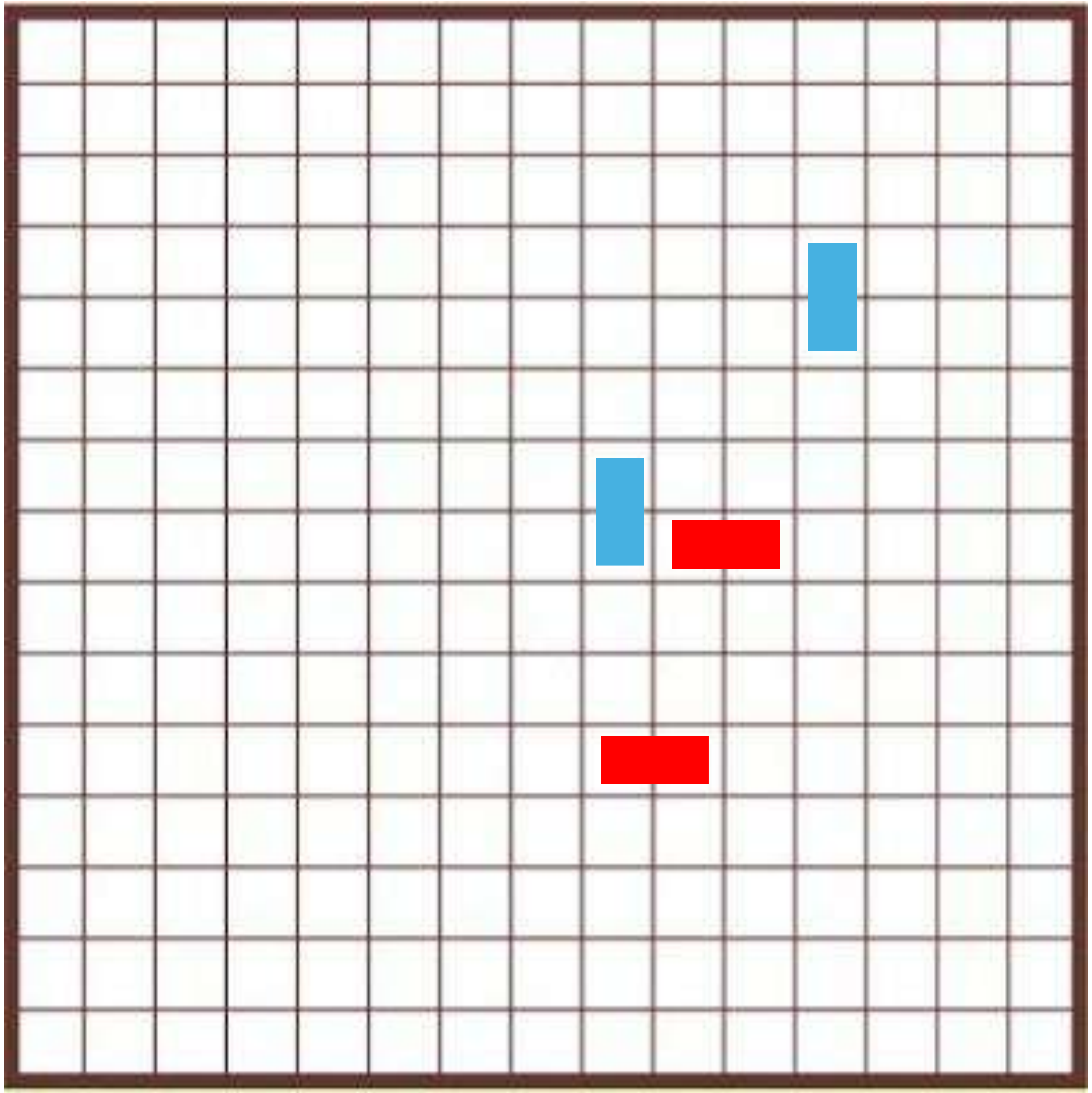
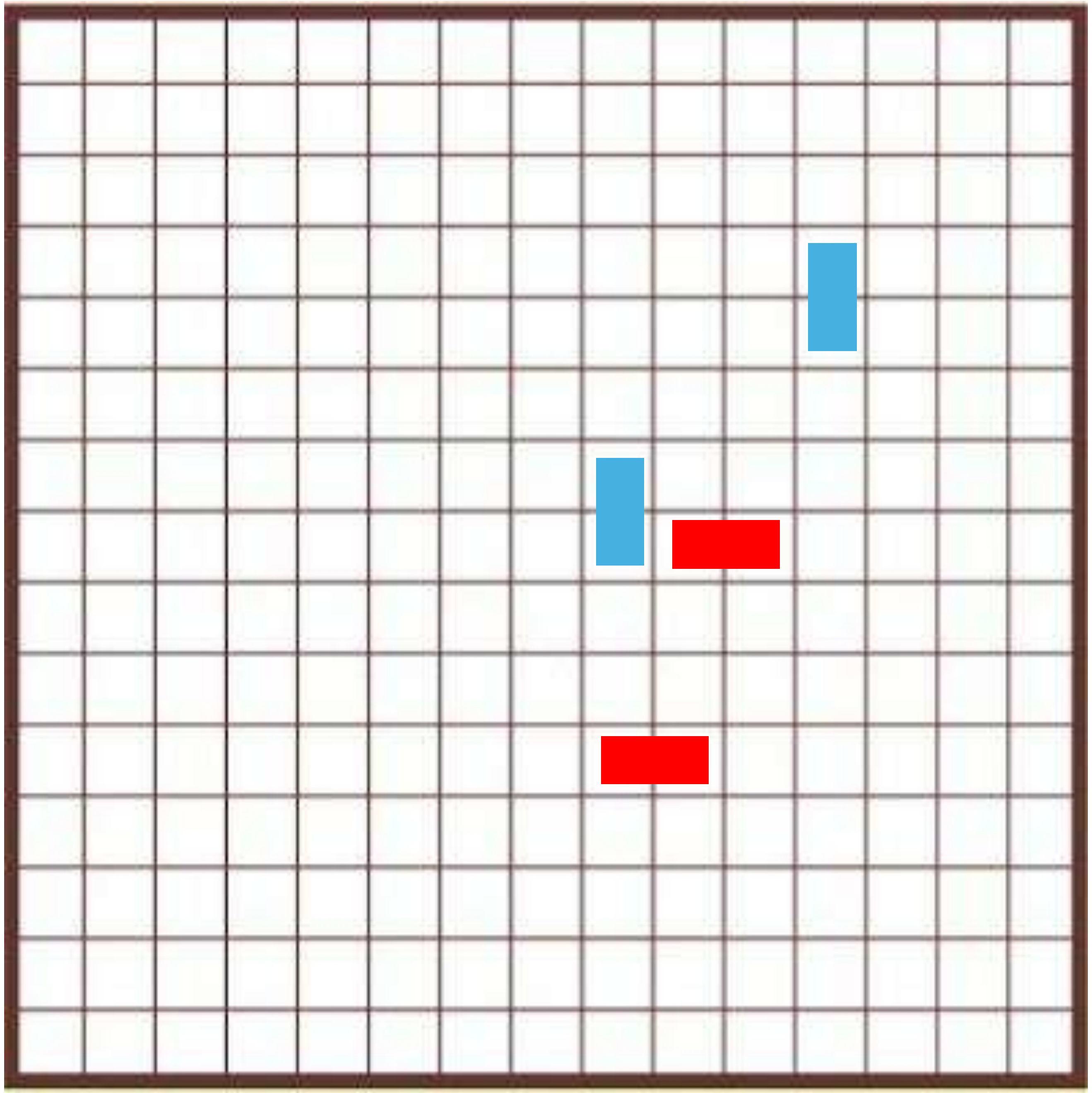


