LECTURE NOTES IN COMBINATORIAL GAME THEORY, IE619 2023

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1. Axioms, Nim and Wythoff Nim

Combinatorial games are games without chance and with no hidden information. The motivation of the field is traditional recreational rulesets such as CHESS, GO, CHECKERS, TIC TAC TOE and more. Games such as POKER, WHIST and BLACK JACK are disqualified because they involve hidden cards, and for example YAHTZEE, PACHISI and MONOPOLY are disqualified because play depends on the outcome of a dice. We follow some more axioms, as listed:

(1) There is a game board (a set of positions) and some ruleset that determines how given pieces are played;
(2) there are two players, and one of them is the starting player;
(3) the players take turns moving;
(4) every game terminates;
(5) these are win/loss games and a player who cannot move loses.

Item (5) is usually called *normal-play*. This convention is based on the goodness of movability. It is never bad to have more move options. The axioms give us a means to predict who is winning a given game in *perfect play*. Namely, we can use a method attributed to Ernst Zermelo [Z1913] (who is also the father of set theory and the axiom of choice etc.) often called *backward induction*. This method will be reviewed in Lecture 3.

The first combinatorial game that appears in the literature is NIM [B1902]. A finite number of beans are split into heaps. For example, a starting position could be four heaps of sizes $2, 3, 4$ and 5 beans respectively. Let us denote this position by $(2, 3, 4, 5)$. The current player choses one of the heaps and removes at lest one bean, and at most the whole heap. This is a normal-play game, so the player with the last move wins.

Bouton discovered a method to find a winning move if there is one. The tool is called nim addition, and it is performed as follows. Write the heap sizes in binary, and add without carry, that is each column adds to 0 if and only if it contains an even number of 1s. Let us compute the nim-sum of our sample game:

$$\begin{array}{cccc} & 1 & 0 & 1 \\ & 1 & 0 & 0 \\ & 0 & 1 & 1 \\ \oplus & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array}$$

The nim-sum is 0. The meaning of this in terms of NIM play is that the player who does not start wins in optimal/perfect play. Every move is losing. Let us say, for example, that the first player removed three beans from the third heap. Then the new position is $(2, 3, 1, 5)$. And, by using nim-addition on that position, we obtain

$$\begin{array}{cccc} & 1 & 0 & 1 \\ & 0 & 0 & 1 \\ & 0 & 1 & 1 \\ \oplus & 0 & 1 & 0 \\ \hline & 1 & 0 & 1 \end{array}$$

Since the nim-sum is non-zero, there is a NIM move to a position such that the nim-sum becomes zero. That is the idea of Bouton's theory. Here, there is only one winning move, namely take all beans from the heap of size five.

Bouton's proof demonstrates that, given any starting position, and given best play by both players, exactly one of the players is able to play to a 0-position in every move (until the game ends). Later, in Theorem 3, we prove this in general.

It is easy to prove this in case of two heaps, and it was discovered in class, namely, if the starting position is (m, n), with say $m < n$, then the winning first move is to (m, m). The next position will be of the form (m, k), for some $0 \leqslant k < m$, which is of the 'same form' as the first position. Namely, the heaps are of different sizes. Exactly one of the players can, by every move, give the two heaps the same size. Note that this implies that the nim-sum is 0, so indeed it is a special case of the above more general idea.

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the non-negative integers.

Later, we will use the common $*$-notation for NIM heaps. That is, $*$ is a NIM heap of size one, $*2$ is a NIM heap of size two, and in general, for $n \in \mathbb{N} = \{1, 2, \ldots\}$, $*n$ is a NIM heap of size n.

The second ruleset is Wythoff's variation of NIM, which is called WYTHOFF NIM [W1907] or sometimes CORNER THE LADY or CORNER THE QUEEN. It is played on two heaps and the rules are as in NIM, or instead, a player may remove the same number of beans from both heaps, at least one from each heap, and and most twice the number of beans of the smaller heap. This game can equivalently be represented by a single Queen of CHESS, which, by each move, must reduce its distance to the lower left square, denoted by $(0, 0)$.

Suppose that the Queen is placed on position $(5, 4)$. The equivalent WYTHOFF NIM position is two heaps, one of size 5 and the other of size 4. The set of options is $\{(5, y), (x, 4), (5-t, 4-t) \mid 0 \leqslant y \leqslant 4, 0 \leqslant x \leqslant 3, 1 \leqslant t \leqslant 4\}$. See Figure 1.

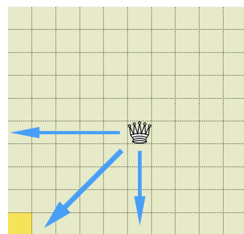


FIGURE 1. The figures illustrate typical move options of CORNER THE QUEEN. The lower left corner represents the terminal position $(0, 0)$.

An elegant method of finding the so-called $\mathscr{P}$-positions ($\mathscr{P}$revious player wins) is to recursively paint the $\mathscr{N}$-positions ($\mathscr{N}$ext or Curre$\mathscr{N}$t player wins), and fill in the smallest un-colored cells with $\mathscr{P}$s. Clearly $(0, 0)$ is a $\mathscr{P}$-position. Thus, each position of the form $(x, 0), (0, x)$ and (x, x) for positive integers x will be $\mathscr{N}$-colored. The method is displayed in Figure 2.

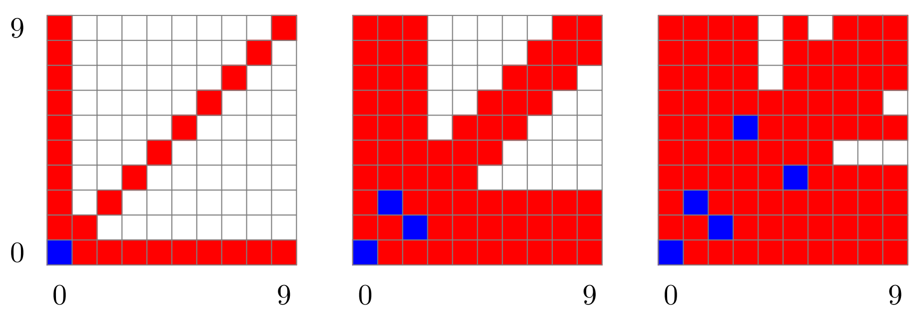


FIGURE 2. A geometric view of the losing positions of WYTHOFF NIM. The $\mathscr{N}$-positions are recursively painted in red, given 'smallest new' $\mathscr{P}$-positions. The terminal position is to the lower left.

This method of painting reveals symmetric $\mathscr{P}$-positons of the form (A_n, B_n) and (B_n, A_n), with the first 8 entries as in Table 1. We note that the classical so-called Fibonacci numbers (they appear long before Fibonacci's study in various Sanskrit works) appear in some of the entries, namely

$(1,2),(3,5),(8,13),\ldots$.[1] The Golden Section is the irrational number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. It is well known that $\frac{F_n}{F_{n-1}} \to \phi$, as $n \to \infty$. Moreover, we note a possible pattern: for all n, $B_n - A_n = n$. The most famous solution of the game is even more elegant.

Theorem 1 ([W1907]). *For all non-negative integers n,*

$$(A_n, B_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor),$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

We will prove this in a later class. And we will include other appealing theorem statements (see Lectures 10 and 11). The main tool will be the so-called Wythoff Properties as defined in Lecture 3.

TABLE 1. The first 8 $\mathscr{P}$-positions of WYTHOFF NIM (modulo symmetry).

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18

2. Example rulesets, and your own rulesets

The two players of a combinatorial game are usually called Left (female and positive) and Right (male and negative). A *sum* of the combinatorial games G, H, K and M, is defined as the composite game where, at each stage of play, the current player picks one of the components G, H, K or M and makes a move in that component (see Lecture 4 for a more formal treatment). We write this sum of games as $G + H + K + M$. For example, if player Left starts and plays in the H component, then the next position is $G + H^L + K + M$. Next, player Right picks one of the components and plays his move, for example to $G + H^L + K^R + M$, and so on. The game continues until there is no move in either component. As usual, in normal play, the player who cannot move loses.

The *rulesets* are at the core of combinatorial game theory. A ruleset does not need to come with a starting position (and given a ruleset one can

[1]"The Fibonacci numbers were first described in Indian mathematics, as early as 200 BC in work by Pingala on enumerating possible patterns of Sanskrit poetry formed from syllables of two lengths." Wikipedia.

usually envision an infinite number of possible starting positions). When we use the word "game position", or just "position", we usually mean a ruleset together with a starting position. The word "game" can be used freely and the surrounding context explains its local meaning. A ruleset is impartial, if for every position in the ruleset, the move options do not depend on who starts. A ruleset is partizan if there exists a position in the ruleset for which the Left and Right options differ. Partizan rulesets include the impartial ones as a subset, but it is quite unusual that a partizan position has the same Left and Right options.

Let us describe some popular rulesets. Here are four partizan rulesets and one impartial:

- Clobber: an n by m game board; Left plays black pieces and Right plays white pieces. A neighbor stone of opposing color can be clobbered and removed from the game board, while your stone takes its position. Starting position: a checker board pattern.
- Toads&Frogs: a 1 by n game board; Left plays Toads and Right plays Frogs. Toads move to the left and Frogs move to the right. A piece can slide to a neighboring empty cell, or jump one of the opponent's pieces. Starting position: t Toads to the right, and f Frogs to the Left.
- Domineering: an n by m game board; Left places horizontal domino tiles, and Right places vertical Domino tiles.
- Toppling Dominoes: Left plays Red dominoes and Right plays blue dominoes. Both players can topple a green domino. Domino pieces are placed in a sequence. Players can topple any direction.
- Non-attacking Queens: an n by m game board; Players place a Queen of Chess in a square, such that they do not attack any previously placed Queen.

To practice the concept of a disjunctive sum of games, in the class we play a tournament game such as: $G + H + K + M$, where G= Toads&Frogs on a 9 by 1 strip, $t = f = 3$, H = Domineering on a 5 by 3 board, K = Toppling Dominoes red blue green red blue blue, and where M = Non-attacking Queens on an 8 by 8 board (or a Nim position). Who wins if Left starts? Who wins if Right starts?

Variation Partizan Non-attacking Queens: this game is played with Black Queens for Left and White Queens for Right? Black plays non-attacking on White and vice versa. Different colored Queens do not attack each other.

This course builds on the idea that students build their own normal-play rulesets, and then study the theory through properties of their rulesets.[2] We will add some guidelines to the initial axioms in Lecture 1.

[2]This is why the lecture are typically not accompanied with exercises. At the end of every lecture though we have quizzes, some of which are mentioned in these lecture notes.

(1) A normal-play ruleset typically should not aim at achieving some 'condition' such as four-in-a-row or similar;
(2) The ruleset should not have been studied before;
(3) The ruleset should have a name.

Item (1) requires an explanation. For example, popular rulesets such as "four-in-a-row" could be envisioned as a normal-play game, by defining such a component dead when this condition has been achieved. But this is one layer more complicated then our typical rulesets that simply end when a player cannot play according to the rules.

Games where ending conditions are given by satisfying given conditions are usually called maker-maker, maker-breaker etc. Other examples are graph theory games with rules such as: one of the players attempts to form a triangle, and the other player is trying to finish the game with a triangle free graph. This is a bit hard to envision as a normal-play game, because the player who makes the last move by forming a triangle might be the losing player.

To test if a candidate ruleset satisfies guideline (1), one may ask the question: given the axioms, is it required to say anything else about the ending/winning of a game? In a typical combinatorial game such as the above tournament game examples, this is not necessary. Given a game board, it suffices to say how to play the pieces, and the winner is given by the normal-play axiom.

Regarding item (2) it is not hard to find new rulesets. CGT is a very young subject; most conceivable normal-play rulesets (satisfying the axioms and the guidelines) have yet to be defined/studied.

3. The first proofs

This section is loaded with proofs. First, we prove Zermelo's theorem in our setting of normal-play with no draw games. This is the first fundamental result of Combinatorial Game Theory (CGT), Theorem 2. Then we move on to a proof of Bouton's classical result on the game of Nim. And after that, in Theorem 4, we prove that the 'not so classical' Wythoff Properties define the $\mathscr{P}$-positions of Wythoff Nim (the best reference I have is a Licentiate-thesis that I wrote when I was a Ph.D. student, but we will explain even better here).

When we prove results about games by induction, we may assume that a desired property is satisfied by all options of a game G, and then we prove that this implies that the property holds for the game G itself. Observe that if G does not have any option, then the desired property vacuously holds. Hence the base case does not require any further mention (!). (Sometimes this induction method is referred to as "Conway Induction", due to one of the founders of the field.). Let us practice this idea in the proof of the First Fundamental Theorem of CGT.

Theorem 2 (The First Fundamental Theorem of CGT)**.** *Consider any normal-play combinatorial game G. Exactly one of the players can force a win.*

Proof. Suppose, by induction, that the statement holds for all options of a game G. Without loss of generality, suppose player Left starts. If, by induction, Right can force a win from each Left option of G, then Left cannot force a win from G. And hence Right can force a win from G. (This holds true also in the case of no Left options of G.) Otherwise Left choses an option form which she, by induction, can force a win. □

Observe that this result implies four well defined outcome classes in combinatorial games. Let us denote them by $\mathscr{L}$ (Left wins independently of who starts), $\mathscr{N}$, $\mathscr{P}$ and $\mathscr{R}$ (Right wins independently of who starts). We will return to these four outcome classes, but let us make more explicit first how we prove results on the outcomes of impartial games (that have only two outcome classes).

Suppose that we are given a candidate set of $\mathscr{P}$-positions of an impartial ruleset. To verify this set, we prove that there is no move from one candidate $\mathscr{P}$-position to another candidate $\mathscr{P}$-position. And we abbreviate this property as "$\mathscr{P} \not\to \mathscr{P}$". Moreover, we must prove that each candidate $\mathscr{N}$-position has an option in the set of candidate $\mathscr{P}$-positions. This we write as "$\mathscr{N} \to \mathscr{P}$". This way of thinking of course bases on the idea of induction, as is often the case in CGT.

Using this idea, we prove Bouton's theorem on the game of Nim. Recall the nim-sum definition "$\oplus$" from Lecture 1.

Theorem 3 ([B1902])**.** *Let h_i denote the heap sizes of a game of* Nim *on n heaps, written in binary power of two expansion, and where $i \in \{1, \ldots, n\}$. The outcome is a $\mathscr{P}$-position if and only if $\bigoplus h_i = 0$.*

Proof. For the property "$\mathscr{P} \not\to \mathscr{P}$", suppose first that

$$\bigoplus h_i = 0. \tag{1}$$

We must prove that every option has non-zero nim sum. Observe that (1) means that there is an even number of 1s in each column of the disjunctive sum, where the rows are the heap sizes h_i written in binary (as in Lecture 1). If there is no option, we are done. Otherwise, a Nim option reduces exactly one heap, corresponding to a row in our binary representation of the heap sizes. Thus there must be a change of parity of the number of ones in at least one column of (1). Thus the new nim-sum is non-zero.

For the property "$\mathscr{N} \to \mathscr{P}$", suppose next that

$$\bigoplus h_i \neq 0, \tag{2}$$

and let $x = \bigoplus h_i$ be the result of the nim addition in (2). We must prove that there is a heap h_k that can be reduced such that the new nim-sum is zero.