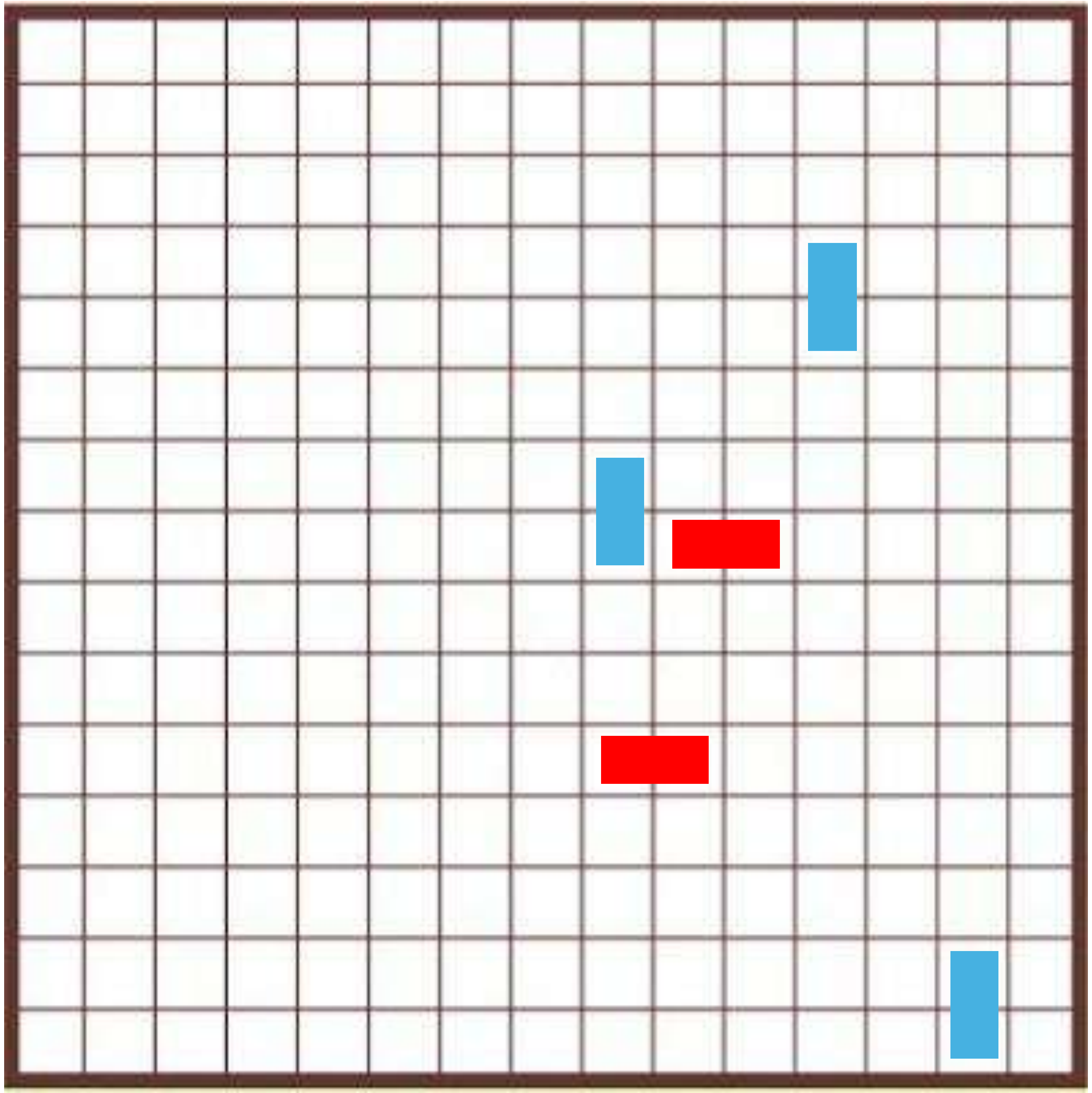
Your turn

DOMINEERING



My turn

DOMINEERING



My turn

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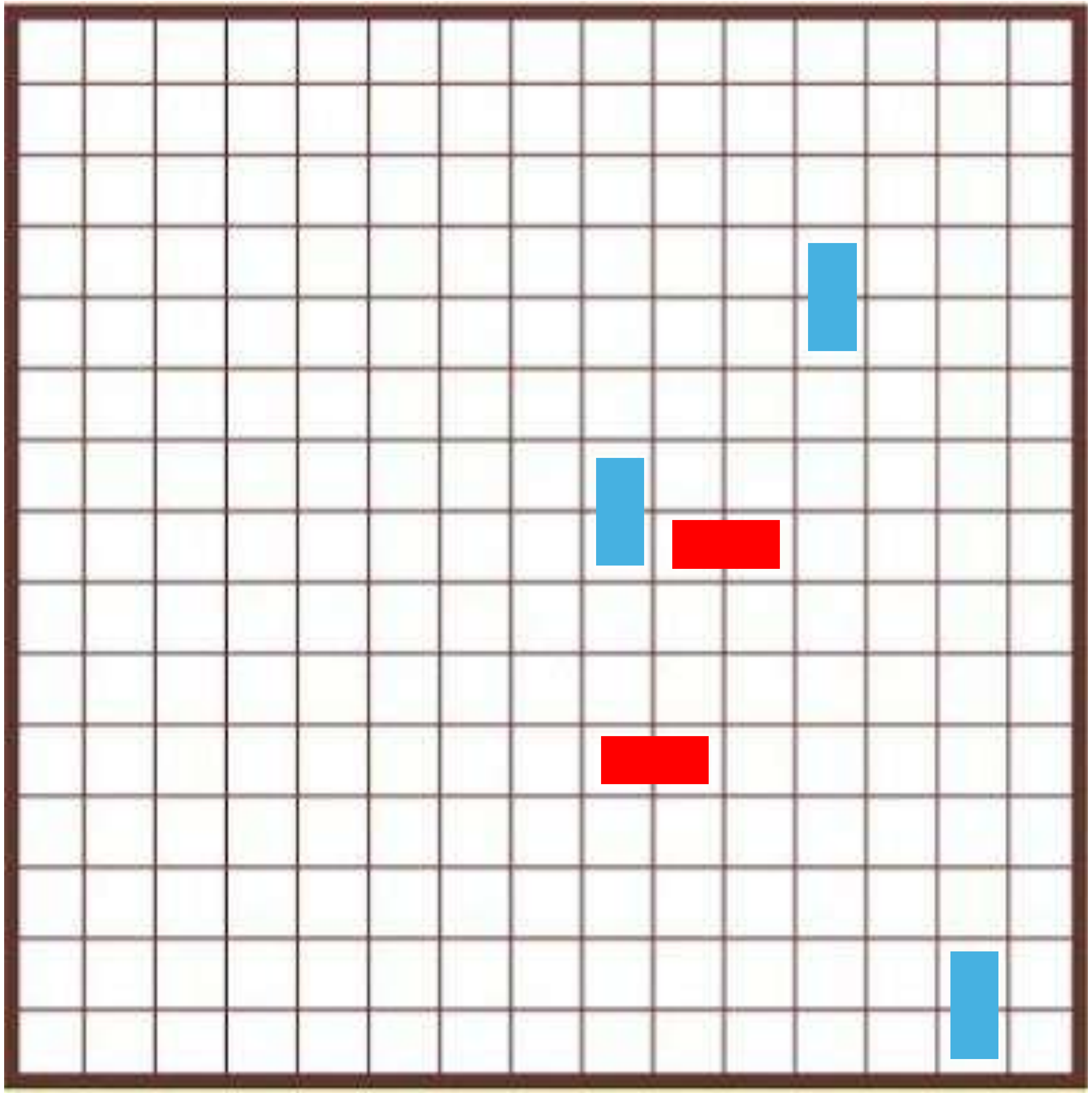


And so on…

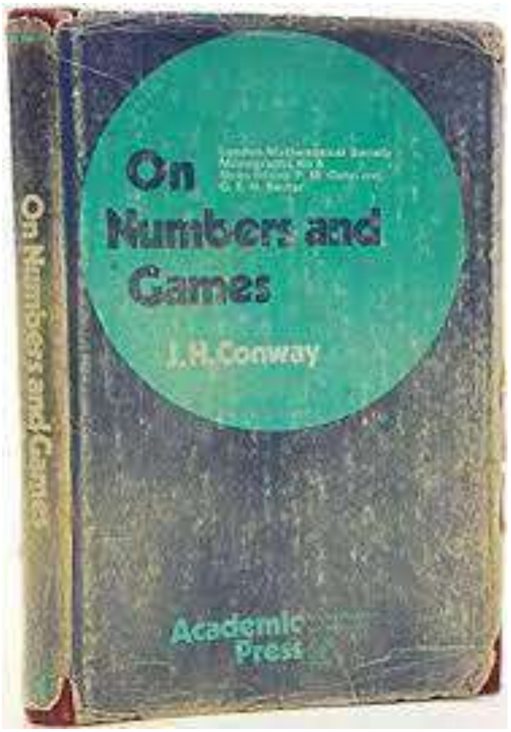
Part II: Partizan Games

I.1: Some famous games
I.2: Contribution of John Conway (1970’s)

John Horton Conway

Conway in June 2005

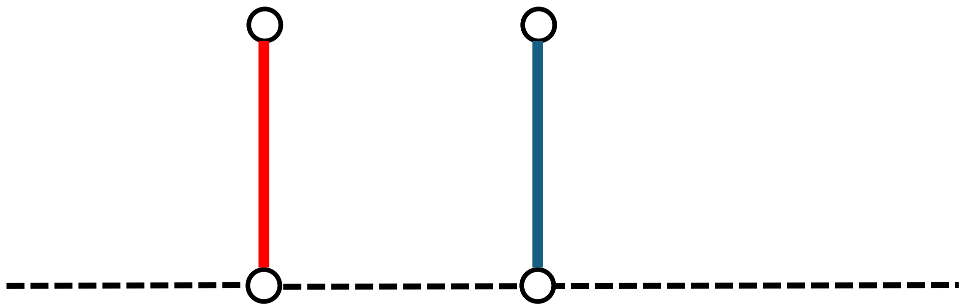
Born	26 December 1937 Liverpool, England
Died	11 April 2020 (aged 82) New Brunswick, New Jersey, U.S.