Write the nim sum x in its binary representation, as $x = \sum 2^i x_i$, where, for all i, $x_i \in \{0,1\}$. By (2), there is a largest index j such that $x_j = 1$. Thus, there must be an odd number of heaps that contain the jth power of 2, 2^j. We claim that either one of these heaps, say heap h_k, suffices.

Similar to the base 2 expansion of x, write $h_k = \sum 2^i h_{k,i}$. Then, by definition of j and h_k, $h_{k,j} = 1$. For all $i < j$ such that $x_i = 1$, let $h'_{k,i} = \overline{x}_i$, where $\bar{\cdot}$ is the binary complement (that is $\overline{0} = 1$ and $\overline{1} = 0$), and otherwise, let $h'_{k,i} = h_i$. For all $i \geqslant j$, let $h'_{k,i} = 0$. A winning move is to reduce h_k to $h'_k = \sum 2^i h'_{k,i}$. That this is a reduction of the heap size h_j follows from the fact that in base two expansion, for all j, $2^j > \sum_{i<j} 2^i$.[3] □

Probably the greatest challenge in this proof is to understand the technical part of the last paragraph. Of course, this part can be expanded by using more sentences in English, similar to the earlier parts of the proof. However, it is also important sometimes to practice reading more 'pure logic' parts of proofs. Why is the "binary complement" introduced in this last paragraph? Is it the best way to do it, or can you find a way to say the same thing without that definition? (There is no ultimate answer to such a question; this question is more meant as a challenge to think of how we write proofs, and why we do what we do in a proof.)

Next, we will prove that the following properties define the $\mathscr{P}$-positions of WYTHOFF NIM. Consider two sequences of integers $(a_n), (b_n)$, $n \in \mathbb{N}_0$. They satisfy the Wythoff Properties if:

(i) $(a_0, b_0) = (0, 0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

Property (ii) is called "*increasing*". Property (iii) we may call a "*shift*". Together with (ii), the properties (iv) and (v) are usually called "*complementarity*".[4] The following result establishes that, in fact, these properties define a unique pair of sequences.

Theorem 4. *The set $W = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}_0\}$ given by the Wythoff Properties is unique, and it is the set of $\mathscr{P}$-positions of* WYTHOFF NIM.

Proof. Observe that it suffices to prove that the set W is the set of $\mathscr{P}$ - positions of WYTHOFF NIM. It then follows that the properties define a unique pair of sequences. Observe that by (ii) and (iii), both $a = (a_n)$ and $b = (b_n)$ are strictly increasing sequences. Clearly $(a_0, b_0) = (0, 0)$ is the terminal $\mathscr{P}$-position. We must prove that every candidate $\mathscr{P}$-position has no $\mathscr{P}$-position as an option, and we must prove that every candidate $\mathscr{N}$-position

[3] This property is of course true in any base n exapnsion if $n \geqslant 2$ is an integer.

[4] Since we are assuming (ii) then (iv) and (v) suffice to define complementary sequences; without (ii), in addition, we should require, for all $n \neq m$, $a_n \neq a_m$.

has a $\mathscr{P}$-position as an option.

"$\mathscr{P} \not\to \mathscr{P}$": We prove that, for any $n > 0$, (a_n, b_n) does not have an option of the same form. We use (iii), (iv) and (v) to exhaust all possibilities. Suppose that a player removes from a single heap to say (a_i, b_n). Then, since b is increasing, $b_i \neq b_n$. If they remove from a single heap to (b_i, b_n), then (iv) contradicts that this be of the desired form. If they remove from a single heap to say (a_n, b_i), then again, since b is increasing, $b_i \neq b_n$. If they remove from a single heap to (a_n, a_i), then by (iv) this option is not of the same form. If they remove the same number m from both heaps, then by (iii) the position cannot be of the form (a_i, b_i). Namely $b_n - m - a_n + m = n > i = b_i - a_i$.

"$\mathscr{N} \to \mathscr{P}$": We prove that, if a position is not of the form (a_n, b_n), then it has an option of this form. Consider first (x, b_n), and $x > a_n$. Then remove $x - a_n > 0$ from the first heap. If $x < a_n$, then, by (v), there are two cases.

1. $x = a_i$, for some $i < n$;
2. $x = b_i$ for some $i < n$.

In case 1, since b is increasing, there is a move to (a_i, b_i). In case 2, there is a move to (b_i, a_i), since, by (iii) and b increasing, $a_i < b_i < b_n$.

Consider next a position of the form (a_n, x), $x \neq b_n$. If $x > b_n$, then (a_n, b_n) is an option. Hence assume $x < b_n$. Then, by (v) $x = b_i$ or $x = a_i$, for some i. In the first case $i < n$ and so (a_i, b_i) is an option, by (ii). In the second case, the position is (a_n, a_i). We have three cases (or two if we regard cases 2 and 3 as the same):

1. $a_n < a_i < b_n$;
2. $a_i < a_n < b_i$;
3. $a_i < b_i < a_n$.

For case 1, observe that $0 < a_i - a_n < b_n - a_n = n$. Therefore there exists $j < n$ such that $a_i - a_n = b_j - a_j = j$. Hence (a_j, b_j) is an option. For case 2, observe that $0 < a_n - a_i < b_i - a_i = i$. Therefore there exists $j < i$ such that $a_n - a_i = b_j - a_j = j$. Hence (a_j, b_j) is an option. For case 3, (a_i, b_i) is an option. □

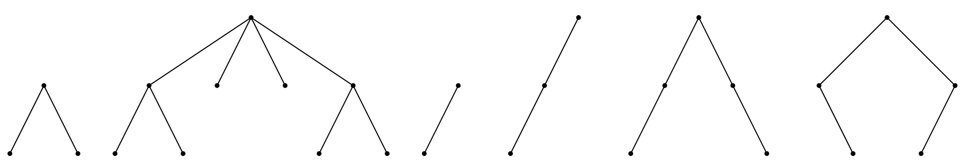
We will use this result in a later lecture to prove Theorem 1, together with some other representations of Wythoff Nim's $\mathscr{P}$-positions. One more component, called Beatty/Lord Rayleigh's Theorem, will be required.

4. Normal play structures

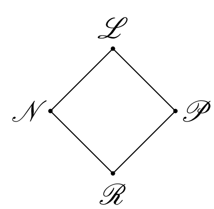
In this lecture we will define the notions of partial order of games, game equivalence and disjunctive sum (addition) of games. Then, in Lecture 5, we prove that the normal-play games, under the disjunctive sum operator, have a group structure. Specifically, in Theorem 10, we prove that every game has an inverse, and we will see that this is a main tool for *constructive* *game comparison* (Theorem 11). In this spirit, we begin here by defining the notion of game comparison in a non-constructive but exhaustive way.

The definition of game comparison (Definition 7) takes into the account game addition (Definition 6), and an inherited partial order of outcomes (see the below "Outcome Diamond"). Moreover it uses a recursively defined bracket notation of a game. We use it in parallel with a standard game tree representation, where Left options are left slanting edges and Right options are right slanting edges. The game $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ represents the set of left and right options of the game G, respectively. If $G^{\mathcal{L}} \neq \varnothing$, a typical Left option is $G^{L} \in G^{\mathcal{L}}$, and similarly, a typical Right option is $G^{R} \in G^{\mathcal{R}}$. By the recursive definition, we would write, for example $G^{L} = \left\{G^{L\mathcal{L}} \mid G^{L\mathcal{R}}\right\}$, and so on.

For example, a Nim heap of size 2 is the game $\{0, * \mid 0, *\}$. The integer games belong to the partizan theory, and they can be defined recursively as $0 = \{\,\mid\,\}$, $1 = \{0 \mid\,\}$ and $n = \{n-1 \mid\,\}$, for $n > 0$. Similarly, for all $n \in \mathbb{N}$, $-n = \{\,\mid -n+1\}$. Let us draw the game trees of the games $*, *2, 1, 2,$ $\{1 \mid -1\}$ and $\{-1 \mid 1\}$.



The standard convention is the total order "Left" > "Right", that is, Left is the "maximizer" and Right is the "minimizer". This induces the Outcome Diamond



with $\mathscr{L} > \mathscr{P}, \mathscr{N}, \mathscr{R}$ and $\mathscr{R} < \mathscr{P}, \mathscr{L}, \mathscr{R}$ but $\mathscr{N} <> \mathscr{P}$. Here '$<>$' denotes '$\not>$' and '$\not<$' and '$\neq$'. That is, the outcomes $\mathscr{N}$ and $\mathscr{P}$ are *confused*, *fuzzy* or *incomparable*. All these three words (and more) appear in the literature. The outcomes do not suffice to understand how to play well a disjunctive sum of games. Table 2 illustrates that.

Suppose that we know the outcomes of the individual games G and H. Now we wish to compute the outcome of the sum of G and H. If one of the outcomes is a $\mathscr{P}$-position, then we know the outcome of the sum; if both outcomes are either $\mathscr{L}$ or $\mathscr{R}$, then we know the outcome of the sum. Otherwise we cannot yet know the outcome of the sum. The notion of outcomes requires a refinement, where alternating play in the separate components is not mandatory.

TABLE 2. Given the outcomes of G and H, when can we know the outcome of $G+H$?

$G \setminus H$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{R}$
$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	?	?
$\mathscr{P}$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{R}$
$\mathscr{N}$	?	$\mathscr{N}$	?	?
$\mathscr{R}$	?	$\mathscr{R}$	?	$\mathscr{R}$

If both outcomes are $\mathscr{L}$ then Left can obviously follow her winning strategies in both components, individually and independently of who starts, and analogously for Right. If one of the components is a $\mathscr{P}$-position, then the other component will determine the outcome of the sum, namely, if the first player plays in the $\mathscr{P}$-position, then the other player can respond there in a manner that they will get the last move in that component. Hence the first player can be forced to 'open' the other component. A $\mathscr{P}$-position cannot affect the outcome in a disjunctive sum.

The following example explains some question marks in Table 2.

Example 5. Suppose $o(G) = o(H) = \mathscr{N}$. This holds if, for example $G = H = *$, a single heap of NIM. Then $o(G+H) = \mathscr{P}$. We could also have $G = *$ and $H = *2$. Then $o(G+H) = \mathscr{N}$. Hence the question-mark is motivated in this case.

If $G \in \mathscr{L}$ and $H \in \mathscr{N}$, we could have $H = \{0 \mid -100\}$ and $G = 1$, with $G+H \in \mathscr{R}$. But we could also have $H = *$ and $G = 1$, which would give $G+H \in \mathscr{L}$.

Suppose that $G \in \mathscr{L}$ and $H \in \mathscr{R}$. We could have $G = 1+*$ and $H = -1$ which gives $G+H \in \mathscr{N}$. On the other hand $G = 1$ and $H = -1$ gives $G+H \in \mathscr{P}$. Similarly $G = 10$ and $H = -1$ gives $G+H \in \mathscr{L}$, while $G = 1$ and $H = -10$ gives $G+H \in \mathscr{R}$. Hence, all outcomes are possible. The other question marks are similar.

So far, we have used the notion of a sum of games in an intuitive way. Now we will present the standard formal way. The disjunctive sum of games is defined in a recursive manner.

Definition 6 (Disjunctive Sum)**.** Consider games G and H. Then $G+H = \left\{G+H^{\mathcal{L}}, G^{\mathcal{L}}+H \mid G+H^{\mathcal{R}}, G^{\mathcal{R}}+H\right\}$, where $\mathcal{X}+G = \{X+G : X \in \mathcal{X}\}$, if $\mathcal{X}$ is a set of games.

Let us define the partial order of games.

Definition 7 (Partial Order)**.** Consider games G and H. Then $G \geqslant H$ if, for all games X, $o(G+X) \geqslant o(H+X)$. And $G = H$ if $G \geqslant H$ and $H \geqslant G$.[5]

This is the desired refinement of the partial order of the outcomes. Namely, it assures Left that the game G is no worse for her than the game H, if played in any arbitrary disjunctive sum. However, it might appear that almost all games would remain incomparable with such a strong notion of a partial order. And moreover, the definition is non-constructive, so there is no algorithm that could determine the relation between two games, unless one can find another equivalent way of expressing the partial order. And indeed, that this is possible will be our second fundamental theorem of combinatorial games. The first major tool is that the games constitute a group structure, and we will prove that in the next lecture. The negative of a game will be the game where the players have swapped roles. Let us give the recursive definition here, and prove its consistency in the next lecture.

Definition 8. Consider a game G. Then the negative of G is $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$.

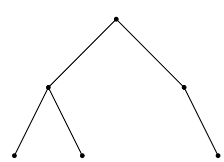
Similar to Definition 6, if $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a set of games, then $-\mathcal{X} = \{-X_1, \ldots, -X_n\}$.

5. The second fundamental theorem

In this lecture we first establish that normal-play games form a group structure (Theorem 10), and then we prove the Second Fundamental Theorem of CGT (Theorem 11) and its corollary (Corollary 12).

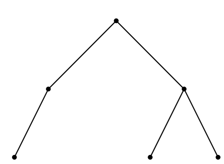
Let us begin with an example of a negative of a game (Definition 8). In terms of game trees, let

$G =$



Then

$-G =$



[5]A partial order is a relation that satisfies 1. Reflexivity (aRa); 2. Antisymmetry (if aRb and bRa then a=b); 3. Transitivity (if aRb and bRc then aRc). It is easy to prove that our definition satisfies these axioms. Then it easily follows that '=' is an equivalence relation, which satisfies Reflexivity, Symmetry (aRb implies bRa) and Transitivity). One has to check first that the axioms 1-3 hold for the partial order of outcomes. But this is also easy to check.

In terms of game forms, these games are $G = \{* \mid -1\}$ and $-G = \{1 \mid *\}$, where, as before, $* = \{0 \mid 0\}$, $1 = \{0 \mid \}$ and $-1 = \{ \mid 0\}$. As an exercise, we may add these two games, and expand this sum as one single game form:[6]

$$\begin{aligned} G + (-G) &= \{* - G, 1 + G \mid * + G, -1 - G\} \\ &= \{\{-G, * - 1 \mid -G, * + *\}, \{G, 1 + * \mid 1 - 1\} \mid \cdot\} \\ &= \{\{-G, \{-1 \mid *, -1\} \mid -G, \{* \mid *\}\}, \{G, \{*, 1 \mid 1\} \mid \{-1 \mid 1\}\} \mid \cdot\}, \end{aligned}$$

where we have omitted to expand the Right options since they are symmetric. This game form can, with some patience, and as an exercise, be drawn as a large game tree. But it should be equivalent to 0, as, starting with $G+(-G)$, the previous player can mimic the current player at each stage, until the current player cannot move. This is covered by Theorem 10, which will be our first application of Definition 7. It will establish that the set of all games together with the disjunctive sum operator constitutes a (partially ordered) group structure.

An abelian group, $(\mathbb{G}, +)$, satisfies five properties.