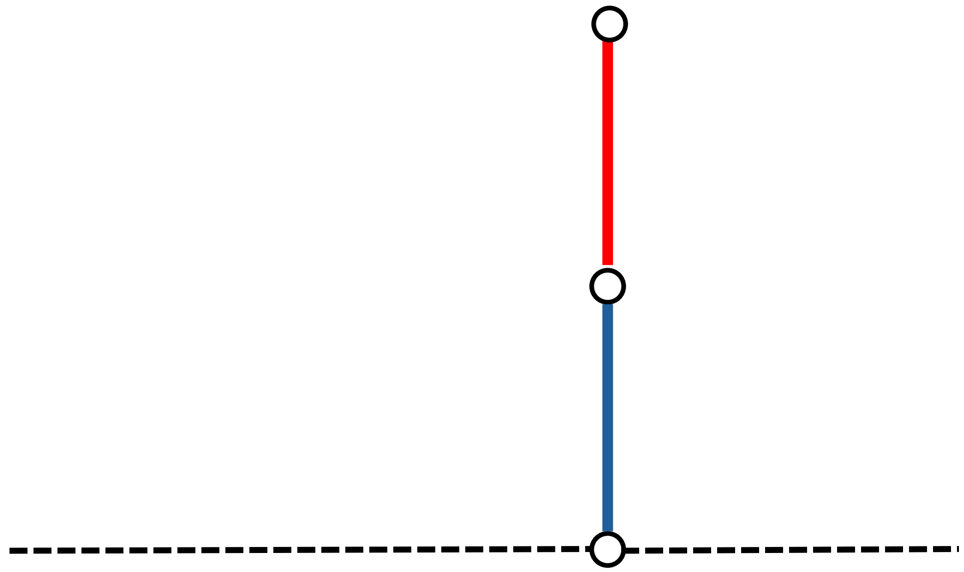
$$\{\ |\ \} = 0$$

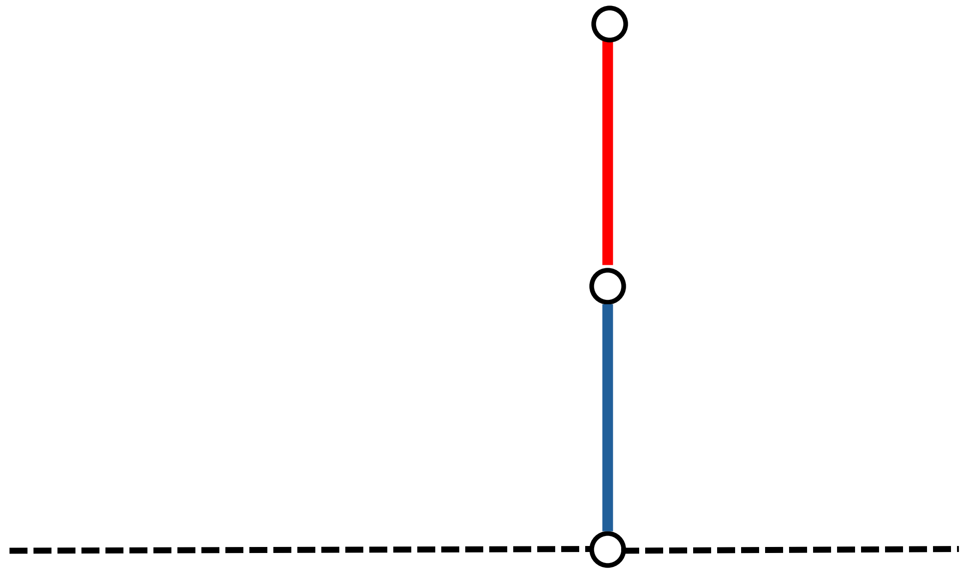
$\{0 \mid \} = 1$

$$\{\ |\ 0\} = -1$$



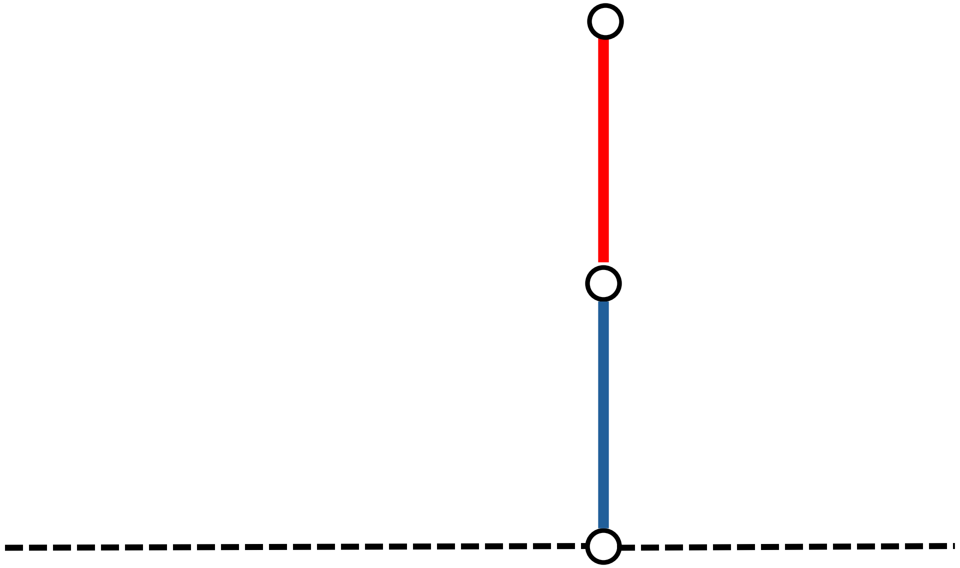
$$-1 + 1 = 0$$



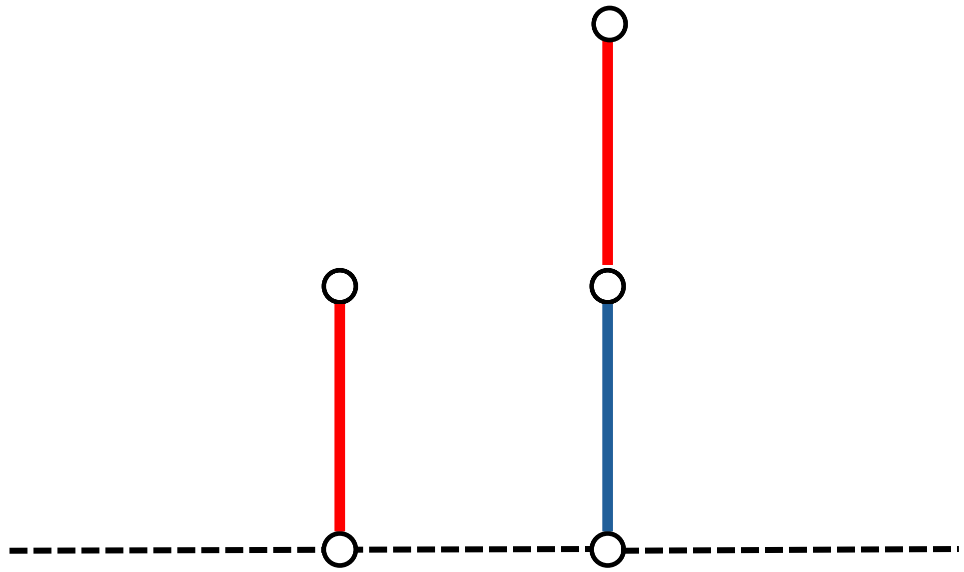


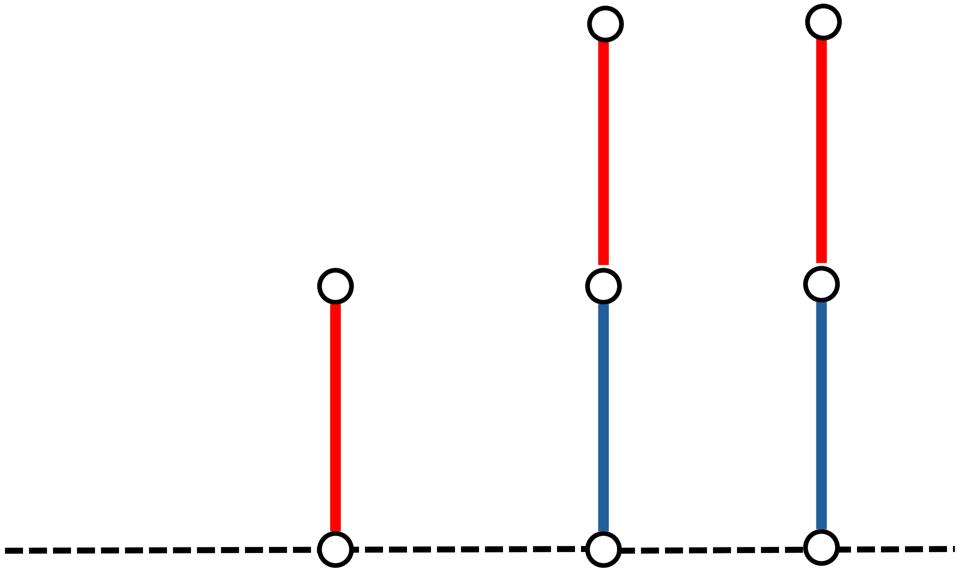
$\{0|1\}$

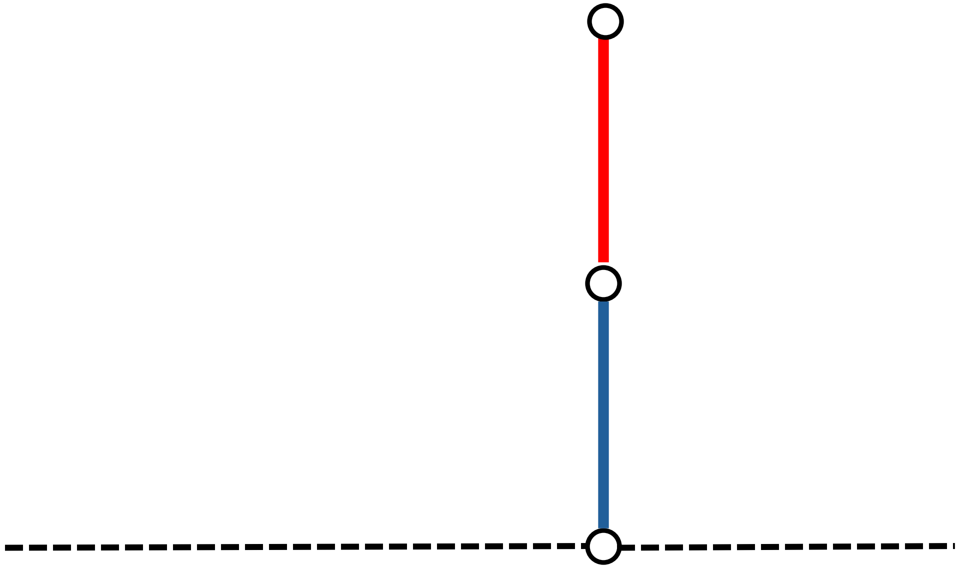
$$\{0 \mid \} = 1$$



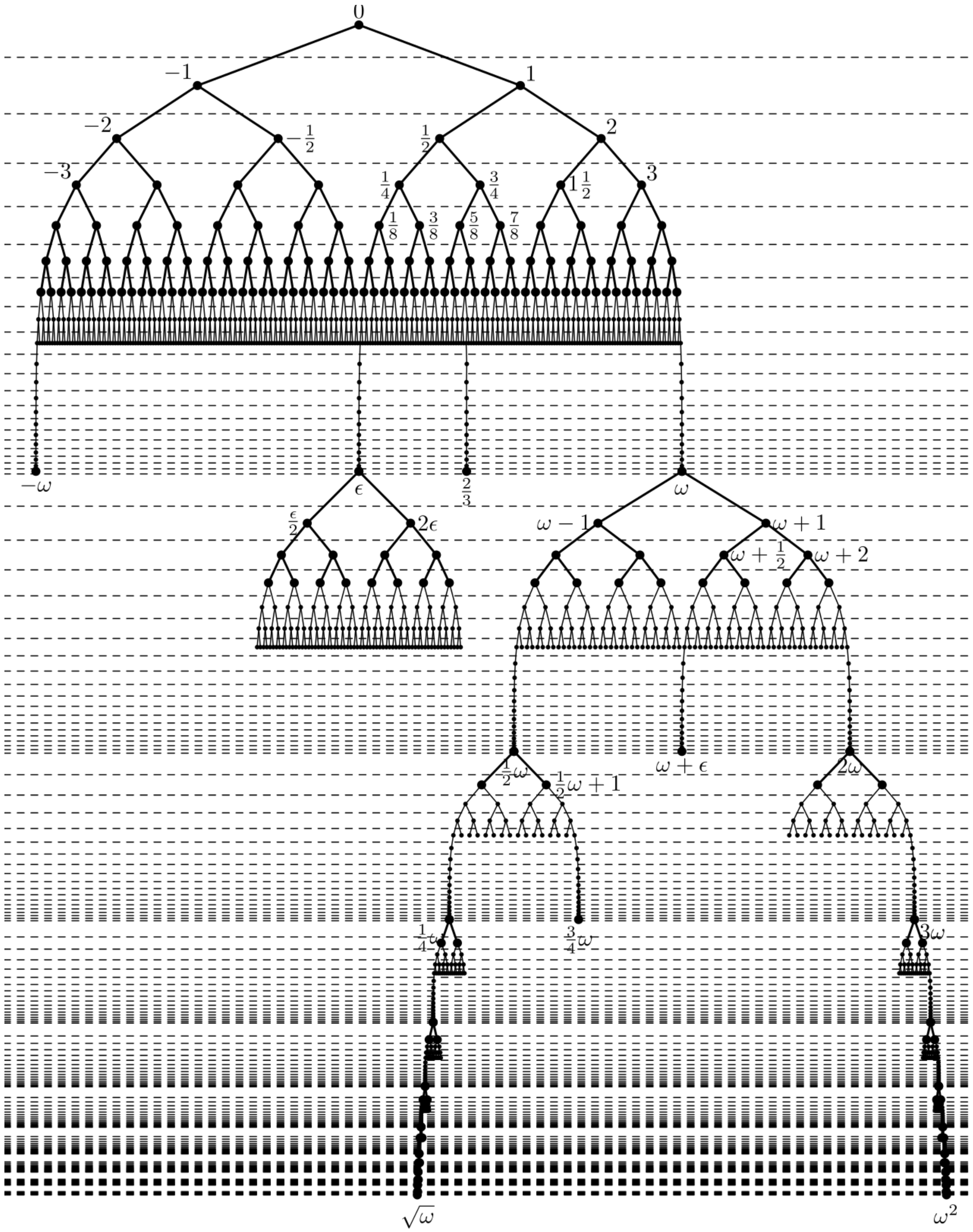
$\{0|1\}$

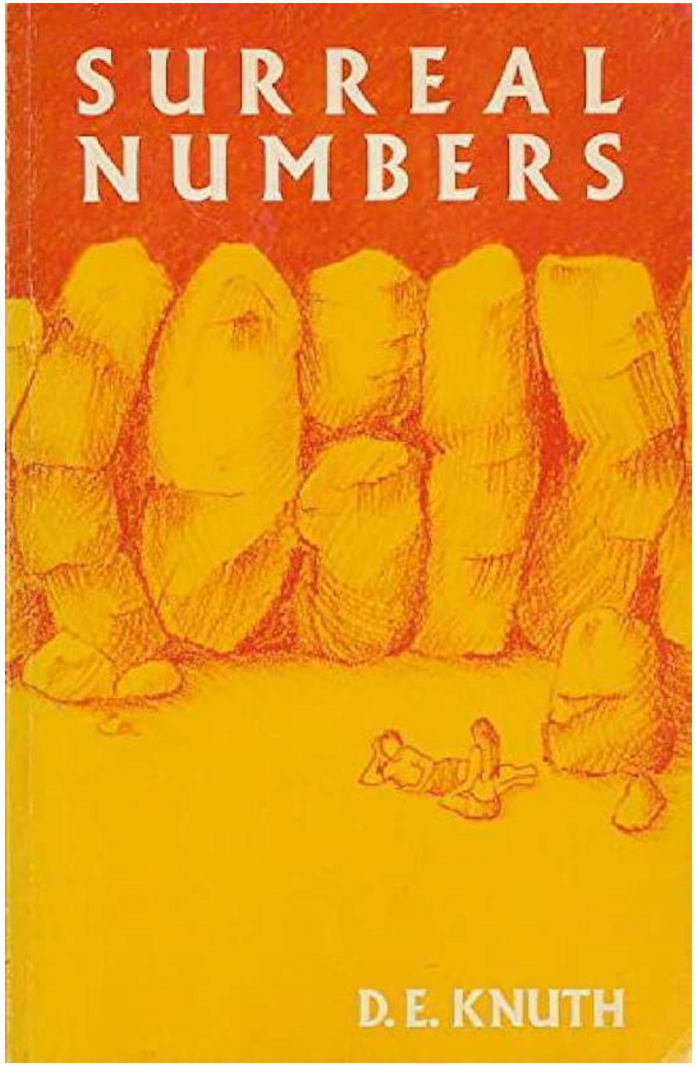






$$\{0|1\} = \frac{1}{2}$$





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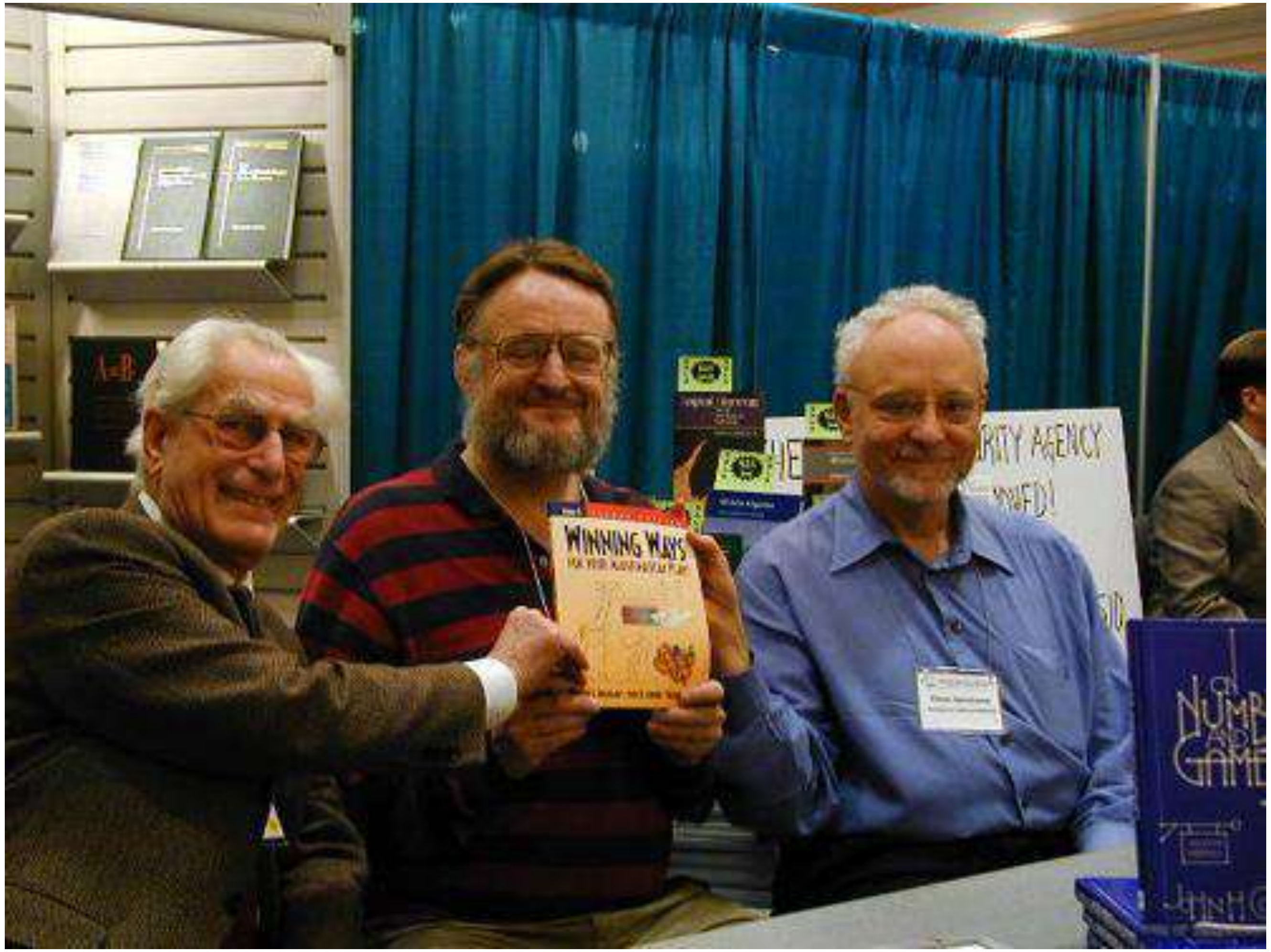
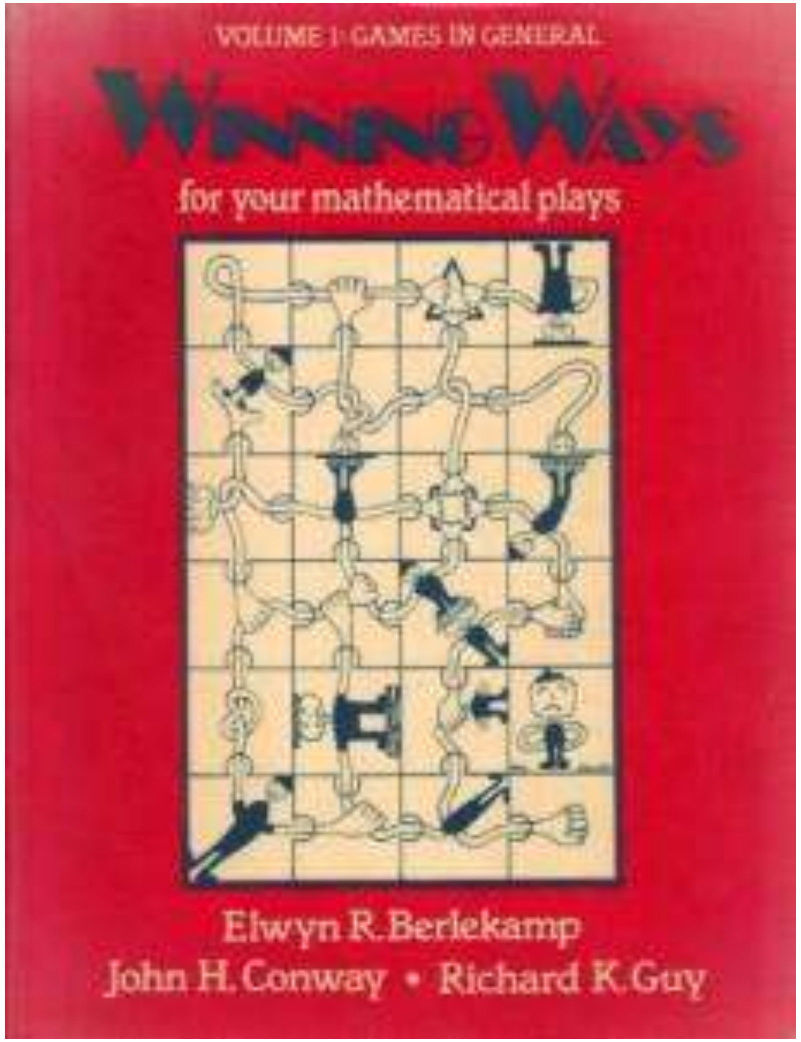
Part II: Partizan Games