- Neutral Element: There exists an element, $0 \in \mathbb{G}$, such that for all $G \in \mathbb{G}, 0 + G = G$;
- Closure: for all $G, H \in \mathbb{G}$, $G + H \in \mathbb{G}$;
- Negative: for all $G \in \mathbb{G}$, there exist an element '$-G$' such that $G + (-G) = 0$;
- Commutativity: for all $G, H \in \mathbb{G}$, $G + H = H + G$;
- Associativity: for all $G, H, K \in \mathbb{G}$, $(G + H) + K = G + (H + K)$.

Suppose now that $(\mathbb{G}, +)$ is our set of games together with the disjunctive sum operator. All properties, except "Negative" are easy exercises.

The following decomposition of the outcomes will be useful.

Definition 9. Let $\mathscr{L} = (\mathrm{L}, \mathrm{L})$, $\mathscr{P} = (\mathrm{R}, \mathrm{L})$, $\mathscr{N} = (\mathrm{L}, \mathrm{R})$ and $\mathscr{R} = (\mathrm{R}, \mathrm{R})$. The first (second) coordinate declares who wins if Left (Right) starts. Similarly, for a given game G, write $o(G) = (o_{\mathrm{L}}(G), o_{\mathrm{R}}(G))$, where $o_{\mathrm{L}}(G), o_{\mathrm{R}}(G) \in \{\mathrm{L}, \mathrm{R}\}$ denotes who wins in perfect play depending on who starts.

Theorem 10 (Negative Game)**.** *For any game G, $G + (-G) = 0$.*

Proof. We have to demonstrate that, for any game G, for all games X, $o(G - G + X) = o(X)$. The proof is by induction on $G - G + X$. If Left cannot play in X, then $o_{\mathrm{L}}(X) = \mathrm{R}$. Similarly, if Left cannot play in X, then $o_{\mathrm{L}}(G - G + X) = \mathrm{R}$, because, if Left can play in $G - G$, then Right can mimic, and so on, which ultimately leads to Right getting the last move in $G - G$. The base case is analogous for the function o_{R}.

[6] This example is admittedly a bit complicated. But its purpose is also to illustrate how easy it is to build complex game trees that are equal to the empty game 0. Take *any* game G and add its inverse, and there you go, a zero-game! If you instead started with a single complex game tree of '$G - G$', it would usually be much harder to see that it equals 0. This is an exercise to do once, and then in a sense 'never again'.

Suppose that the statement holds for all options of G. For example, if G^R is a Right option of G, then $G^R - G^R = 0$. If Left has an option in X, then there are two cases to consider, namely $o_{\mathrm{L}}(X) = \mathrm{L}$ or $o_{\mathrm{L}}(X) = \mathrm{R}$.

Suppose first that $o_{\mathrm{L}}(X) = \mathrm{L}$, and consider the game $G - G + X$. Suppose that Left's winning move in X is to X^L. We claim that $o_{\mathrm{R}}(G - G + X^L) = \mathrm{L}$. Namely, if Right plays to $G - G + X^{LR}$, then Left wins by induction (Right does not have a winning move in X^L). And if Right plays in the '$G - G$' part, then Left can mimic. This would result in $G^R - G^R = 0$ or $G^L - G^L = 0$, by induction.

Suppose next that $o_{\mathrm{L}}(X) = \mathrm{R}$, that is, Left does not have a winning option in X played alone. In the game $G - G + X$, if Left plays her losing move in the X-component, then Right can respond locally to $G - G + X^{LR}$ and win by induction. If Left starts by playing in the '$G - G$' component, then Right can mimic, and the argument is the same as in the previous paragraph.

The proofs for Right playing first are analogous (symmetric). □

The second fundamental theorem of combinatorial games is as follows. We utilize that games form a group structure. That is, every game has an inverse (Theorem 10).

Theorem 11 (The Second Fundamental Theorem)**.** *Consider games G and H. Then $G \geqslant H$ if and only if Left wins the game $G - H$ playing second, that is if and only if $o_{\mathrm{R}}(G - H) = \mathrm{L}$.*

Before the proof, we give two examples of how to use this result.

Let $G = \{* \mid -1\}$ and let $H = *2$. We use Theorem 11 to show that $G \not\geqslant H$. That is, it suffices to show that Left does not win the game $G - H$ playing second. If Right plays to $-1 + *2$, then Left can only respond to -1 or $-1 + *$, and loses either way. How about the reverse inequality? Is $H \geqslant G$? That is, can Left win if Right starts in the game $H - G = *2 + \{1 \mid *\}$? If Right plays to $*2 + *$, then Left can respond to $* + *$; if Right plays to $* + \{1 \mid *\}$ or to $\{1 \mid *\}$, then Left can respond to 1 or $1 +^{*}$ and wins in either case. Altogether this proves that $H > G$.

Let $G = \{0 \mid 1\}$ and $H = *$. It is easy to check that $o_{\mathrm{R}}(G - H) = \mathrm{L}$. Hence $G \geqslant H$. In addition, since $o_{\mathrm{L}}(G - H) = \mathrm{L}$, then in fact $G > H$.

Proof of Theorem 11. By Theorem 10, we may study the game $G - H$. We must prove that $G - H \geqslant 0$ is the same as Left wins $G - H$ playing second. By definition, $G - H \geqslant 0$ means that, for all X, then $o(G - H + X) \geqslant o(X)$.

Suppose first that for all X, then $o(G - H + X) \geqslant o(X)$. This holds in particular for $X = \{\,\mid\,\}$. And then $o_{\mathrm{R}}(X) = \mathrm{L}$ implies $o_{\mathrm{R}}(G - H + X) = o_{\mathrm{R}}(G - H) = \mathrm{L}$. This proves this direction.

Next, suppose that Left wins $G - H$ playing second. We must prove that, for all X, then $o(G - H + X) \geqslant o(X)$.

If $o_{\mathrm{L}}(X) = \mathrm{R}$ there is nothing to prove, so let us assume that $o_{\mathrm{L}}(X) = \mathrm{L}$. In particular, this means that Left has a winning move in X played alone, to say X^L. She can play this move in the game $G - H + X$. If Right responds in the '$G-H$' part, then, by assumption, Left has a winning response inside that part. And if he plays to $G-H+X^{LR}$, then Left wins by induction (since $o_{\mathrm{L}}(X^{LR})=\mathrm{L}$). Hence, $o_{\mathrm{L}}(G-H+X)=\mathrm{L}$.

Similarly, assume that $o_{\mathrm{R}}(X)=\mathrm{L}$, and we must prove that $o_{\mathrm{R}}(G-H+X)=\mathrm{L}$, under the assumption that $o_{\mathrm{R}}(G-H)=\mathrm{L}$. If Right starts in the '$G-H$' part, then Left has a winning response, by assumption, and otherwsie, if Right starts in the 'X' part, then, by $o_{\mathrm{R}}(X)=\mathrm{L}$, Left can respond locally and win, since by induction $o_{\mathrm{R}}(X^{RL})=\mathrm{L}$. $\square$

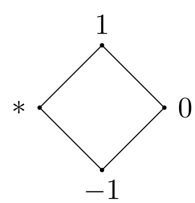
It is convenient to be explicit about all relations in the inherited partial order.

Corollary 12. *Consider games G and H. Then*

- $G=H$ *if and only if* $G-H\in\mathscr{P}$;
- $G>H$ *if and only if* $G-H\in\mathscr{L}$;
- $G<H$ *if and only if* $G-H\in\mathscr{R}$;
- $G \parallel H$ *if and only if* $G-H\in\mathscr{N}$.

Proof. For the first item, apply Theorem 11 for $G\geqslant H$ and $H\geqslant G$. The other items are similar, namely apply Theorem 11 for $G\geqslant H$ and $H\not\geqslant G$, $G\not\geqslant H$ and $H\geqslant G$, and $G\not\geqslant H$ and $H\not\geqslant G$ respectively. $\square$

In Corollary 12, it is instructive to revisit the outcome diamond, and instead of outcomes put games *born* by day 1 as representatives of the outcome classes.



A game's birthday is defined recursively. We will do this after Theorem 15.

6. Still to do

Here we will insert the approved projects from this workshop type lecture. Some of the lectures in this course are of the form workshop/tournaments etc. Anyone has a latex version of their rulesets with some analysis? Nov 11 2024: we still plan to do this, but it had to be after your projects presentations are finished, and preferably (but not necessarily) before final exam. Please prepare any latex code and submit online, if you have the time!

7. Game reductions and canonical form

Here we discuss the two reduction theorems on combinatorial games; they concern *domination* and *reversibility*. We will show that together they imply, for any game G, the existence of a unique reduced form, usually referred to as the *canonical form*, the *game value*, or just the *value* of G. We state the results in terms of Left options, and the symmetric statement in terms of Right options has an analogous statement and proof. These two results are nice applications of the Second Fundamental Theorem and its corollary.

Let us start with some examples. If $G = \{1, 2, 3 \mid *\}$, then Left should ignore the Left options 1 and 2, and hence $G = \{3 \mid *\}$. This guess can be verified by using Corollary 12 as follows. The previous player wins $\{1, 2, 3 \mid *\} + \{* \mid -3\}$, by mimic strategy, unless Left starts and plays in the first component to $1 - G$ or $2 - G$. Then Right responds to $1 - 3$ or $2 - 3$ and wins. This is all good, but note that we found a simpler form of G by guess work. Domination is better in that it achieves the same result but without guessing a simpler equivalent form.

Theorem 13 (Domination)**.** *Consider any game G. If there are Left options A and B, such that $A \leqslant B$, then $G = \left\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\right\}$.*

Proof. Let $H = \left\{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\right\}$. By Corollary 12, it suffices to prove that $G - H \in \mathscr{P}$. Observe that the Left options of H are the negations of the Right options of G. Hence any play in those options can be mimicked, and then $G^R - G^R = 0 \in \mathscr{P}$ settles those cases.

Similarly, if Left plays to $G^L + H$, where $G^L \neq A$, then Right can respond to $G^L - G^L = 0 \in \mathscr{P}$. The only remaining case is if Left is the starting player and she plays to $A - H$. But then Right can respond to $A - B \leqslant 0$, and he wins (playing second in $A - B$). □

By this result, we see immediately that $G = \{1, 2, 3 \mid *\} = \{3 \mid *\}$, because $3 \geqslant 2 \geqslant 1$.

The next result concerns the reduction *reversibility*. We have seen already that the game $G = \{* \mid *\} = 0$, and one can argue directly that it is true because the game is a $\mathscr{P}$-position. The game's simplest reduced form is 0, but it cannot reduce via domination, because there is only one option. We obviously need another tool. Luckily, one can argue by using 'reversibility', that this holds true: If Left plays to $*$, then Right has on option that is no worse than the original game $\{* \mid *\}$, that is, it reverses Left's move. Therefore Left's move is meaningless and should be reduced to whatever remains after Right's 'automatic' response. We prove that this idea holds in general, and give some more examples after the result.

Theorem 14 (Reversibility)**.** *Consider any game G. If there is a Left option G^L with a Right option $G^{LR} \leqslant G$, then $G = \left\{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\right\}$.*

Proof. Let $H = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}}\} \mid G^{\mathcal{R}}\}$. We prove that $o(G - H) = \mathscr{P}$. Similar to the proof of Theorem 13, moves that can be mimicked cannot disprove the result. Hence it suffices to analyze two cases:

- Left starts by playing to $G^L - H$;
- Right starts by playing to $G - G^{LRL}$, for some $G^{LRL} \in G^{LR\mathcal{L}}$.

Assume first that Left starts by playing to $G^L - H$. Then Right can play to $G^{LR} - H$. By assumption, $o_{\mathrm{L}}(G^{LR} - G) = \mathrm{R}$. Since $-H$ and $-G$ have the same Left options, Right wins if Left plays in the $-H$ component to $G^{LR} - H^R$. Thus assume she plays to $G^{LRL} - H$. But then, by definition of H, Right can respond to $G^{LRL} - G^{LRL} = 0$.

For the second item, assume that Right plays to $G - G^{LRL}$. But $G - G^{LR} \geqslant 0$ means that $o_{\mathrm{R}}(G - G^{LR}) = \mathrm{L}$, and so $o_{\mathrm{L}}(G - G^{LRL}) = \mathrm{L}$. □

Let us give two examples of 'similar looking' games, one of which reduces by reversibility but the other does not reduce. The games are $G = \{0, * \mid 0\}$ and $H = \{0, * \mid *\}$. For both games, the only Left option with a Right option is $G^L = H^L = *$, and $H^{LR} = 0 \leqslant H$ but $G^{LR} = 0 \not\leqslant G$, by Theorem 11, namely Left wins H^{LR}, but not G^{LR}, playing second. Observe that $H^{LR\mathcal{L}} = \varnothing$. Hence $H = \{0 \mid *\}$.

We can prove directly by using Theorem 11 that $G \neq \{0 \mid 0\}$; hence, the Left option $*$ cannot be reversible. Namely Left does not win $G - * = G + *$ playing second. Right starts by playing to $*$ and Left loses.

Theorem 15 (Canonical Form Game Value). *Suppose that domination and reversibility have been applied to a game G until any further reduction results in the same literal form game. If two literal form games G' and G'' are the end results of such reductions then they are identical.*

Proof. Suppose, by induction, that this holds true for all game trees of rank smaller than n. Let us prove that, then it holds for a game G of rank n. By the induction hypothesis, we may assume that every option of G is in its unique reduced form.

Claim: We can pair the options of G' and G'', such that for each option G'^L there is a G''^L such that $G'^L = G''^L$ (and similarly for Right).

Proof of Claim: The proof is by way of contradiction. Suppose that there is an option G'^L with no equal option in G''. We show that this contradicts $\mathbb{G}'' - G' \in \mathscr{P}$. Suppose that Right plays to $G'' - G'^L$. Since G' and G'' do not have any dominated or reversible options, we get that Left must play to either $G''^L - G'^L$, or $G'' - G'^{LR} \lhd\!\!\vert\; 0$ (we must have $\not\geqslant$ since the option is not reversible). In the second case, she cannot win playing second. Thus she could win playing second if and only if G''^L is such that $G''^L > G'^L$. Suppose next that Left starts and plays to $G''^L - G'$. Since G'' and G' are equal, Right must have a winning move. Again, this must be of the first form. Thus, for some Left option, we obtain inequalities $G'^{L_1} > G''^L > G'^L$.

This contradicts that G' does not have any dominated options.

But then, since these options are in reduced forms, by induction, since they are equal they must be identical. Hence, G' and G'' must also be identical. □

Definition 16 (Birthday)**.** A game is *born by* day $n > 0$ if every option of its canonical form game value is born by day $n-1$. A game is born by day 0 if its canonical form game value has no options. A game is *born at* day n if it is born by day n but not by day $n-1$.

For example, the game 0 is born by day 0, $*$, -1 and 1 are born at day 1. Together with 0, they form the same partial order as the above Outcome Diamond. There are 22 games born by day 2 (See Figure 3). There are 1474 games born by day 3. The number of games born by day 4 is huge, recently estimated between 10^{28} and 10^{185}, by Koki Suetsugu. The number of games born by day 5 is unknown.

8. A chocolate bar game and Pingala Nim

Chomp is an impartial game played with an m by n chocolate bar (see Figure 4). The lower left piece is poisoned, and the player who chomps it loses (it is a normal play game: think that the poisoned piece is not present). The game is played as follows: point at a remaining piece and chomp off everything above and to the right of that piece. A classical *strategy stealing* argument shows that the first player has a winning strategy for Chomp played on a rectangular grid. However, nobody fully understands optimal play, unless the grid is a square.