I.1: Some famous games
I.2: Contribution of John Conway (1970’s)
I.3: Elwyn Berlekamp, John Conway, and Richard Guy: the birth of a theory (1982)


WINNING WAYS


VOLUME 1: GAMES IN GENERAL
WINNING WAYS
for your mathematical plays
Elwyn R. Berlekamp
John H. Conway • Richard K. Guy

Part II: Partizan Games

I.1: Some famous games
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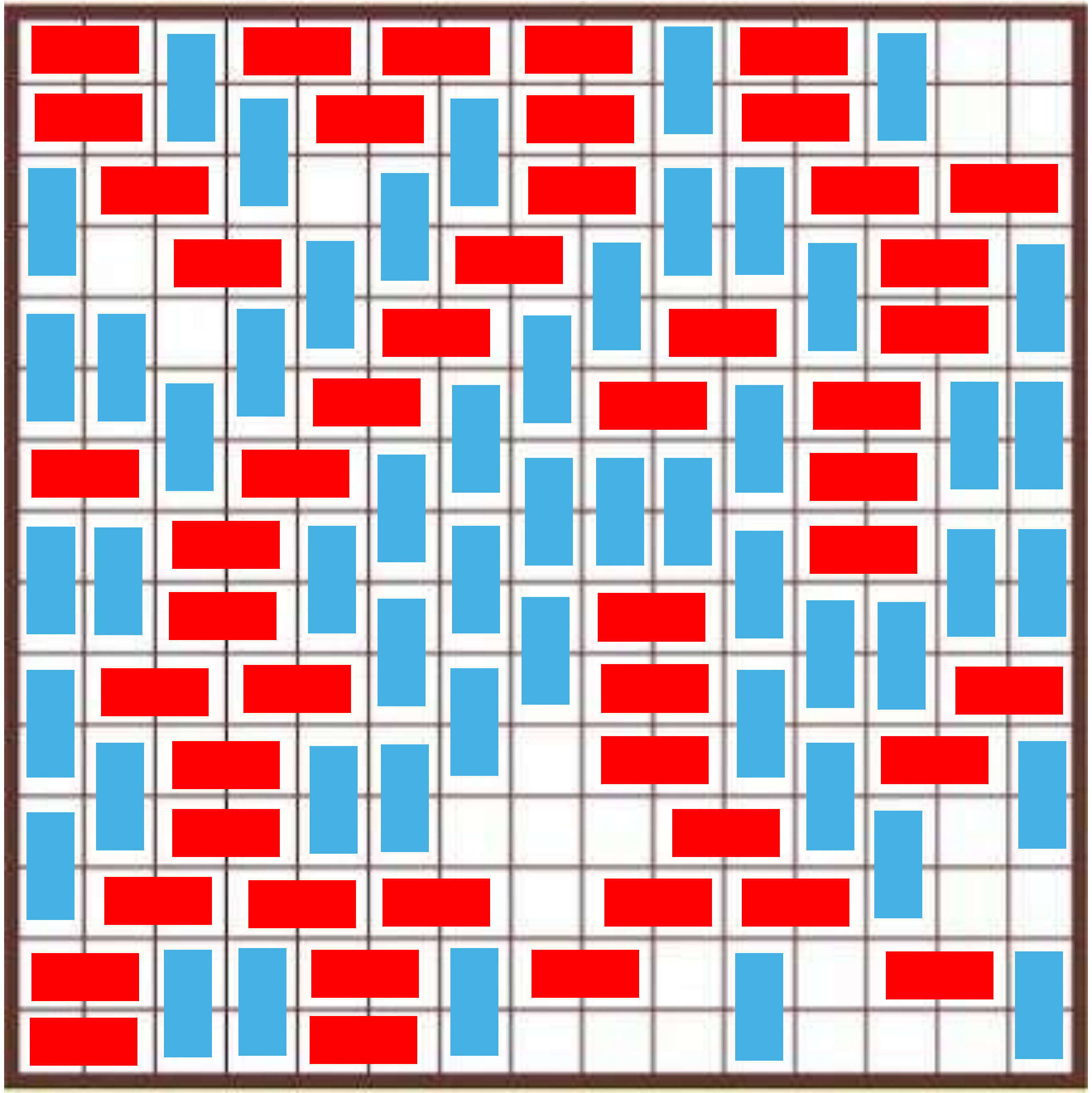
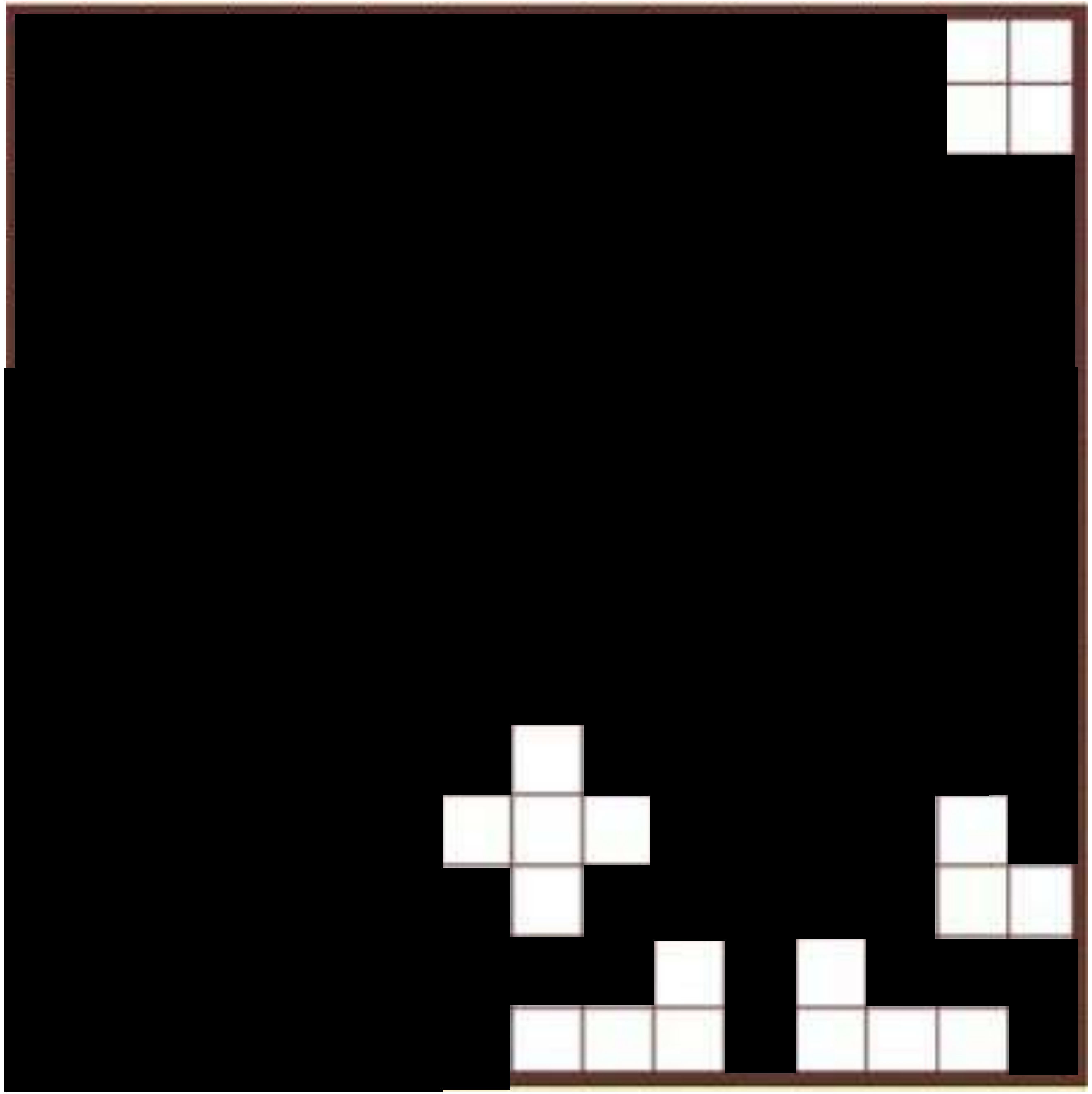
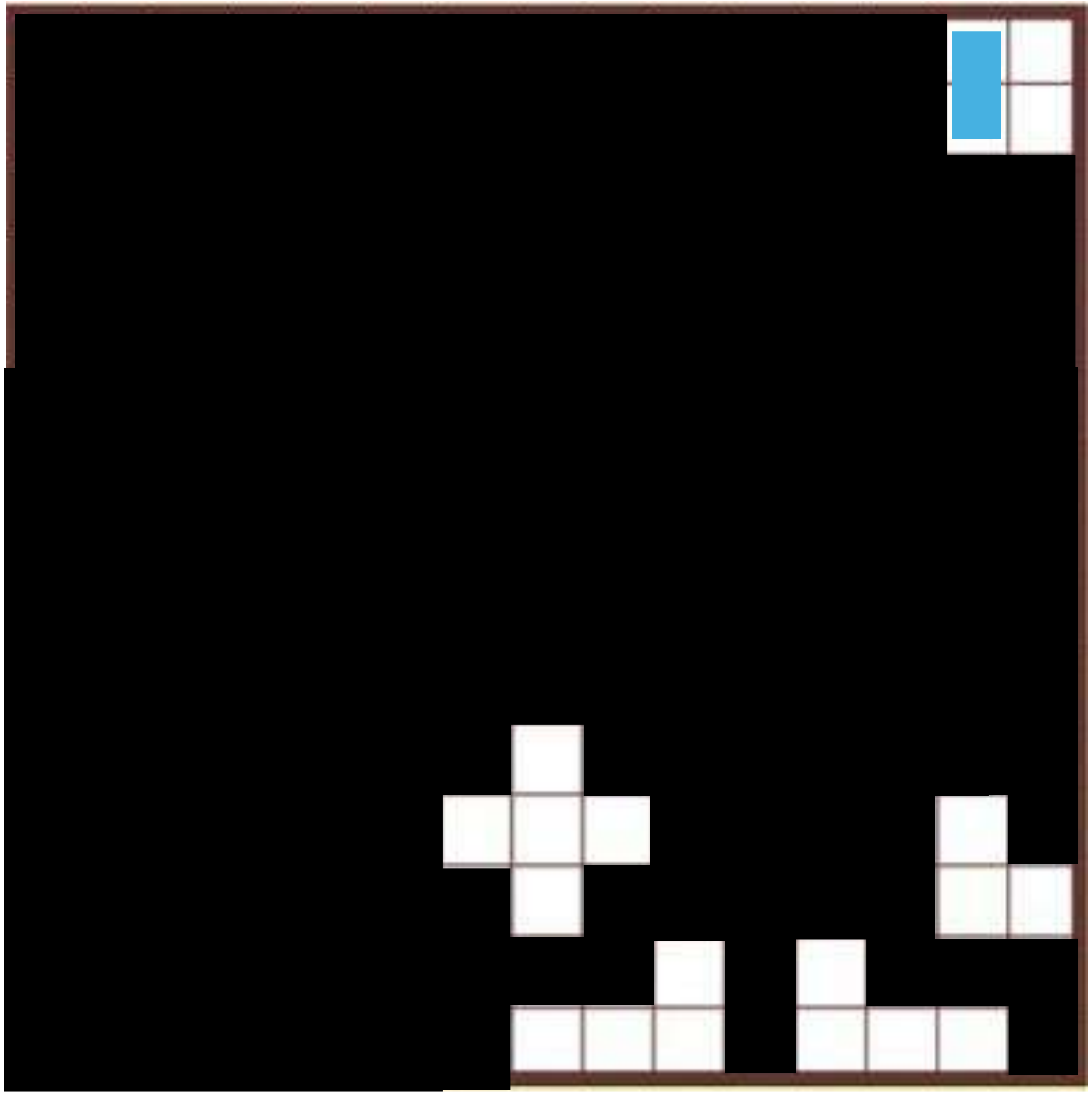
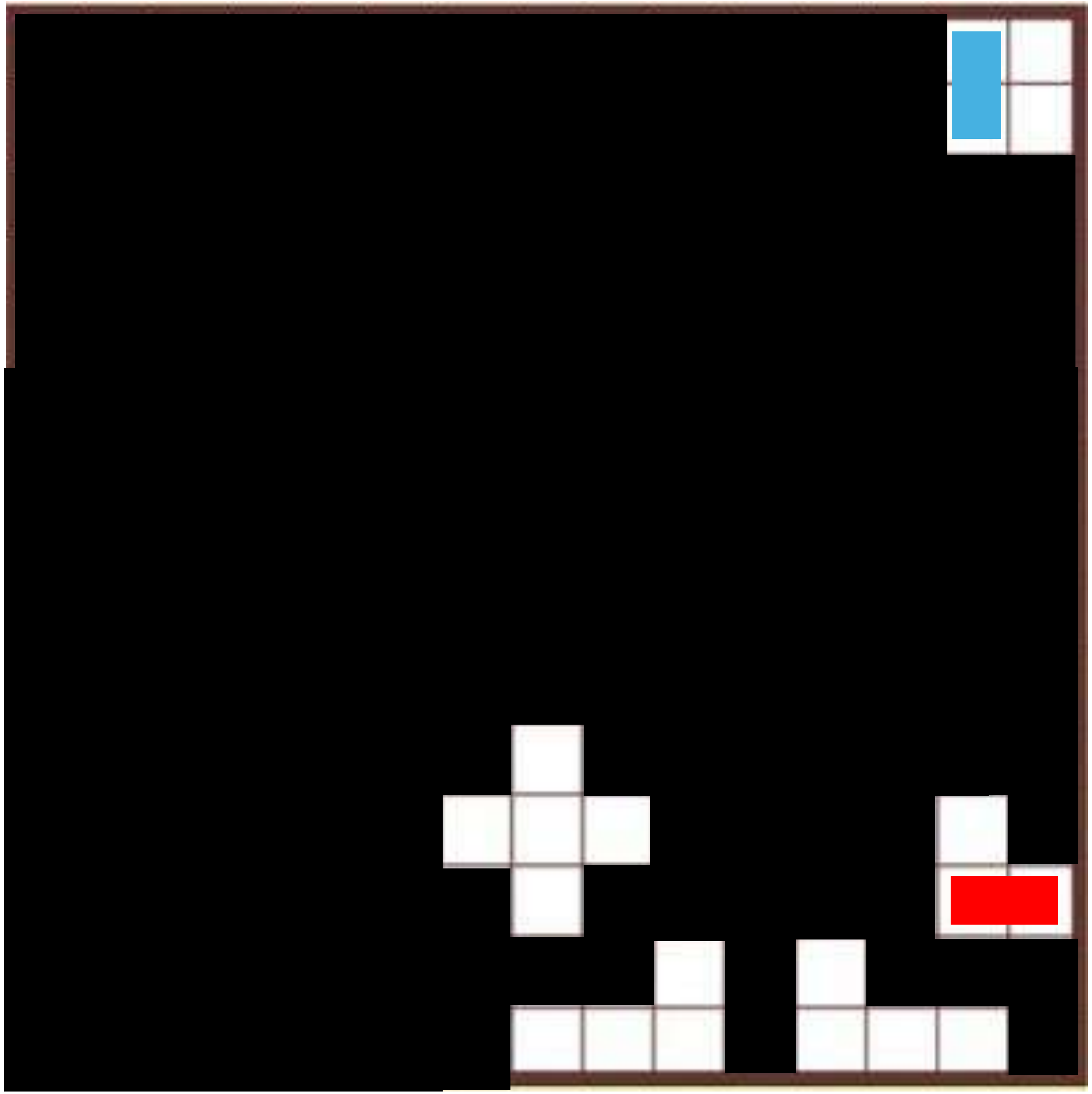
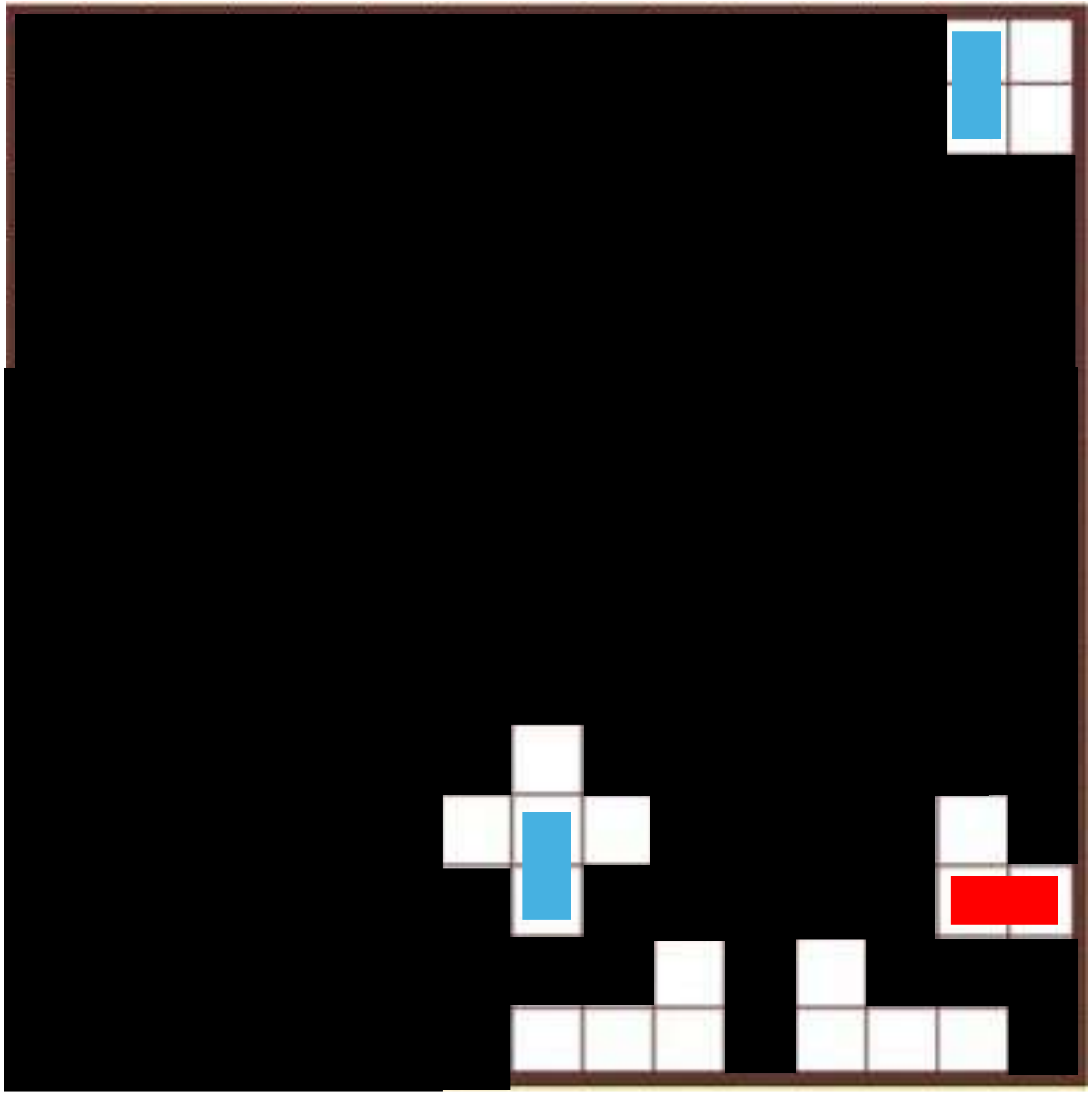
I.3: Elwyn Berlekamp, John Conway, and Richard Guy: the birth of a theory (1982)
I.4: How can you apply the theory?

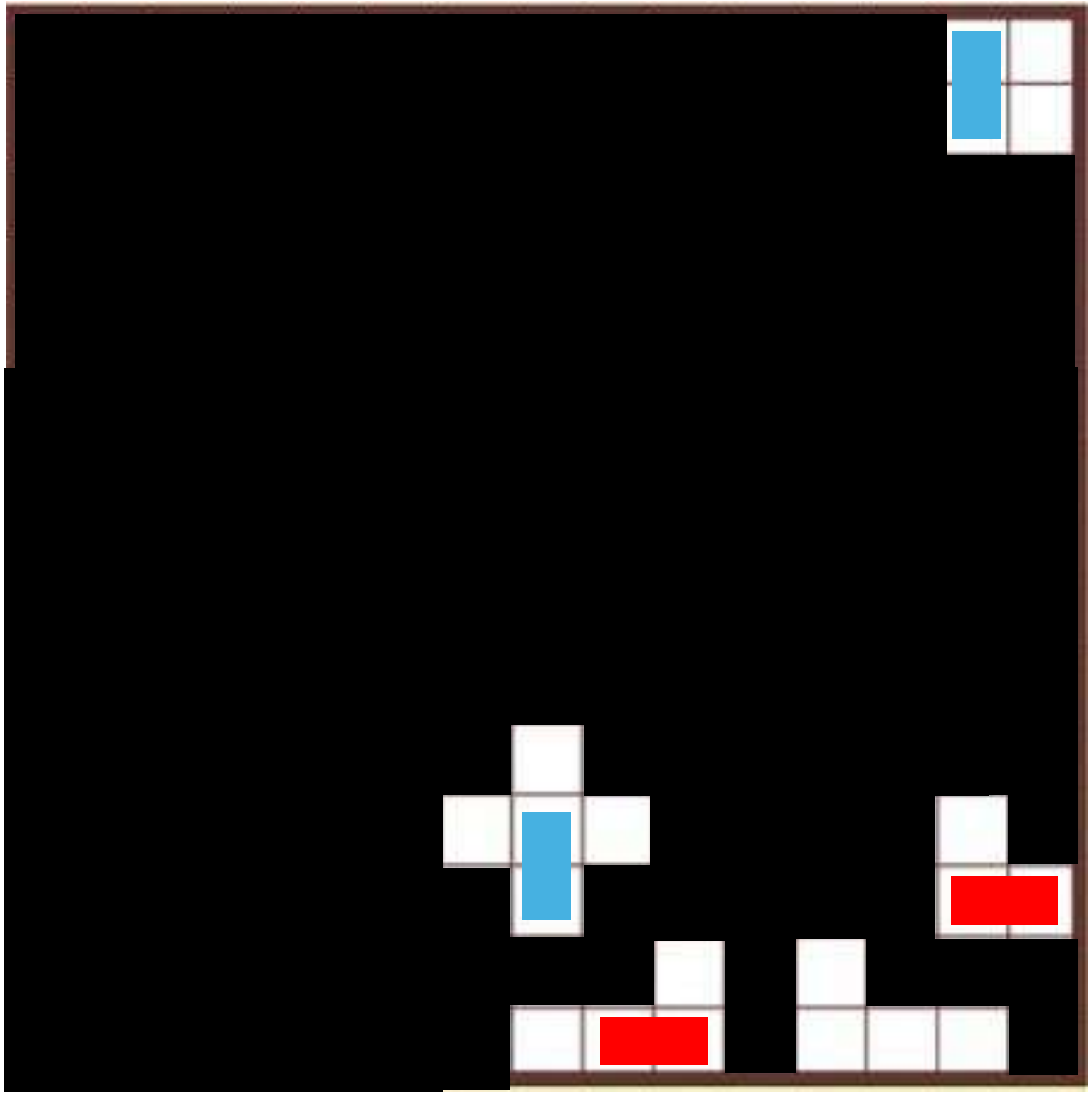


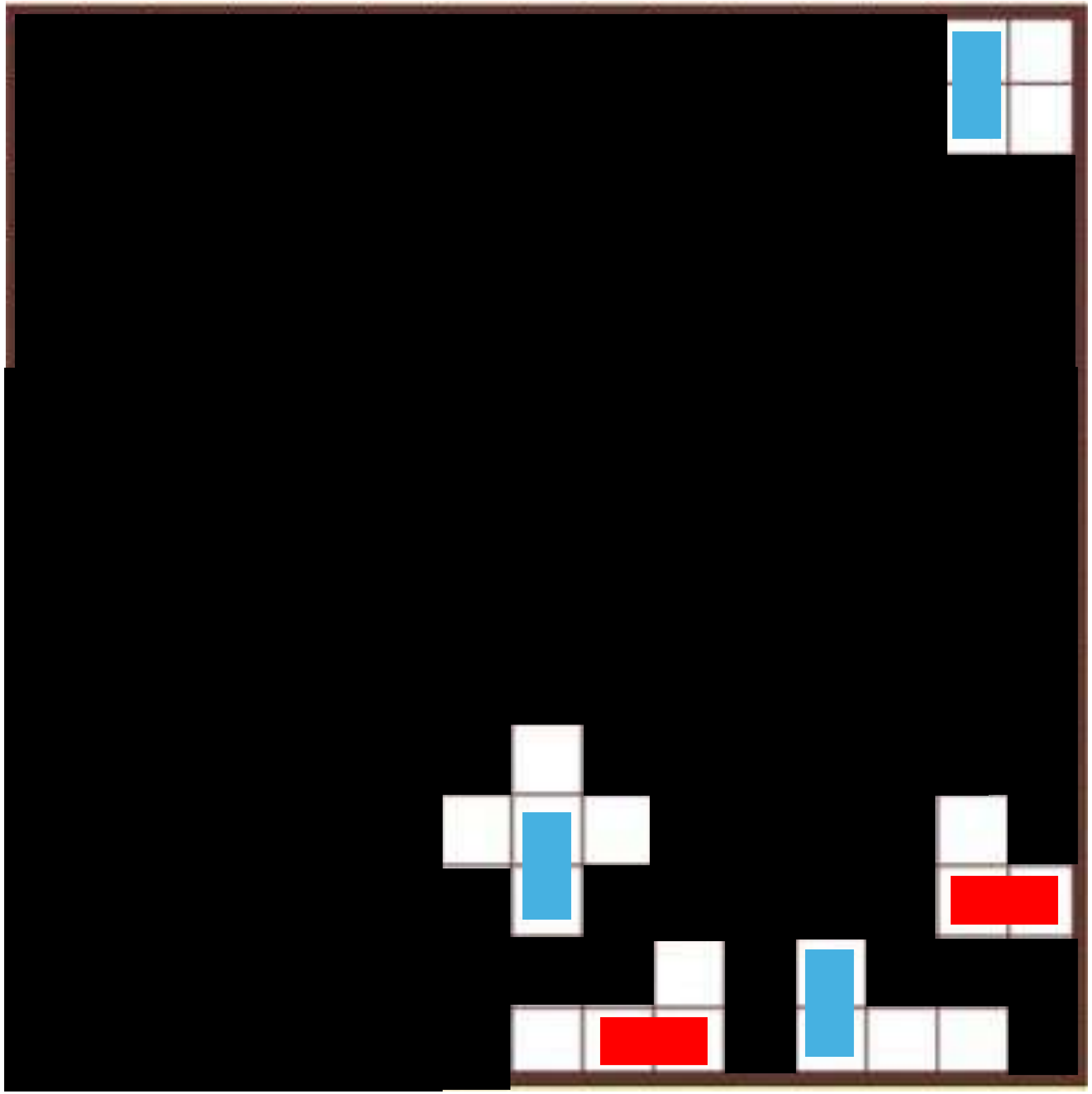


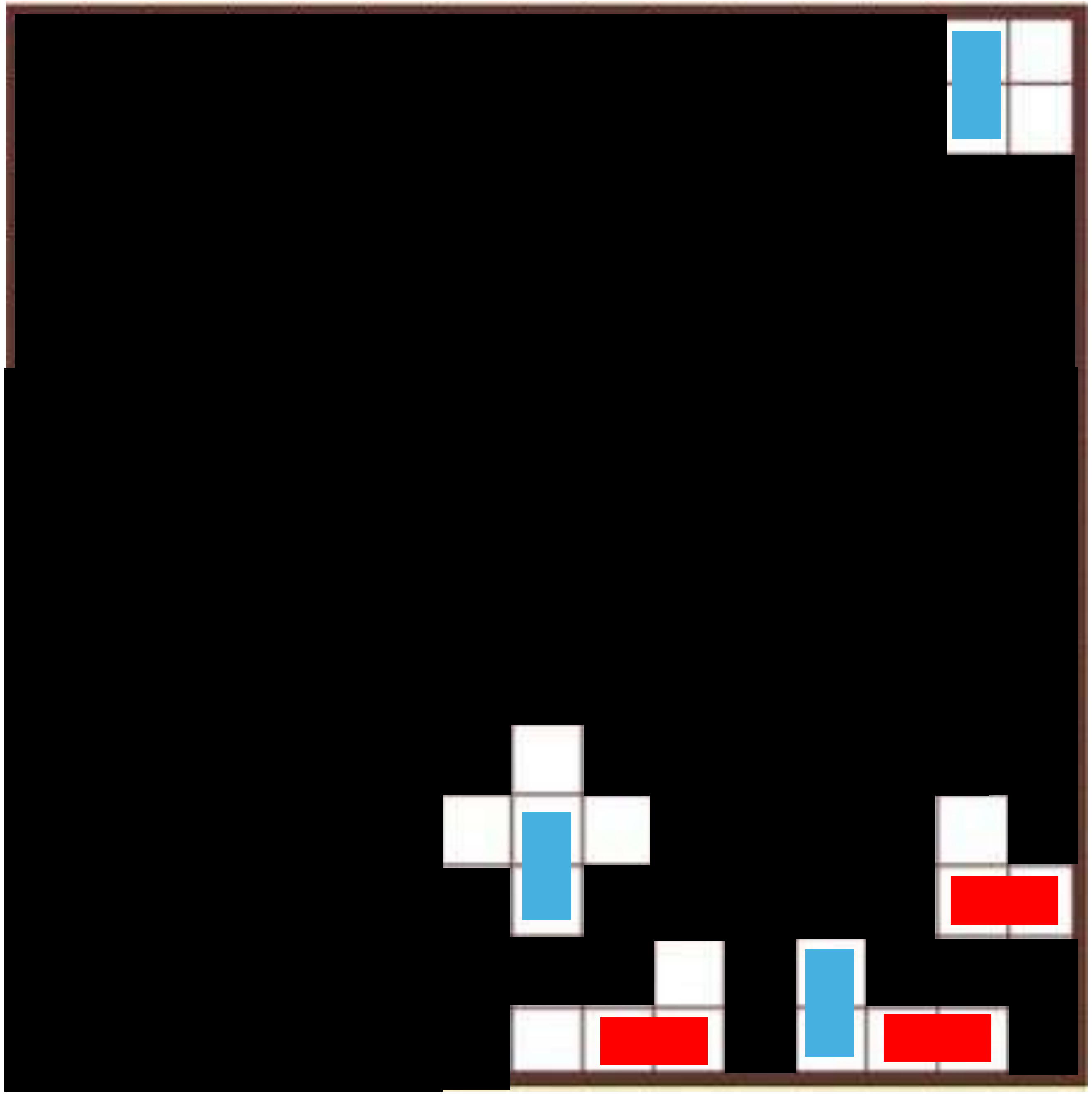


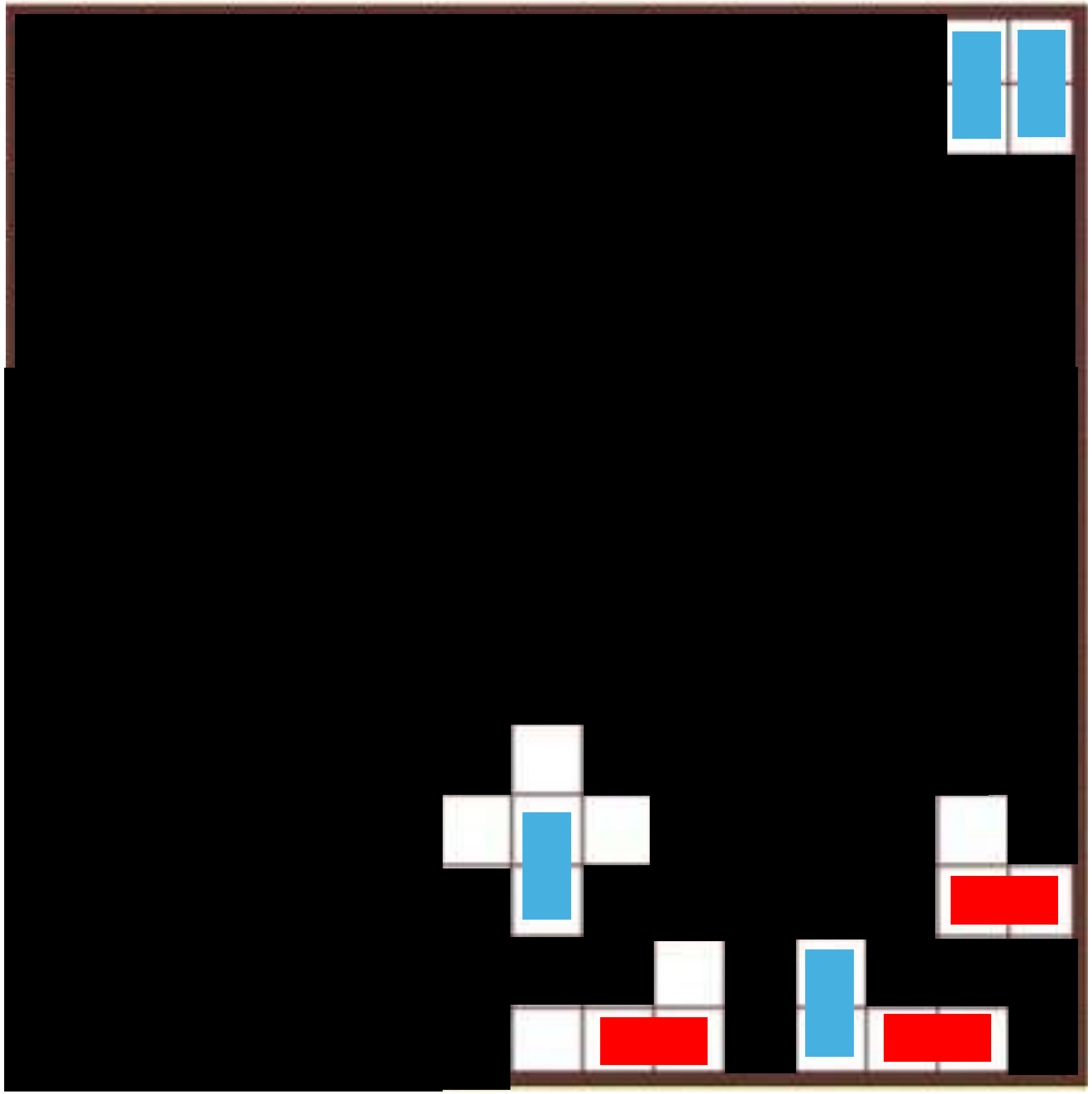












Annex: Why study Combinatorial Game Theory?

An Intellectual Introduction was written by renowned Professor Elwyn Berlekamp (one of the founding fathers of CGT). We highlight

Most of the initial theoretical results of combinatorial game theory were achieved by exploiting the power of recursions. Combinatorial game theory has that in common with many other mathematical topics, including fractals and chaos. Combinatorial game theory also has obvious and more detailed overlaps with many other branches of mathematics and computer science, including topics such as algorithms, complexity theory, finite automata, logic, surreal analysis, number theory, and probability.

Playing Games with Algorithms: Algorithmic Combinatorial Game Theory*

Erik D. Demaine† Robert A. Hearn‡

Abstract

Combinatorial games lead to several interesting, clean problems in algorithms and complexity theory, many of which remain open. The purpose of this paper is to provide an overview of the area to encourage further research. In particular, we begin with general background in Combinatorial Game Theory, which analyzes ideal play in perfect-information games, and Constraint Logic, which provides a framework for showing hardness. Then we survey results about the complexity of determining ideal play in these games, and the related problems of solving puzzles, in terms of both polynomial-time algorithms and computational intractability results. Our review of background and survey of algorithmic results are by no means complete, but should serve as a useful primer.

1 Introduction

Many classic games are known to be computationally intractable (assuming $P \neq NP$): one-player puzzles are often NP-complete (as in Minesweeper) or PSPACE-complete (as in Rush Hour), and two-player games are often PSPACE-complete (as in Othello) or EXPTIME-complete (as in Checkers, Chess, and Go). Surprisingly, many seemingly simple puzzles and games are also hard. Other results are positive, proving that some games can be played optimally in polynomial time. In some cases, particularly with one-player puzzles, the computationally tractable games are still interesting for humans to play.

Combinatorial Game Rulesets

Combinatorial games are two-player, perfect-information games with no randomness. This table contains many rulesets and known properties. I update it as I learn new things.

New: Hover over the name cell of a ruleset to see a brief description. I don't have descriptions for all of them yet.

Rulesets

Ruleset	Image and Variants	Partiality	Length	Initial Position Outcome Classes	Computational Complexity	Other Properties
Arimaa -- link here --		Strictly partisan	Loopy	?	In EXPTIME	
Amazons (blog post) -- link here --		Strictly partisan	Short	? (I don't know what is considered an Initial Position.)	PSPACE-complete even with only one amazon apiece	
Atropos Play it: HTML -- link here --		Impartial	Short	Open! Conjecture: N ("fuzzy") iff there are an even number of open circles.	PSPACE-complete	
	Unrestricted Atropos -- link here --	Impartial	Short	Open! Conjecture: N ("fuzzy") iff there are an even number of open circles.	Open; in PSPACE	
Chess -- link here --	Know of a better image for Chess? Please let Kyle know!	Strictly partisan	Loopy	Open	EXPTIME-complete	
Chomp Play it: Java -- link here --		Impartial	Short	First Player	In PSPACE	

Lexicographic Codes: Error-Correcting Codes from Game Theory

JOHN H. CONWAY AND N. J. A. SLOANE, FELLOW, IEEE