Theorem 17. Chomp *on a rectangular grid is a first player win.*

Proof. If the grid is a square, then point at position $(1, 1)$, and mimic the rest of play. Otherwise, suppose that the second player has a winning strategy. Take the upper right piece. If that is a winning move we are done. Otherwise, wait and see what the second player does. If the first move is not winning, then he has a winning strategy. But the first player could have played that move in the first move. Hence she has a winning strategy. □

Pingala Nim is played on one heap of pebbles.[7] The first player can remove any positive number of pebbles, except the whole heap. Any other move is restricted by taking at most twice the number of pebbles that the previous player removed.

The Pingala (Fibonacci) numbers are defined by $F_0 = 1, F_1 = 1$, and if $n \geqslant 2, F_{n+2} = F_{n+1} + F_n$. Thus, the sequence is $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

Theorem 18. *The first player loses* Pingala Nim *if and only if the starting heap size is a Pingala (Fibonacci) number.*

[7]This game is also known as Fibonacci Nim.

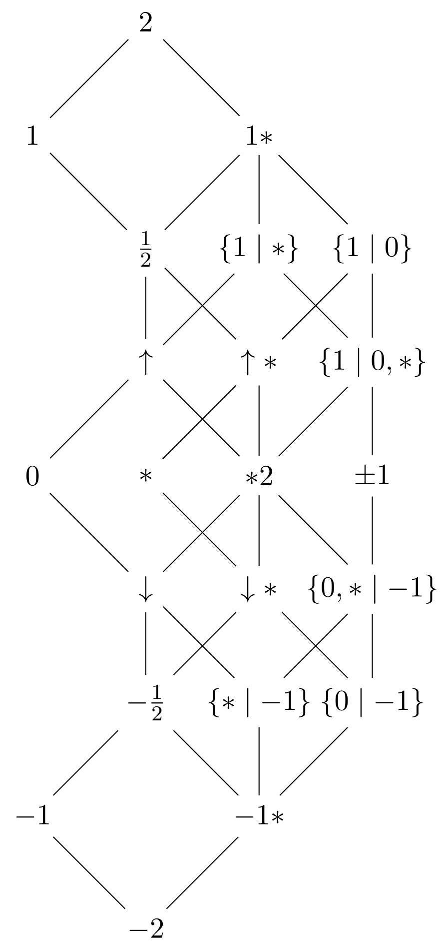


FIGURE 3. There are 22 games born by day 2. The picture shows the partial order of these 22 games, where an edge represents 'upper node > lower node'. The structure is a lattice: every disjoint pair of nodes has a least upper bound (a join) and a greatest lower bound (a meet).

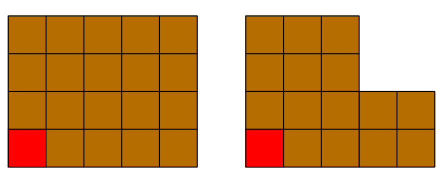


FIGURE 4. Two CHOMP positions. The red piece in the lower left is poisoned and cannot be eaten. The first player pointed at $(3, 2)$ and chomped off four pieces.

The proof uses a classical result on Pingala numbers, namely: every positive integer decomposes uniquely as a sum of non-consecutive Pingala numbers. For example $11 = 8 + 3$, $23 = 21 + 2$, $30 = 21 + 8 + 1$. We write this unique Fibonacci representation as a binary word $\zeta(x) = \zeta_n \cdots \zeta_1$ where $x = \sum_{i\geqslant 1} \zeta_i F_i$, and we call this representation ZOL, because it was independently discovered by Ostrowski (in 1922), Lekkerkerker (in 1952), and Zeckendorf (in 1972). Note that F_0 is not used in this representation. In our examples, using a binary word notation, thus $\zeta(11) = 10100$, $\zeta(21) = 1000010$ and $\zeta(30) = 1010001$. The representation is obtained by greedily at each step including the largest Pingala number. It is obvious by the definition of the Pingala numbers that there cannot be two consecutive 1s in ζ. We leave it as an exercise to prove the uniqueness.

One more basic property of ZOL-numeration is that, by adding 1 to a word of alternating 1s and 0s of length k results in the word of length $k+1$ of the form $10\ldots0$. For example $101 + 1 = 1000$ and $1010 + 1 = 10000$. That is, in this numeration, we have $10^k - 1 = (10)^{k/2}$, if k is even, and $10^k = (10)^{(k-1)/2}$, if k is odd. Let us call this property *ZOL-carry*.

Theorem 19. *The first player wins if and only if they can remove the smallest number in the ZOL-decomposition of the heap size.*

Notice that this statement includes the previous one (Theorem 18), since the starting player is not allowed to remove the whole heap. In the example $11 = 8 + 3$, the first player removes 3, and then the second player has to move from the Fibonacci number 8. They can remove $1 \leqslant r \leqslant 6$. If they remove 3 or more they lose in the next move. Otherwise the next player can play to 5, again, a Fibonacci number. In their next move they can either win directly, or, if the other player removed 1, they can take 1 from 4 and reach 3. Now, because the other player can only reach 1 or 2, they win in their next move.

There is a better notation. We write (x, r) for a heap of size x, where the next player is allowed to remove at most r pebbles. Thus, the first position is of the form $(x, x - 1)$, and later $r < 2x$ is some even number.

Proof. Consider a position (x, r). Denote by $Z(x)$ the set of Pingala numbers in the ZOL-representation of x. We must justify the following statements.

(i) If $\min Z(x) > r$, then any removal m satisfies $\min Z(x - m) \leqslant 2m$.
(ii) If $m = \min Z(x) \leqslant r$, then $\min Z(x - m) > 2m$, or $x - m = 0$.

Observe that (i) states that if the current player cannot remove the smallest Pingala component of x, then, for all options $(x - m, 2m)$, the other player can. And (ii) is the opposite statement.

Item (ii) is almost automatic by the definition of ZOL-decomposition. Namely, when the smallest ZOL-component of a number has been removed, then, the second smallest becomes the smallest, but it is distanced by at least one Fibonacci number. And by the definition of the Fibonacci numbers, for all $n \geqslant 2$, with the removal of $F_n = m = \min Z(x)$, say,

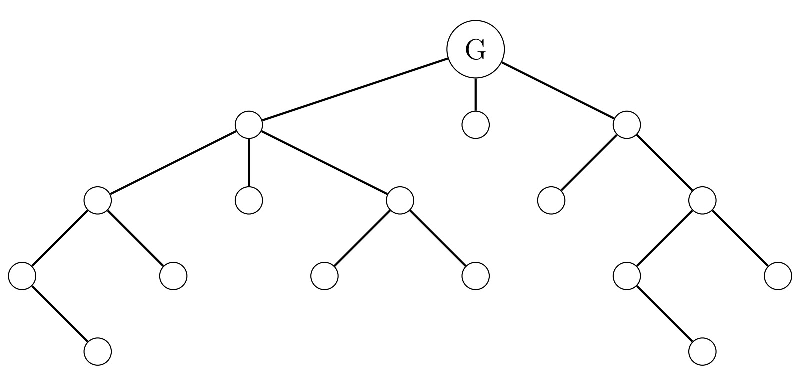
$$2F_n < F_n + F_{n+1} \tag{3}$$
$$= F_{n+2} \tag{4}$$
$$\leqslant Z(x - m), \tag{5}$$

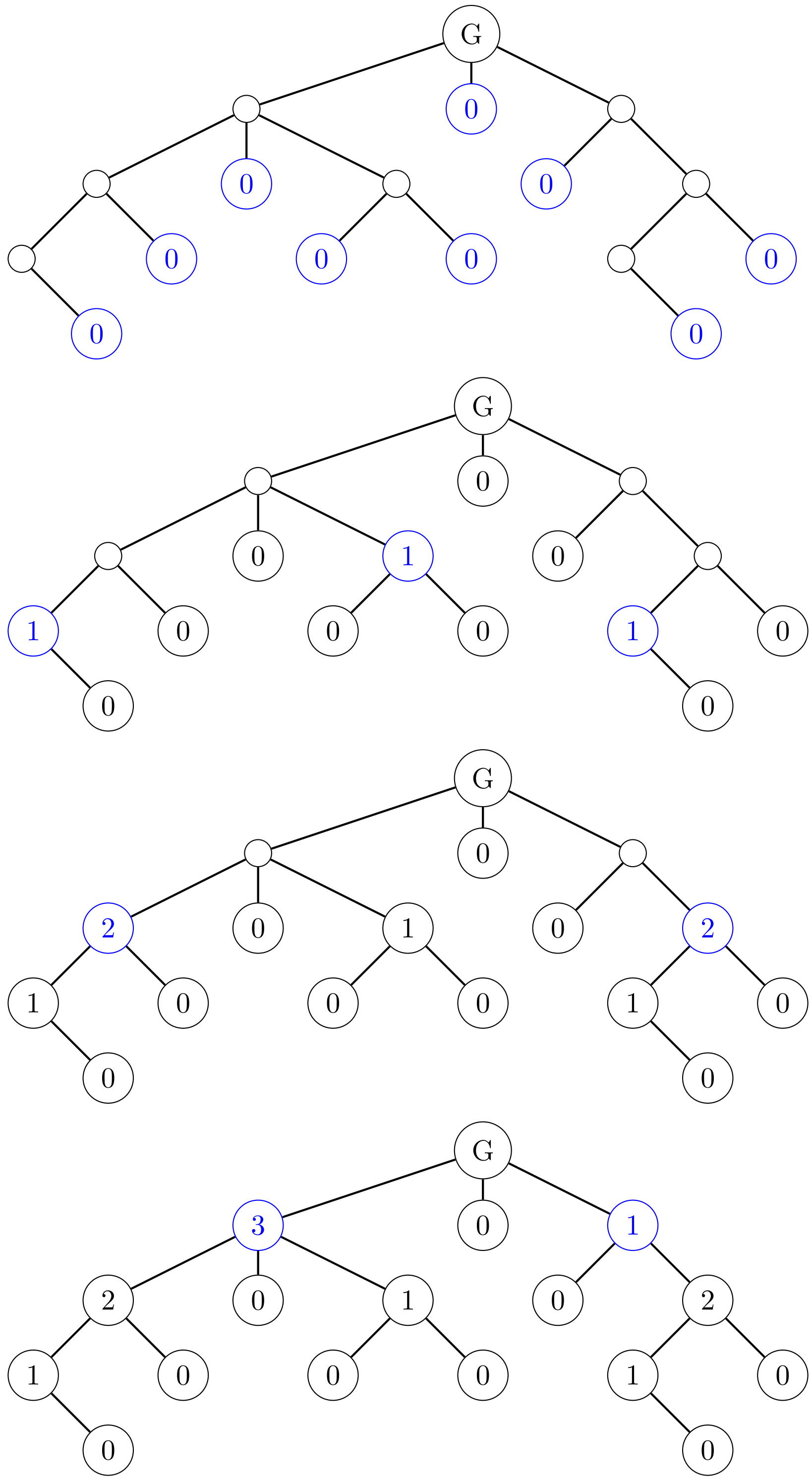
by definition of ZOL-numeration, unless $x - m = 0$.

The resolution of item (i) is a bit more hidden at a first sight, but it will follow by the uniqueness of the ZOL-numeration. Since the removal m is smaller than, say $F_n = \min Z(x)$, then $\min Z(x-m) < F_n$. We may assume that $m < \min Z(F_n - m)$ (because otherwise we are done). Then, because $F_n = m + \sum_{z \in Z(F_n - m)} z$, this implies that m is at least the size of the greatest Fibonacci number smaller than $\min Z(x - m)$, because otherwise the decomposition of F_n would not be unique (as F_n). Hence, the inequality holds. $\square$

9. Sprague and Grundy's contribution

We review the famous Sprague-Grundy Theory. It says that, for any impartial normal play game G, there is a nim-heap h such that played together $G + h \in \mathscr{P}$. Moreover, the proof provides a constructive way to find that nim heap. The *minimal exclusive* function, abbreviated 'mex' finds the nim-value of a given game. The mex-function is defined as follows. Let $X \subset \mathbb{N}_0$ be a strict subset of the non-negative integers. Then $\text{mex}X = \mathbb{N}_0 \setminus X$. For example $\text{mex}\{0, 1, 3, 5, 17\} = 2$ and $\text{mex}\{1\} = 0$. As a motivation, before the proof, let us draw a game tree and compute its equivalent nim-value via the mex-algorithm. Recursively, it computes the nim-values on every sub-position, via the mex-rule, until it finds the root, and assigns its nimber. This is an impartial game so the directions of the slopes do not matter.

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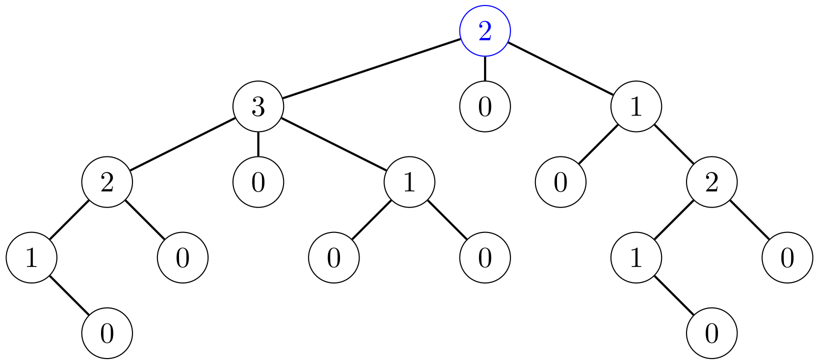


Definition 20 (Subgame)**.** Consider a game G. Then H is a subgame of G if there is a sequence of moves from G to H, not necessarily alternating and perhaps empty.

In the literature, subgame is often called "subposition" or "follower" with exactly the same meaning.

Theorem 21. *Every impartial normal play game is equivalent to a nimber.*

Proof. Consider any impartial normal play game G. The statement holds if G does not have any options, so assume that G has options. Assign each sub position of G without any option nim-value 0. Suppose that the statement is true for all games of birthday less than G. That means in particular that each option of G, say G_i, equals a nim-heap, say $*h_i$. We will demonstrate that G equals the nim-heap $*\text{mex}\{h_i\}$, where i ranges over the options of G. To this purpose we play the game $G + *\text{mex}\{h_i\}$, and demonstrate that it is a $\mathscr{P}$-position. Suppose that the first player plays to $*h_j + G$, for some $h_j < \text{mex}\{h_i\}$ (this is possible by the definition of a nim-heap). Then the second player can respond to $*h_j + G_j = *h_j + *h_j \in \mathscr{P}$. Suppose next that the first player plays in the G component to $G_j + *\text{mex}\{h_i\}$. There are two possibilities:

(i) $h_j > \text{mex}\{h_i\}$;
(ii) $h_j < \text{mex}\{h_i\}$.

In case (i), the second player can respond to $*\text{mex}\{h_i\} + *\text{mex}\{h_i\}$, and win. In case (ii), the second player can respond to $*h_j + *h_j$ and win. □

10. Subtraction games and a ZOL-solution of Wythoff Nim

Subtraction is a generalization of Nim defined by a subtraction set $S \subset \mathbb{N}$. The move options from a heap of size x is the set $\{x-s \mid s \in S, x-s \geqslant 0\}$. For example, if $S = \{1, 3, 4, 7\}$, then the first few outcomes and nimbers are as follows:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$o(x)$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$
nim(x)	0	1	0	1	2	3	2	3	0	1	0	1	2	3	2	3

We conclude that both the outcomes and the nimbers are periodic, with period length 8, and starting at the heap of size 0. How do we know this? It follows by observing a repetition of the content of 7 consecutive squares in either row. Why this 'window' size of 7? This is because this is the maximum size of a subtraction. This idea assists us in the proof that any subtraction game on a finite subtraction set has an *eventually periodic* outcome.[8]

Theorem 22. *Every subtraction game S has eventually periodic outcomes.*

Proof. There are $2^{\max S}$ possible combinations of outcomes $\mathscr{P}$ or $\mathscr{N}$ in a 'window' of size $\max S$. Since this is a finite number, there must exist a smallest heap size x' for which the outcome window $(o(x'),\dots,o(x'+\max S))$ is the same as $(o(y'),\dots,o(y'+\max S))$, for some $y'>x'$. This defines the period. □

Similarly we can prove that the nim-sequence is eventually periodic. The argument is the same, just replace the 2 in the number of outcomes for $|S|+1$ as the number of possible nimbers.

Corollary 23. *Every subtraction game S has eventually periodic nimbers.*

Proof. By the mex rule, since a ruleset S has (at most) $|S|$ options, the largest nimber that can occur is $*(|S|+1)$. The rest of the proof is analogous to that of Theorem 22. □

Many 'small' rulesets have period length equal to the sum of two of the possible subtractions. In our example: $7+1=8$. But there are exceptions. For example, the ruleset $S=\{2,5,7\}$ has period length 22. Moreover many games in SUBTRACTION have a preperiod before the eventual behavior settles. An early example is $S=\{2,4,7\}$. What is its preperiod and period?

Let us return to WYTHOFF NIM. We start with a recap that includes a to do list. There are several representations of the $\mathscr{P}$-positions of WYTHOFF NIM. Let us list a few.

(1) Geometric Approach: A recursive painting of $\mathscr{N}$-positions illuminates smallest missing $\mathscr{P}$-positions. (done)
(2) A mex-Algorithm: This was briefly mentioned in the classes, but not yet included in these notes. (will do)
(3) Golden Section: We have stated the result in Lecture 1, but not yet proved it. (will do)
(4) Wythoff Properties: We listed five properties that uniquely define the Wythoff sequences. Proof included in Lecture ?, but not yet done in class.

[8]Consider a sequence $s=(s_i)$ indexed by the non-negative integers. Then s is 'eventually periodic' if one can decompose the sequence in a finite part $(s_i)_{i\leqslant k}$, the *preperiod*, and an infinite part $(s_i)_{i>k}$, the *periodic* part, for which there is a p such that for all $i>k$, $s_i=s_{i+p}$. If $k=0$ we say that the sequence is *purely periodic*, or just *periodic*.

(5) ZOL-numeration: The $\mathscr{P}$-positions have a nice interpretation in the ZOL-numeration mentioned in the discussion of FIBONACCI NIM (will do here)
(6) A Morphism on Words: Lecture 11.

Let us do (5). First we construct a small table of the first positive integers in ZOL-numeration. We use the binary notation with $F_2 = 1$ as the smallest 'digit', so for example $7 = 5+2 = F_5+F_3 = 1010$ and $13 = F_7 = 100000$.

n	ZOL(n)
1	1
2	10
3	100
4	101
5	1000
6	1001
7	1010
8	10000
9	10001
10	10010

The bold numbers are those that end in an even number of 0s. That is, $1, 3, 4, 6, 8, 9, \ldots$. We note that those coincide with those in the A-sequence of WYTHOFF NIM's $\mathscr{P}$-positions. The B-sequence is obtained by adjoining a '0' to the right of the numbers in the A-sequence. And indeed, this is our next theorem.

Theorem 24. *In ZOL-numeration A_n (B_n) is the nth number that ends with an even (odd) number of 0s, and, in this numeration, for all n, $B_n = A_n0$.*

Proof. Recall the Wythoff properties:

(i) $(a_0, b_0) = (0, 0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

By Theorem 4, it suffices to justify each item. But all items except (iii) are immediate by definition. It remains to prove that for all n, in ZOL-numeration, $A_n0 = A_n + n$. Let us describe A_n as a function of n.

Claim: if ZOL(n) ends in an odd number of 0s, then ZOL(A_n) = ZOL(n)0, and otherwise, ZOL(A_n) = ZOL($n - 1$)1.

In these examples, instead of writing ZOL, we use quotation: $5 =$ "1000", $A_5 = 8 =$ "10000", $B_5 =$ "100000"; $3 =$ "100", $A_4 = 6 =$ "1001", $B_4 = 10 =$ "10010".

Before we prove this claim, let us see that it suffices to prove the result. In the first case, by using the definition of Pingala recurrence,

$$\begin{aligned}
\mathrm{ZOL}(B_n) &= \mathrm{ZOL}(A_n) + \mathrm{ZOL}(n)\\
&= \mathrm{ZOL}(n)0 + \mathrm{ZOL}(n)\\
&= \mathrm{ZOL}(n)00\\
&= \mathrm{ZOL}(A_n)0.
\end{aligned}$$

And notice that $\mathrm{ZOL}(B_n)$ ends in an odd number of 0s, because $\mathrm{ZOL}(n)$ does so. The second case is similar, but it requires a small trick, namely

$$\begin{aligned}
\mathrm{ZOL}(B_n) &= \mathrm{ZOL}(A_n) + \mathrm{ZOL}(n)\\
&= \mathrm{ZOL}(n-1)1 + \mathrm{ZOL}(n)\\
&= \mathrm{ZOL}(n-1)1 + \mathrm{ZOL}(n-1) + 1\\
&= \mathrm{ZOL}(n-1)0 + \mathrm{ZOL}(n-1) + 2\\
&= \mathrm{ZOL}(n-1)00 + 2\\
&= \mathrm{ZOL}(n-1)10\\
&= \mathrm{ZOL}(A_n)0.
\end{aligned}$$

Here it is important to observe that each line is a valid ZOL-representation. In particular, $\mathrm{ZOL}(n-1)10$ is valid, by the following implication: if $\mathrm{ZOL}(n)$ ends in an even number of 0s, then $\mathrm{ZOL}(n-1)$ does not end in a "1". Namely, if $\mathrm{ZOL}(n)$ ends in a "1" then remove it to obtain $\mathrm{ZOL}(n-1)$. Otherwise argue by ZOL-carry as explained in Lecture 8.

Proof of Claim. It is clear that all positive integers that end in an even number of 0s in the ZOL-representation are represented. Therefore it suffices to establish that going from n to $n+1$ implies that the claimed ZOL-representation of A_n is increasing. If $\mathrm{ZOL}(n)$ and $\mathrm{ZOL}(n+1)$ end in the same parity of 0s, there is nothing to prove. Thus there are two cases to check:

(i) $\mathrm{ZOL}(n)$ odd and $\mathrm{ZOL}(n+1)$ even;
(ii) $\mathrm{ZOL}(n)$ even and $\mathrm{ZOL}(n+1)$ odd.

For item (i) it is immediate by the statement that $A_{n+1} - A_n = 1$. For item (ii), going from even to odd when $n \to n+1$, it must be the case that $\mathrm{ZOL}(n)$ ends in a "1", that is the rules of ZOL-carry applies. SImilar to the second case above one can see that in this case $A_{n+1} - A_n = 2$.

The corresponding table begins like this:

n	ZOL(n)	A_n	ZOL(A_n)
1	1	1	1
2	10	3	100
3	100	4	101
4	101	6	1001
5	1000	8	10000
6	1001	9	10001

□

11. More solutions of Wythoff Nim and a brief introduction to game temperature

In this lecture we will start talking about game temperature.

But let us first complete the Wythoff story with items (2) and (6) in Lecture 10.

Theorem 25. *Let the A and B be the increasing sequences that define* Wythoff Nim*'s $\mathscr{P}$-positons. For all $n \in \mathbb{N}_0$, let*

$$\begin{cases} a_n &= \text{mex}\{a_i, b_i \mid 0 \leqslant i < n\}; \\ b_n &= a_n + n. \end{cases}$$

Then, for all n, $a_n = A_n$ and $b_n = B_n$.

Proof. The Wythoff Properties are all satisfied by definition, (i) is immediate. That the a-sequence is increasing as in (ii) follows by the definition of mex. Item (iii) is obvious. Item (iv) can be verified by an inductive argument. Namely, suppose that b_{n-1} is the largest element in $\{a_i, b_i \mid 0 \leqslant i < n\}$. Then $b_n = a_n + n > b_{n-1}$, by using also (ii). Hence there can be no collision. □

The Fibonacci Morphism $\varphi : \{0,1\}^* \to \{0,1\}^*$ is defined by,[9]

$$\begin{cases} \varphi(0) &= 01; \\ \varphi(1) &= 0. \end{cases}$$

If the initial seed is 0, then φ generates the infinite *Fibonacci word*, ω. This word is generated recursively as follows:

$$\varphi(0) = 01;$$
$$\varphi(01) = 010;$$
$$\varphi(010) = 01001;$$
$$\varphi(01001) = 01001010,$$

and so on. Note that $\varphi(010) = \varphi(0)\varphi(01)$, and so on. That is $\omega = \lim_{n\to\infty} \varphi^n(0)$, where, for all $n > 0$, for all $x \in \{0,1\}^*$, $\varphi^n(x) = \varphi^{n-1}(\varphi(x))$, where $\varphi^0(x) = x$. We index the letters in ω by $\mathbb{N}$. We get $\omega_1 = 0, \omega_2 =$

[9]The set of possible finite words on a finite set of letters X is denoted by X^*. A function $f : X^* \to X^*$ is a morphism, under concatenation, if for all $v, w \in X^*$, $f(vw) = f(v)f(w)$.

$1, \omega_3 = 0, \omega_4 = 0, \omega_5 = 1$, and so on. The morphism φ has many interesting properties. For example, for all $n \in \mathbb{N}_0$, the lengths of the words $\varphi^n(0)$ correspond to the Fibonacci numbers F_{n+2}.

The following result relates the occurrences of the 0s and 1s in this word with the $\mathscr{P}$-positions of WYTHOFF NIM.

Theorem 26. *For all $n \in \mathbb{N}$, A_n equals the index of the n^{th} "0" in ω, and for all n, B_n equals the index of the n^{th} "1" in ω.*

Proof. Set $A_0 = B_0 = 0$ to satisfy (i) in the Wythoff Properties. Clearly A is increasing, so (ii) holds. Similarly complementarity, that is (iv) and (v) hold by definition. It remains to justify (iii), the shift property. But this follows by definition of φ; namely, the nth "1" is written when we read the nth "0", and it is shifted n steps, since, at each translation, we write "01" (instead of just "1"). $\square$

Let us prove that the sequences from WYTHOFF NIM, $(\lfloor n\phi \rfloor)$ and $(\lfloor n\phi^2 \rfloor)$, are complementary. That is every positive integer appears in exactly one of these sequences. This can be done in full generality, by instead proving that the sequences $(\lfloor n\alpha \rfloor)$ and $(\lfloor n\beta \rfloor)$ are complementary whenever α and β are irrationals with $1/\alpha + 1/\beta = 1$.

Theorem 27 (Beatty's/lord Rayleigh's Theorem)**.** *Let α and β be irrational numbers with $1/\alpha + 1/\beta = 1$. Then $(\lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n\in\mathbb{N}}$ are complementary.*

Proof. Collision: Since α and β are irrationals, if $\lfloor n\alpha \rfloor = \lfloor m\beta \rfloor = x$ then $x < n\alpha < x + 1$ and $x < m\beta < x + 1$. Thus $x/\alpha + x/\beta < n + m < (x + 1)/\alpha + (x + 1)/\beta$. But then $x < n + m < x + 1$, where all expressions are integers, which is impossible.

Anticollision: Since α and β are irrationals, if $\lfloor n\alpha \rfloor < x < \lfloor (n + 1)\alpha \rfloor$ and $\lfloor m\beta \rfloor < x < \lfloor (m + 1)\beta \rfloor$, for an integer x, we get $n\alpha < x < (n + 1)\alpha - 1$ and $m\beta < x < (m + 1)\beta - 1$. Thus, $n + m < x/\alpha + x/\beta < n + m + 2 - 1/\alpha - 1/\beta$. That is, $n + m < x < n + m + 1$, where all entries are integers, which is impossible. $\square$

Let us prove Theorem 1 by using the Wythoff Properties from page 8. Let us recall them here:

(i) $(a_0, b_0) = (0, 0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

Proof of Theorem 1. By Theorem 1 and Theorem 4, it suffices to justify that, for all non-negative integers n, $(a_n, b_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$. Item (i) is immediate by setting $n = 0$. Item (ii) is immediate, because $1 < \phi$. And (iii) follows from $\phi^2 = \phi + 1$. Items (iv) and (v) follow from Theorem 27 $\square$

Similarly the mex-description in Theorem 25 easily follows from the Wythoff Properties (check this!). For Theorems 24 and 26 the interesting Wythoff Property is the shift property (iii).

Game temperature. There are *cold* games, there are *tepid* games and there are *hot* games. HACKENBUSH is an example of a ruleset for which all games are cold. The game is played with red or blue pieces stacked upon each other in various directions. Red can remove red pieces and Right can remove blue pieces. Any piece that ceases to have a connection to the ground falls off and is no longer part of the game. Let us list a few examples together with their game values (they will studied later in this section).

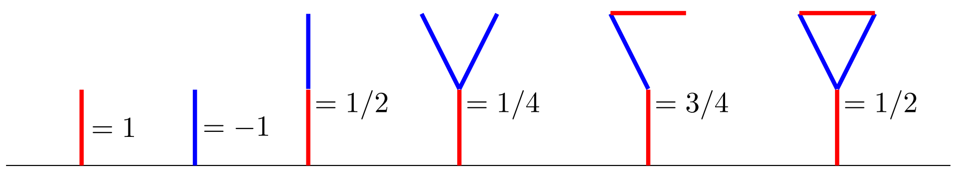


FIGURE 5. Some HACKENBUSH positions and their values.

CLOBBER is an example of a ruleset with only tepid positions (in fact the games are so-called *all-small*). TOPPLING DOMINOES is a ruleset with a variety of positions, in particular, we can build arbitrarily hot positions; imagine a stretch of pieces throughout the room, with same colored pieces to the left and right of the middle respectively. Such positions can be made arbitrarily hot, that is, the first player can gain a huge number of 'free moves'. On the other hand, we can show that HACKENBUSH does not even have any $\mathscr{N}$-positions. (This problem might be a quiz at some point, so you may want to start thinking about it.)

The right most picture in Figure 5 reduces to $1/2$. This can be proved in a similar fashion to what we usually do (that is, by using domination and/or reversibility or, by justifying this guess, by showing that $G - 1/2 \in \mathscr{P}$. But here we will discover an easier way that only applies to games that are *numbers*.