Abstract—Lexicographic codes, or lexicodes, are defined by various versions of the greedy algorithm. The theory of these codes is closely related to the theory of certain impartial games, which leads to a number of surprising properties. For example, lexicodes over an alphabet of size $B = 2^a$ are closed under addition, while if $B = 2^{2^a}$ the lexicodes are closed under multiplication by scalars, where addition and multiplication are in the nim sense explained in the text. Hamming codes and the binary Golay codes are lexicodes. Remarkably simple constructions are given for the Steiner systems $S(5,6,12)$ and $S(5,8,24)$. Several record-breaking constant weight codes are also constructed.

I. Introduction

THIS PAPER is concerned with various classes of lexicographic codes, that is, codes that are defined by a greedy algorithm: each successive codeword is selected as the *first* word not prohibitively near (in some prescribed sense) to earlier codewords. For example, the very simplest class of lexicographic codes is defined as follows. We specify a base B and a desired minimal Hamming distance d. The first codeword accepted is the zero word. Then we consider all base-B vectors in turn, and accept a vector as a codeword if it is at Hamming distance at least d from all previously accepted codewords. (An example with $B = 3$ and $d = 3$ can be seen in Table XI.)

One of our goals is to point out the essential identity between this kind of lexicographic coding theory and the theory of certain impartial games (see Section II). Then the Sprague-Grundy theory of games has a number of interesting and surprising consequences for lexicographic codes (or *lexicodes*).

1) Unrestricted binary lexicodes are linear (Theorems 1, 3).

2) For base $B = 2^a$, unrestricted lexicodes are closed

Two other results worth mentioning here are the following.

5) Several well-known codes unexpectedly turn out to be lexicographic codes, including Hamming codes and the binary Golay codes of length 23 and 24 (Section III-B).

6) The constant weight binary lexicode of length 24, distance 8 and weight 8 is the Steiner system $S(5,8,24)$ (Theorem 12). By imposing an additional constraint on a constant weight lexicode (see Section IV-E), Ryba obtained an almost equally simple construction for the Steiner system $S(5,6,12)$ (Theorem 13). The corresponding game, called Mathematical Blackjack (or Mathieu's Vingt-et-un) is described at the end of Section IV-E.

7) A number of constant weight codes with minimal distance 10 and containing a record number of codewords are given in Table XIII.

Some of the game-theoretic aspects of this work are described in [1] and [2]. The relations between the theories of games and of lexicographic codes, and in particular the multiplicative theorem, underly some of the results in [1]. However, most of the results are published here for the first time. This work may be regarded as a coding-theoretic analog of the laminated lattices described in [5], [6].

The paper is arranged as follows. The connections with game theory are discussed in Section II, unrestricted lexicodes are treated in Section III, and Section IV deals with constant weight and constrained lexicodes. Tables IV–VIII and XII give the parameters of a number of lexicodes.

II. The Connections with Game Theory

A. Grundy's Game

Games!

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Combinatorial Game Theory

Videos from BIRS Workshop

Olivier Teytaud, Université Paris-Sud
Monday Jan 10, 2011 09:10 - 10:00

Monte Carlo Tree Search

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