12. GAMES THAT ARE NUMBERS

Let us start by defining *cold* games; they are all *numbers*. We use the word 'numbers' interchangeably in two ways, either as usual, or as defined here, where 'numbers' are a special type of games. This abuse (or double use) of language is due to that games that are numbers will inherit their basic properties from our standard total order and arithmetics.

Definition 28 (Number)**.** The game G is a number, if for all options G^L and G^R, $G^L < G^R$, and all options are numbers.

Theorem 29. *Consider a number G. Then, for all options, $G^L < G < G^R$.*

Proof. Note that $o_L(G - G^L) = \mathrm{L}$, because she can play to $G^L - G^L$ (this is true for all games, not just numbers). We must prove that $o_R(G - G^L) = \mathrm{L}$, if G is a number. Right can play to some $G^R - G^L > 0$ (a Left win), by assumption, so assume instead that he plays to some $G - G^{LL}$. Then, since G^L is a number, induction gives $o_R(G^L - G^{LL}) = \mathrm{L}$. □

Consider two games G and H. Then G is *simpler* than H, if its canonical form has smaller birthday than that of the canonical form of H.

We will show that the non-zero canonical form numbers have a one-to-one correspondence with the *dyadic rationals*, that is all rationals of the form $\frac{m}{2^k}$, for odd m, $k \in \mathbb{N}$. That is, later on we will be able to identify the games that are numbers with the dyadic rationals and 0. The dyadic game forms are defined as follows.

Definition 30 (Dyadic Game)**.** For all $k \in \mathbb{N}_0$, let the game $1/2^k = \left\{0 \mid 1/2^{k-1}\right\}$. The game $m/2^k$ is m copies of $1/2^k$ in a disjunctive sum.

If $k = 0$, then this definition provides all integer games (note that $\left\{0 \mid 1/2^{-1}\right\} = \{0 \mid 2\} = 1$). If $k \geqslant 1$, then this says that $1/2 = \{0 \mid 1\}$, $1/4 = \{0 \mid 1/2\}$, and so on.

We can prove a couple of nice properties of the dyadic games.

Theorem 31 (Dyadic Properties)**.** *For all $k \in \mathbb{N}$,*

(a) $1/2^k > 0$*;*
(b) $1/2^k > 1/2^\ell$*, if* $\ell > k$*;*
(c) $1/2^k + 1/2^k = 1/2^{k-1}$*;*
(d) *for all* m*,* $\frac{m}{2^k} = \left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\}$*, and this game is in canonical form.*

Proof. For (a) we prove that Left wins playing first or second. Playing first, she wins in her first move. Playing second, she wins by induction, since Right plays to $1/2^{k-1}$. The base case is the game 1 when $k = 0$.

For (b), we prove that Left wins $1/2^k - 1/2^\ell$, if $\ell > k$. Left wins playing first to $1/2^k - 1/2^{\ell-1}$, by induction, or by mimic. If Right starts, he loses, by (a), by playing to $1/2^k$, and he loses by induction, by playing to $1/2^{k-1} - 1/2^\ell$.

For (c), we prove that $1/2^k + 1/2^k - 1/2^{k-1} \in \mathscr{P}$. If Right starts, he plays either to $1/2^k + 1/2^k > 0$, by (a), or he plays to $1/2^k + 1/2^{k-1} - 1/2^{k-1} = 1/2^k > 0$, by (a). If Left starts, she plays either to $1/2^k - 1/2^{k-1} < 0$, or $1/2^k - 1/2^{k-2} < 0$, both by (b).

For (d), we prove that

$$\left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\} - \frac{m}{2^k}$$

is a $\mathscr{P}$-position. If Left starts by playing to $\frac{m-1}{2^k} - \frac{m}{2^k}$, then Right can respond to $\frac{m-1}{2^k} - \frac{m-1}{2^k} = 0$. If Left starts by playing to $\left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\} - \frac{m-1}{2^k} - \frac{1}{2^{k-1}}$, then Right can play to $\frac{m+1}{2^k} - \frac{m-1}{2^k} - \frac{1}{2^{k-1}} = \frac{2}{2^k} - \frac{1}{2^{k-1}} = 0$. Similarly, if Right starts by playing to $\frac{m+1}{2^k} - \frac{m}{2^k}$, then Left can play to $\frac{m}{2^k} - \frac{m}{2^k} = 0$. And if Right plays in the second sum-component, then Left responds to $\frac{m-1}{2^k} - \frac{m-1}{2^k} = 0$.

															0
							1								
			2								1/2				
	3				3/2				3/4				1/4		
4		5/2		7/4		5/4		7/8		5/8		3/8		1/8	

FIGURE 6. The birth of integers and dyadic rationals coincide with the birth of games that are numbers; the first row is birthday 0 games, the second row is birthday 1 games, and so on. In both cases, the negatives are symmetrically obtained.

Next we prove that this is the canonical form. There is no domination since there is only one option. The Left option is reversible, since its Right option $\frac{m-2}{2^k} + \frac{1}{2^{k-1}} = \frac{m}{2^k}$, but when we replace it with its set of Left options, we find by induction that gives again the same option. Hence, no further reduction is possible. □

Let us define the nonnegative integers and dyadic rationals recursively via their birthdays as in Figure 6 (the negative ones are analogously defined). First think of them in the usual way as we know from high school. We will establish that this construction follows the birthdays of the corresponding games that are numbers. We start with $D_0 = \{0\}$, $D_1^+ = \{0, 1\}$, $D_2^+ = \{0, 1/2, 1, 2\}$, $D_3^+ = \{0, 1/4/, 1/2, 3/4, 1, 3/2, 2, 3\}$,

$$\begin{aligned} D_4^+ &= D_3^+ \cup \{1/8, 3/8, 5/8, 7/8, 5/4, 7/4, 5/2, 4\} \\ &= \{0, 1/8, 1/4/, 3/8, 1/2, 5/8, 3/4, 7/8, 1, 5/4, 3/2, 7/4, 2, 5/2, 3, 4\}. \end{aligned}$$

In general $D_n^+ = D_{n-1}^+ \cup \left\{n, \frac{d_i + d_{i+1}}{2} \mid d_i, d_{i+1} \in D_{n-1}^+\right\}$. For all n, let $D_n^- = \{-x : x \in D_n^+\}$. If we let $n \to \infty$, then this construction gives all integers and dyadic rationals. Let $D = \bigcup_n D_n$.

Observe, that, by construction, if $x, y \in D_n$, with $x < y$, then there is a unique z, such that $x < z < y$, and $z \in D_i$ for some smallest $i < n$. This is the *simplest* dyadic between x and y.

Theorem 32 (Number Simplicity)**.** *Consider a number game G. Then G equals the simplest dyadic x such that, for all options G^L and G^R, $G^L < x < G^R$. And x is the canonical form of G.*

Proof. By induction, we may assume that all options of G are simplest form dyadic games. Then, we may use domination to single out one Left and one Right option such that $G = \left\{G^L \mid G^R\right\}$. Since G is a number, we know that $G^L < G < G^R$.

Suppose next that x is the simplest dyadic such that

$$G^L < x < G^R. \tag{6}$$

We demonstrate that $G - x \in \mathscr{P}$.

We begin by proving that $o_{\mathrm{L}}(G - x) = \mathrm{R}$. By assumption $G^L - x < 0$. Hence, suppose instead that Left starts by playing to $G - x^R$. Since x^R is simpler than x, then, by (6), either,

(a) $x^R \not> G^L$, or
(b) $x^R \not< G^R$.

Observe that all games are numbers, so no two games are fuzzy. If (a) then, because x is a number, then $x < x^R \leqslant G^L < x$, which is impossible. If (b), then Right can respond to $G^R - x^R \leqslant 0$, and win.

Secondly, let us prove that $o_{\mathrm{R}}(G - x) = \mathrm{L}$. If Right plays to $G^R - x > 0$, he loses. Suppose therefore that Right plays to $G - x^L$. Since x^L is simpler than x, by (6) (and by no two numbers fuzzy), either

(a) $x^L \leqslant G^L < x$, or
(b) $x^L \geqslant G^R > x$.

In case of (a), Left can respond to $G^L - x^L \geqslant 0$, and win. The case (b) contradicts that x is a number.

For the last part we may write $x = \frac{m}{2^k} = \left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\}$, so $x^L = \frac{m-1}{2^k} = \frac{n}{2^j} = \left\{\frac{n-1}{2^j} \mid \frac{n+1}{2^j}\right\}$, for n odd. Hence $x^{LR} = \frac{n+1}{2^j} \geqslant \frac{m}{2^k}$, with equality only if $j = k - 1$ and $m - 1 = n/2$, in which case $x^{LRL} = \frac{m-1}{2^k}$. And otherwise reversibility is not possible. And there is no domination. □

13. Number translation and avoidance

Number avoidance means that, in a disjunctive sum of games, every player avoids playing in a number component, unless there is nothing else to do. There is a slick symbol for "Left wins playing first", that is "Right does not win playing second": $0 \lhd\!\mid G$ has the same meaning as $0 \not\geqslant G$. The following are general lemmas.

Lemma 33. *Let $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, and let $H = \{G^{\mathcal{L}}, A \mid G^{\mathcal{R}}\}$, with $A \lhd\!\mid G$. Then $G = H$.*

Proof. A mimic strategy suffices to prove that $G - H \in \mathscr{P}$, unless Right plays to $G - A$. But then the result follows by $G - A \mid\!\rhd 0$; Left wins playing first. □

Lemma 34. *For any game G, and all left options G^L, $G^L \lhd\!\mid G$.*

Proof. The game $G^L - G \lhd\!\mid 0$, since Right wins playing first to $G^L - G^L$. □

We are aiming to prove a "translation property" for numbers. This is also a strong version of "number avoidance": you do not play in a number unless there is nothing else to do.

Theorem 35 (Weak Number Avoidance)**.** *If Left can win $G+x$, where x is a number and G is not a number, then she has a winning move of the form G^L+x.*

Proof. Quiz. □

This result is a consequence of the following result, but it turns out that we need Theorem 35 to prove an important lemma. We must be careful to not run into cycles.

Theorem (Number Translation - Number Avoidance). Suppose that x is a number game, and G is a game that is not a number. Then, $G+x=\left\{G^{\mathcal{L}}+x \mid G^{\mathcal{R}}+x\right\}$.

It is not possible to prove this result by using only what we learnt so far. (We tried it out unsuccessfully last lecture.) Let us develop some more theory to prove this result, and we will restate it as Theorem 41.

Combinatorial game theory has min/max type functions that are also common in classic game theory. Here they are called the Left- and Right stops respectively:

Definition 36. The Left- and Right stops are,

$$\operatorname{Ls}(G)=\begin{cases} x, & \text{if } G=x \text{ is a number,}\\ \max \operatorname{Rs}(G^L), & \text{otherwise.}\end{cases}$$

$$\operatorname{Rs}(G)=\begin{cases} x, & \text{if } G=x \text{ is a number,}\\ \min \operatorname{Ls}(G^R), & \text{otherwise.}\end{cases}$$

Here max and min ranges over all the Left and Right options, respectively. These functions are useful in many way to analyze our games. In particular, we get a very slick proof of the Number Translation Theorem, Theorem 41. It will depend on some more lemmas though.

Lemma 37. *For any game G, $\operatorname{Ls}(G) \geqslant \operatorname{Rs}(G)$.*

Proof. The proof is by way of contradiction. Suppose $\operatorname{Ls}(G)<\operatorname{Rs}(G)$. Then there is a dyadic number x such that $\operatorname{Ls}(G)<x<\operatorname{Rs}(G)$. Here we can use weak number avoidance. If Left starts in the game $G-x$, then, if she has a winning move it is to some G^L-x. Similarly, unless G^L is a number, if Right has a winning move it must be of the form $G^{LR}-x$. And so on. But then, since $\operatorname{Ls}(G)-x<0$ by assumption, Right wins when Left starts $G-x$. Similarly, by using $0<\operatorname{Rs}(G)-x$ we obtain that Left wins when Right starts $G-x$. Therefore $G=x$, a number, which contradicts $\operatorname{Ls}(G)<\operatorname{Rs}(G)$. □

Lemma 38. *For any games G and H, $\operatorname{Rs}(G+H) \geqslant \operatorname{Rs}(G)+\operatorname{Rs}(H)$.*

Proof. In the game $G+H$, if Right starts by playing to G^R+H, then, unless G^R is a number, Left can respond to $G^{RL}+H$, and if Right starts by playing to $G+H^R$, then Left can respond to $G+H^{RL}$. Then use induction. If $G^R = x$ is a number, then $\mathrm{Rs}(G+H) = x + \mathrm{Ls}(H) \geqslant \mathrm{Rs}(G) + \mathrm{Rs}(H)$, by Lemma 37. $\square$

Lemma 39. *For any game G that is not number, there is a Left option G^L such that $\mathrm{Rs}(G^L - G) \geqslant 0$.*

Proof. We get

$$\mathrm{Rs}(G^L - G) \geqslant \mathrm{Rs}(G^L) + \mathrm{Rs}(-G) \tag{7}$$
$$= \mathrm{Rs}(G^L) - \mathrm{Ls}(G) \tag{8}$$
$$= \mathrm{Ls}(G) - \mathrm{Ls}(G) = 0, \tag{9}$$

where (7) is by Lemma 38, and (9) is by assuming that G^L is such that $\mathrm{Ls}(G) = \mathrm{Rs}(G^L)$. $\square$

Lemma 40. *Suppose that G is such that $\mathrm{Rs}(G) \geqslant 0$. Then Left wins $G+x$, if x is a positive number.*

Proof. Clearly this holds if G is a number. Suppose Right starts by playing in the G component. Then, unless G^R is a number, Left has a response G^{RL} such that $\mathrm{Rs}(G^{RL}) \geqslant 0$. Use induction. If G^R is a number, then this number is no smaller than 0, and so Left wins $G+x$. If Right starts by playing in the x component, to say $G + x^R$, then we can use induction, since $x^R > x$ (or use weak avoidance). If Left starts, observe that Lemma 37 implies that $\mathrm{Ls}(G) \geqslant \mathrm{Rs}(G) \geqslant 0$. That is, she has a move such that $\mathrm{Rs}(G^L) \geqslant 0$. Now use induction. $\square$

Let us restate the result we are aiming at.

Theorem 41 (Number Translation - Number Avoidance)**.** *Suppose that x is a number game, and G is a game that is not a number. Then, $G + x = \left\{G^{\mathcal{L}} + x \mid G^{\mathcal{R}} + x\right\}$.*

Proof. By definition of disjunctive sum,

$$G + x = \left\{G^{\mathcal{L}} + x, G + x^{\mathcal{L}} \mid G^{\mathcal{R}} + x, G + x^{\mathcal{R}}\right\}.$$

Thus, by domination, it suffices to prove that there exists a G^L such that $G^L + x \geqslant G + x^L$, for any x^L. We combine Lemma 39 with Lemma 40; together they establish that there exists G^L such that $G^L - G + x - x^L > 0$. $\square$

14. In a very small world

If x is a positive number (which means that Left wins playing first or second in x), then $x > *$.

Lemma 42. *For any number game $x > 0$, then $x > *$.*

Proof. Consider the game $x + *$, where x is a number. Left playing first to x wins. If Right starts by playing to x, then Left wins. If Right plays to $x^R + *$, then Left wins by induction since $x^R > x$ is a number. $\square$

Note that $* \parallel 0$. There are also positive games so small that, however many copies you add, you will not reach one move for Left, the game 1. One such game is $\uparrow = \{0 \mid *\} > 0$. A related game is the fuzzy game $\uparrow* = \uparrow + * = \{0, * \mid 0\}$. Fuzzy means that it is an $\mathscr{N}$-position, that is, incomparable with 0.

Definition 43. The game g is an infinitesimal, with respect to numbers, if for all positive numbers x, $-x < g < x$.

Mostly we omit the part "with respect to numbers", but as we will see later there is an infinite hierarchy of games that are infinitesimals with respect to each other. We have already observed that the the game $*$ is an infinitesimal. However $*$ is confused with 0. The following two results are perhaps more remarkable.

Theorem 44. *There are positive infinitesimals, with respect to numbers.*

Proof. The game $\uparrow$ is positive. We will demonstrate that, for all $n \in \mathbb{N}$, $n \cdot \uparrow < 1$. Suppose that Left starts in the game $1 + n \cdot \downarrow$. She plays to $1 + * + (n-1) \cdot \downarrow$. If Right responds to $1 + (n-1) \cdot \downarrow$, then she wins by induction, and if he plays to $1 + * + (n-2) \cdot \downarrow$, then Left can play to $1 + (n-2) \cdot \downarrow$, and win by induction. Suppose next that Right starts in the game $1 + n \cdot \downarrow$. Then he plays to in the game $1 + (n-1) \cdot \downarrow$, and Left wins by induction. $\square$

In one lecture we mentioned briefly another choice, the class of so-called *uptimals*, a sequence of the form $\uparrow > \uparrow^2 > \uparrow^3 > \ldots > 0$. They build infinite hierarchies of yet smaller positive games. By definition, the game $\uparrow^n = \left\{0 \mid * - \uparrow - \uparrow^2 - \cdots - \uparrow^{n-1}\right\}$. For example $\uparrow^2 = \{0 \mid \downarrow*\}$. Moreover, the uptimals are infinitely small with respect to each other.

Theorem 45. *Fix a positive integer n. Then, for all positive integers m $\uparrow^n > m \cdot \uparrow^{n+1}$.*

Proof. Exercise! $\square$

This result motivates a special notation for uptimals. We use a standard positional numeration system, but where the digit denotes the number of the given uptimal, respectively. For example $0.10\overline{2}3 = \uparrow + 2 \cdot \downarrow^3 + 3 \cdot \uparrow^3$ (here we use $\overline{x}$ to denote the negative of a positive uptimal x).

The next result concerns even smaller games, called Tiny,

$$+_1 = \{0 \mid \{0 \mid -1\}\}$$

and Miny,

$$-_1 = \{\{1 \mid 0\} \mid 0\}$$

Theorem 46. *There are positive infinitesimals, with respect to any uptimal $\uparrow^n$.*

Proof. Exercise! (Try with Miny and Tiny.) $\square$

This is the content of the next lecture. There is some overlap.

14.1. **More on Infinitesimals - a TOPPLING DOMINOES approach, by Anjali.**

Definition 47. A short game G is infinitesimal if $x > G > -x$ for every positive number x.

Consider the combinatorial game of TOPPLING DOMINOES. There are three types of dominoes in the game; red which can be toppled by Left, blue which can be toppled by Right and green which can be toppled by both. Take the following game with two red dominoes as in Figure 7. This game has the value of 2.

Now, let's add a blue domino in between the two red domino in figure 8. Let this game be G_1. This has the value

$$G_1 = \{0, * \mid 1\}$$

The second left option is reversible, and hence

$$\begin{aligned} G_1 &= \{0 \mid 1\} \\ &= \frac{1}{2} \end{aligned}$$

Adding one more blue domino in the middle gives us the game in Figure 9. Let this game be G_2. This game has the value

$$G_2 = \{0, \{0| - 1\} \mid 1, *\}$$

The Left option $\{0| - 1\}$ is reversible and $1 < *$. Thus,

$$\begin{aligned} G_2 &= \{0 \mid *\} \\ &= \uparrow \end{aligned}$$

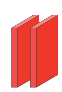


FIGURE 7. $G = 2$

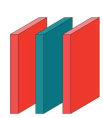


FIGURE 8. $G_1 = \frac{1}{2}$

FIGURE 9. $G_2 = \uparrow$

We can compare this game with any number, say 1. We find that $1 > \uparrow$. We want to find out how many $\uparrow$s it might take for the value to be more than 1, if possible at all. Actually, by induction, we can prove that, $\forall n \geqslant 1$,

$$1 > n \cdot \uparrow.$$

The game $\uparrow$ is an infinitesimal. It has a very small game value, infinitesimally small with respect to the dyadic rationals. Consider the sequence $1, 1/2, 1/4, 1/8, \ldots$. It rapidly tends to 0, right? Well, in the amazing world of combinatorial games, there is some space between this infinite sequence that 'converges to 0', and the game 0. And now we will demonstrate what this means.

Now, let's add one more blue domino in the middle so that, now there are three blue dominoes in between two red domino. See figure 10. Let this game be G_3.

$$G_3 = \{0, \{0| - 2\} \mid \{0| - 1\}, *\}$$

The second Left option is reversible and the second Right option is dominated by $\{0| - 1\}$.

$$\begin{aligned} G_3 &= \{0 \mid \{0| - 1\}\} \\ &= ⧾_1 \end{aligned}$$

This type of games are called *Tiny* and it is a type of infinitesimal. This particular game is Tiny(1) written as $⧾_1$.

Definition 48. For all $G > 0$, the games $⧾_G$ and $⧿_G$ are defined by

$$⧾_G = \{0 \mid \{0 \mid -G\}\} \text{ and } -(⧾_G) = ⧿_G = \{\{G \mid 0\} \ \mid \ 0\}.$$

We can prove that $\uparrow > ⧾_1$. Moreover, we have the following theorem.

Theorem 49. *$⧾_1$ is infinitesimally small with respect to $\uparrow$. That is, $\forall n \geqslant 1$,*

$$\uparrow > n \cdot ⧾_1.$$

Proof. We want to find the outcome of $\uparrow + n \cdot ⧿_1$ where $⧿_1 = \{\{1 \mid 0\} \ \mid \ 0\}$.

Case 1: Left starts. If Left moves in $\uparrow$ to go to $0 + ⧿_1$ then Right would win the game since, $⧿_1 < 0$. Hence, Left moves in one of the $⧿_1$ to go to

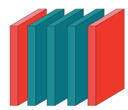


FIGURE 10. $G_3 = \boldsymbol{+}_1$

$\uparrow + \{1 \mid 0\} + (n-1) \cdot \boldsymbol{-}_1$. Then, Right moves to $* + \{1 \mid 0\} + (n-1) \cdot \boldsymbol{-}_1$. By induction we prove that $\uparrow + n \cdot \boldsymbol{-}_1$ is won Left when Left starts.

Case 2: Right starts. Right has advantage in $\boldsymbol{-}_1$ as $\boldsymbol{-}_1 < 0$ so Right moves in $\uparrow$ to go to $* + n \cdot \boldsymbol{-}_1$. This is a $\mathcal{N} - position$ since Right can undo all Left moves in $\boldsymbol{-}_1$. Left will not move in $*$ unless necessary and win the game. If at certain point Right moves in $*$ then Left will gain an advantage by going to $1 + m \cdot \boldsymbol{-}_1$, where $m \leqslant n$ and Left would win similarly.

Therefore, the game $\uparrow + n \cdot \boldsymbol{-}_1$ is an $\mathcal{L} - position$ which implies $\uparrow > n \cdot \boldsymbol{+}_1, \forall n \geqslant 1$. □

We can now easily make the game $\boldsymbol{+}_2$ by adding one more blue domino in the middle, making it 4 instead of 3 blue dominoes in Figure 10, and so on. We have the following result.

Lemma 50. *The game $\boldsymbol{+}_2$ is infinitesimally small with respect to $\boldsymbol{+}_1$. That is, $\forall n \geqslant 1$,*

$$\boldsymbol{+}_1 > n \cdot \boldsymbol{+}_2.$$

Proof. To prove this, we first show that $\boldsymbol{+}_1 + \boldsymbol{-}_2 > 0$ which implies $\boldsymbol{+}_1 > \boldsymbol{+}_2$.

$$\boldsymbol{+}_1 = \{0 \mid\mid 0 \mid -1\}$$
$$\boldsymbol{+}_2 = \{0 \mid\mid 0 \mid -2\}$$
$$\boldsymbol{-}_2 = \{2 \mid 0 \mid\mid 0\}$$

Case 1: Left moves first

Left makes a move to go to the game $\boldsymbol{+}_1 + \{2 \mid 0\}$ since $\boldsymbol{+}_1$ is positive game and it is advantage for the Left. Next, the Right's best move to replicate Left's move in $\boldsymbol{+}_1$ and the game becomes $\{0 \mid -1\} + \{2 \mid 0\}$. Now, the Left can move to the game $\{0 \mid -1\} + 1$ which gives advantage of free move to the Left. This game has only one move for the Right so the game becomes 1 and thus, Left wins.

Case 2: Right moves first

Right best move is disrupt the positive part of the game by moving to the game $\{0 \mid -1\} + \boldsymbol{-}_2$. Next, Left replicates the last move by Right and the game becomes $\{0 \mid -1\} + \{2 \mid 0\}$. Now, Right's best move to have a free move by going to $-1 + \{2 \mid 0\}$. Left now moves to, remove the all the Right domino that Left is able to, in order to force Right to let go of the free move. Thus, the Left moves to $-1 + 1$. Now, Right has only one move left and the game becomes 1. Thus, Left wins.

Therefore, we see that Left can force a win regardless of who moves first. Hence, we have

$$\begin{aligned} \boldsymbol{+}_1 + \boldsymbol{-}_2 &> 0 \\ \implies \boldsymbol{+}_1 &> \boldsymbol{+}_2 \end{aligned}$$

The proof then follows through induction. □

This lemma gives the following theorem.

Theorem 51. *Let G and H be two game such that $G > H > 0$. Then, $\boldsymbol{+}_H$ is infinitesimally small with respect to $\boldsymbol{+}_G$ i.e.*

$$\boldsymbol{+}_G \geqslant n \cdot \boldsymbol{+}_H, \ \forall n \geqslant 1.$$

Proof. The proof follows through induction using the previous lemma. □

15. An overview of atomic weight theory

In this section we do not prove any result. But you will get to know "the raison d'être" for atomic weight theory.

This is an introduction to the *atomic weights* of *all-small* combinatorial games. In an all-small combinatorial game, for all subgames, either both players can move or neither can. Let us begin with some ruleset example, a disjunctive sum of a FLOWER GARDEN with a single strip of BIPASS. The ruleset FLOWER GARDEN is a subset of HACKENBUSH; it has a green stalk of integer length, and on top a flower, with blue and red petal leaves. See Figure 13 to the right.

A bi-collective of one-directional micro organisms, consisting of a red tribe and a blue tribe, live in close proximity, and they take turns moving. The red tribe moves by letting one of its members crawl rightwards across a number of blue amoebae, while settling in the spot of a blue amoeba, and thus pushing each bypassed amoeba one step to the left, whereas the blue tribe moves by letting one of its members crawl leftwards across a bunch of red amoebae, while shifting the position of each bypassed amoeba one step to the right. Amoebae cannot bypass their own kind. When an amoeba reaches end of line, it cannot be played, and thus dies (of boredom). See Figure 11. The exception is if no more moves are possible in the full collective; in this case the last moving tribe wins, and is rewarded eternal life, as in Figure 12. This ruleset is called BIPASS.

Who wins the game in Figure 13, and why? To understand this, let us dwell a bit on the theory of atomic weights (from the books).

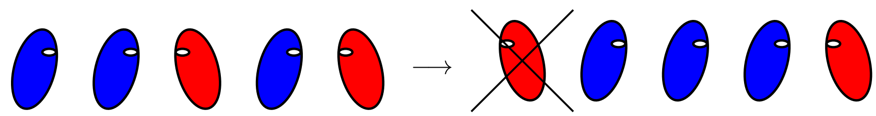


FIGURE 11. The middle amoeba crawled to the left end. When an amoeba does not face any opponent, even at a far distance, it gets removed, because it cannot be used in the game by either player.

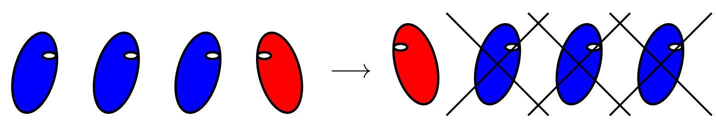


FIGURE 12. By moving, the single white amoeba bypassed all remaining amoebae, and will be celebrated as a hero by its resurrected tribe.

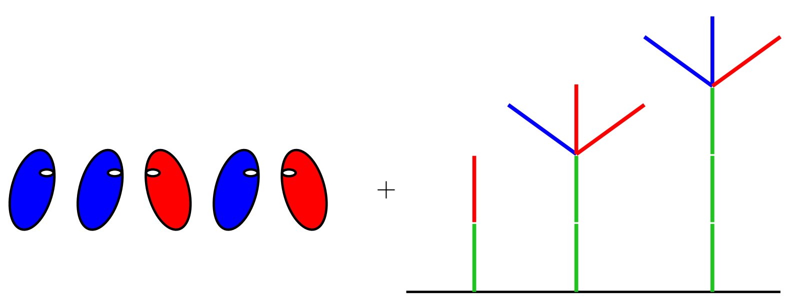


FIGURE 13. A BIPASS strip in disjunctive sum with a FLOWER GARDEN.

Definition 52 (Far Star)**.** The *far star*, denoted ☆, is an arbitrarily large NIM heap; that is both players can move to any NIM heap from ☆. It has the additional property that $\text{☆} + \text{☆} = \text{☆}$.

Equivalence modulo ☆ is obtained as follows.

Definition 53 (Equivalence Modulo ☆)**.** Let G, H be normal play games. Then $G \geqslant_{\text{☆}} H$, if, for all games X, $o(G + X + \text{☆}) \geqslant o(H + X + \text{☆})$, and $G =_{\text{☆}} H$ if $G \geqslant_{\text{☆}} H$ and $H \geqslant_{\text{☆}} G$.

The 'game' ☆ may be treated as a standard combinatorial game, since, for each game G, for any sufficiently large n, $o(G + {*n})$ is constant. We may take n greater than the birthday of G.

Theorem 54 (Constructive ☆-equivalence)**.** *Let G, H be normal play games. Then $G \geqslant_{\text{☆}} H$ if and only if $G - H > \downarrow + \text{☆}$, and $G =_{\text{☆}} H$ if and only if $\downarrow + \text{☆} < G - H < \uparrow + \text{☆}$.*

For example, we have $G =_{☆} 0$ if and only if Left wins $G + \uparrow + *n$ and Right wins $G + \downarrow + *n$ for some sufficiently large n, and n is sufficiently large if $*n$ does not equal any follower of $G + \uparrow$ or $G + \downarrow$. This motivates the naming as a constructive ☆-equivalence. This method is useful also in proofs, for example whenever we have a good guess of one of the games G or H, then we use this result to verify whether our candidate games are equal. See Theorem 59 for how it applies to atomic weight theory.

Let X be a set. Then $X + y = \{x + y : x \in X\}$. If X is a set of all small games, let $\mathrm{aw}(X) = \{\mathrm{aw}(x) : x \in X\}$.

The product of a game G and $\uparrow$ is: $0 \cdot \uparrow = 0$; $n \cdot \uparrow = \uparrow + (n-1) \cdot \uparrow$, in case $G = n$ is an integer. Otherwise $G \cdot \uparrow = \{G^{\mathcal{L}} + \Uparrow * \mid G^{\mathcal{R}} + \Downarrow *\}$.

Lemma 55. *Consider any normal play game G.*

- *If $G \cdot \uparrow \geqslant_{☆} 0$ then $G \geqslant 0$.*
- *If g is all-small then there is a unique G such that $g =_{☆} G \cdot \uparrow$.*

In a proof of this lemma, the first item implies the second, and we use the uniqueness to define atomic weight.

Definition 56 (Atomic Weight)**.** The atomic weight of an all-small game g is the unique game $G = \mathrm{aw}(g)$ such that $G \cdot \uparrow =_{☆} g$.

Example 57. Let $g = *n$. By Theorem 54, $\mathrm{aw}(*n) = 0$. Namely, we claim that Left wins $*n + \uparrow + ☆$ (and symmetrically Right wins $*n + \downarrow + ☆$). If Left starts, she can play to $\uparrow > 0$. If Right starts by playing to $*m + \uparrow + ☆$, then Left wins (by induction). If he plays to $*(n+1) + ☆$, then Left can respond to $*(n+1) + *(n+1) = 0$. If he plays to $*n + \uparrow + *m \mathrel{|\!\!>} 0$, then Left wins playing first (Right can win playing first if and only if $*n + *m = *$).

Example 58. Let $g = \uparrow$ and let $h = \uparrow *$. By Theorem 54, $\mathrm{aw}(g) = \mathrm{aw}(h) = 1$. Namely $☆ < \uparrow$ and $☆ < \uparrow *$. By using also symmetry, the verification is similar to the one in Example 57.

In this example we had to guess the atomic weight 1 and then verify. There is a constructive/recursive method for computing atomic weights.

Theorem 59 (Constructive Atomic Weight)**.** *Let g be an all-small game, and let*

$$G = \left\{\mathrm{aw}(g^{\mathcal{L}}) - 2 \mid \mathrm{aw}(g^{\mathcal{R}}) + 2\right\}.$$

Then $\mathrm{aw}(g) = G$, unless G is an integer. In this case, compare g with the far star. If

- *$g \parallel ☆$, then $\mathrm{aw}(g) = 0$;*
- *$g < ☆$, then $\mathrm{aw}(g) = \min\{n \in \mathbb{Z} : n \mathrel{|\!\!>} G^L\}$;*
- *$g > ☆$, then $\mathrm{aw}(g) = \max\{n \in \mathbb{Z} : n \mathrel{<\!\!|} G^R\}$.*

Theorem 60 (Atomic Weight Properties)**.** *Let g and h be all-small games. Then*

(i) $\mathrm{aw}(g + h) = \mathrm{aw}(g) + \mathrm{aw}(h)$;

(ii) $\mathrm{aw}(-g) = -\mathrm{aw}(g)$;
(iii) *If* $\mathrm{aw}(g) \geqslant 1$, *then* $g \,|\!\!\rhd 0$ *(Left wins playing first);*
(iv) *if* $\mathrm{aw}(g) \geqslant 2$, *then* $g > 0$ *(Left wins).*

As usual '$|\!\!\rhd$' denotes greater than or confused with. In particular (iv) is the raison d'être for atomic weight, and it is popularly called "the two-ahead-rule". We will have plenty use for it.

For example, we argue that the game in Figure 13 is an $\mathscr{L}$-position. Namely, the chosen rulesets satisfy very elegant properties with respect to the atomic weight theory. If a flowers in the garden has more red (Left plays red) than blue petal leaves, then this flower has atomic weight one. And since atomic weights are additive, we can simply compute the number of red flowers minus the number of blue flowers to get the atomic weights of a FLOWER GARDEN (see for example [BCG1982] or [S2013]). BIPASS has the inverse property in a sense: the more pieces of the opponent the better. Let $\Delta(g)$ be the number of blue amoebae minus the number of red amoebae in a BIPASS strip. Then $\mathrm{aw}(g) = \Delta(g)$. See [LN2023] for a proof of this result. By these two results, we can compute the atomic weight of the disjunctive sum game in Figure 13, and indeed, it is two, so the two ahead rule applies, Left is two atomic units ahead of Right, and so she will win independently of who starts. (How?)

15.1. **A quiz solution:** EUCLID'S GAME. The ruleset EUCLID is played on two non-empty heaps of pebbles. A player must remove a multiple of the size of the smaller heap from the larger heap. We represent a position by a pair of positive integers (x, y), where say $x \leqslant y$. Note that if $x = y$, then the position is terminal. Example: $(2,7) \to (2,3) \to (1,2) \to (1,1)$. Since we put the requirement that (both) heaps remain non-empty, then no more move is possible. Note that the losing moves are forced.

Optimal play reduces to minimizing the relative distance of the heaps. Recall the golden section, $\phi = \frac{1+\sqrt{5}}{2}$.

Theorem 61 ([CD1969])**.** *A player wins* EUCLID *if and only if they can remove a multiple of the smaller heap such that the ratio of the heap sizes* (x, y), *satisfies* $1 \leqslant y/x < \phi$.

Proof. Suppose that the current player is in a position of the form (a, b) with $b/a > \phi$. We must prove that they can find a move to a position (c, d) of the form

$$1 \leqslant d/c < \phi. \tag{10}$$

We claim that there is a positive integer k such that either

- $(c, d) = (b - ka, a)$, or
- $(c, d) = (a, b - (k-1)a)$

satisfies the desired inequality (10). If $1 \leqslant \frac{a}{b-ka} < \phi$, we are done, so suppose that

$$\frac{a}{b-ka} > \phi. \tag{11}$$

Then $\frac{b-ka}{a} < \phi^{-1}$. And so

$$\begin{aligned}\frac{b-(k-1)a}{a} &= \frac{b-ka}{a} + 1 \\ &< \phi^{-1} + 1 \\ &= \phi.\end{aligned}$$

Suppose next that the current player is in a position of the form (a, b), with $1 \leqslant b/a < \phi$. Then there is only one move option, namely $(b-a, a)$, and it follows that

$$\begin{aligned}\frac{b-a}{a} &= \frac{b}{a} - 1 \\ &< \phi - 1 \\ &= \phi^{-1}.\end{aligned}$$

And hence, $\frac{a}{b-a} > \phi$. □

16. An overview of reduced canonical form, and a bit about temperatures and hotstrat

Recall the definitions of Left- and Right stops, Ls and Rs respectively, from Definition 36. A game G is *hot* if $\mathrm{Ls}(G) > \mathrm{Rs}(G)$, it is *tepid* if $\mathrm{Ls}(G) = \mathrm{Rs}(G)$, and it is an *infinitesimal* if $\mathrm{Ls}(G) = \mathrm{Rs}(G) = 0$.[10]

Example 62. For example $G = \{\{2 \mid 1\} \mid -1\}$ is hot, because $1 = \mathrm{Ls}(G) > \mathrm{Rs}(G) = -1$. But $H = \{\{2 \mid 1\} \mid 1\} = 1 + \{\{1 \mid 0\} \mid 0\} = 1 + \mathbf{-}_1$ is tepid. On the other hand $g = \{\{1 \mid \uparrow\} \mid \downarrow\}$ is infinitesimal, because $\mathrm{Ls}(g) = \mathrm{Rs}(\{1 \mid \uparrow\}) = \mathrm{Ls}(\uparrow) = 0$ and $\mathrm{Ls}(g) = \mathrm{Rs}(\downarrow) = 0$.

In a sense, the "one move for Left", which is hidden to the left of the Left option g^L, will mostly be irrelevant in play. Similar to Miny and Tiny though, the option does not reverse out. The hinge here of course is the word "mostly". In an environment with similar infinitesimals, there are situations where the hidden "one move for Left" can become alive. But those situations are rare, and the concept of a Reduced Canonical Form removes the appearance of infinitesimals for all subgames. It becomes an important tool to analyze games that otherwise seem intractable.

In fact, there are two approximation techniques related to these concepts, namely Temperature Theory (see Section 17), which outputs a temperature and a mean value, and Reduced Canonical Form (rcf), which outputs a coarsening of the usual canonical form (value) of a game. By taking the reduced canonical form approximation of a game, the temperature and mean

[10] A special case of infinitesimal games are the all small games. But there are many infinitesimals that are not all small. For example, we have already seen $\mathbf{+}_1$ and $\mathbf{-}_1$.

value remains the same. Hence, one may view Temperature Theory as a last resort, if both the game value and the rcf are intangible for a human eye. As usual, we are looking for information of how to play well games in a disjunctive sum. There is a classical strategy called *hotstrat*, which says: play in the hottest component. This is often the best strategy, but not always. Take for example the disjunctive sum game

$$G = \{1/2 \mid -100\} + \{100 \mid 1/2\} + \{0 \mid \{-1 \mid -101\}\}$$

Hotstrat fails here. The game is an $\mathscr{N}$-position, but if Left starts in the hottest component, she loses.[11] A brief introduction to Temperature Theory is the topic of the next section. In fact G is problematic also from a point of view of reduced canonical form: in that case G remains the same, and its canonical form appears more complicated for a human eye than the displayed disjunctive sum. We will see other examples when rcf is more helpful.

The reduced canonical forms for the games in Example 62 are $\mathrm{rcf}(G) = \{\{2 \mid 1\} \mid -1\}$, $\mathrm{rcf}(H) = 1$, and $\mathrm{rcf}(g) = 0$. We will study the meaning of these statements in what comes, and we will define all concepts accordingly.

First we define the equivalence classes modulo inf, and state a main theorem with tools of efficient computation of inf-equivalence. Then, we define the reductions modulo inf, and at last we state a result of uniqueness of the end result of these reductions. By the way, there are many hot games for which $\mathrm{rcf}(G) \neq G$. For example $G' = \{\{2 \mid 1*\} \mid -1\}$ is hot and in canonical form, but $\mathrm{rcf}(G') = \{\{2 \mid 1\} \mid -1\}$.

The equivalence relation modulo infinitesimals is defined as follows.

Definition 63 (Equivalence Mod Inf)**.** Consider game G and H. Then $G \geqslant_{\inf} H$, if, for all positive numbers x, $G - H > -x$. And $G =_{\inf} H$ if $G \geqslant_{\inf} H$ and $H \geqslant_{\inf} G$.

That is, $G =_{\inf} H$ if $-x < G - H < x$ for all positive numbers x. The games G and H are *infinitesimally close* if $G =_{\inf} H$, and this is also called "equivalent modulo infinitesimals" or simply "inf-equivalent". The relation $\geqslant_{\inf}$ is a partial order, the partial order modulo infinitesimals. We may use terms such as "numberish" if a game is infinitesimally close to a number.

The first result of this section simplifies game comparison modulo infinitesimals, since showing $G \geqslant_{\inf} H$ is equivalent to showing $G - H \geqslant_{\inf} 0$.

Theorem 64 (Constructive Inf-inequality)**.** *The following are equivalent.*

(1) $G \geqslant_{\inf} 0$*;*
(2) $\mathrm{Rs}(G) \geqslant 0$*;*
(3) $G \geqslant \epsilon$ *for some infinitesimal* ϵ*.*

[11] Other instructive examples of when it fails uses the concept of over heating (a type of 'inverse' of cooling), but we do not have the time to cover that topic in this course. Please see [S2013].

The first item is Definition 63; the second and third items are efficient tools, and they often appear in proofs and examples. It is worthwhile to rewrite this in terms of inf-equivalence.

Corollary 65 (Constructive Inf-equivalence)**.** *The following are equivalent.*

(1) $G =_{\text{inf}} 0$*;*
(2) $\text{Rs}(G) = \text{Ls}(G) = 0$*;*
(3) $\epsilon \leqslant G \leqslant \epsilon'$ *for some infinitesimals* ϵ, ϵ'*.*

For example, $\uparrow =_{\text{inf}} \downarrow$, by (2), since $\text{Rs}(\Uparrow) = \text{Ls}(\Uparrow) = 0$.
For example, $\{1 \mid 1\} =_{\text{inf}} 1$, by (3), since $\{1 \mid 1\} = 1{*}$ and $1{*} - 1 = {*}$.

Definition 66 (Inf-reduction)**.** Let G be a game.

(1) Suppose $A, B \in G^{\mathcal{L}}$. Then A inf-dominates B, if $A \geqslant_{\text{inf}} B$.
(2) The Left option G^L is inf-reversible (through G^{LR}), if $G \geqslant_{\text{inf}} G^{LR}$ for some Right option G^{LR}.

Theorem 67 (Domination Mod Inf)**.** *Assume that G is not equal to a number and suppose that G' is obtained by removing some inf-dominated option (either Left or Right). Then $G =_{\text{inf}} G'$.*

Example: the game $\{1, 1{*} \mid 0\} =_{\text{inf}} \{1 \mid 0\}$, since 1 inf-dominates $1{*}$.

Theorem 68 (Reversibility Mod Inf)**.** *Let G be a game and suppose that G^L is inf-reversible through G^{LR}. Let*

$$G' = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\},$$

If G' is not a number, then $G =_{\text{inf}} G'$.

It is important that G' in Theorem 68 is not a number. For example, with $H = \{\{2 \mid 1\} \mid 1\}$, as in Example 62, we get that $H^{LR} \leqslant_{\text{inf}} H$, by noting that $v \leqslant H - 1 = \text{-}_1$. And so, if the result would apply to numbers, then H' would be $\{0 \mid 1\} = 1/2 \neq_{\text{inf}} 1$.

Luckily, we have other tools, and the Number Translation Theorem (Theorem 41) tells us that $H = 1 + \text{-} =_{\text{inf}} 1$, by using also Theorem 64 (3). Or, we can use directly Theorem 64 (2), by noting that $\text{Ls}(H) = \text{Rs}(H) = 1$.

For an example when we can use the reversibility theorem, take $g = \{\{1 \mid \uparrow\} \mid \downarrow\}$ as in Example 62, and we prove that $g =_{\text{inf}} 0$. Again, we can apply either Theorem 64 (2) or (3). If we use (2), we must prove that both $\text{Ls}(g) = \text{Rs}(g) = 0$. But this is easy to see (why?). If we use (3), we must find infinitesimals ϵ_1, ϵ_2 such that $g \geqslant \epsilon_1$ and $g \leqslant \epsilon_2$. Take $\epsilon_1 = \Downarrow$ and $\epsilon_2 = \Uparrow$. Please verify these choices!

Definition 69. A game G is in *reduced canonical form*, if, for every subgame H of G, either H is in canonical form and is a number, or H is hot and G does not contain any inf-dominated or inf-reversible options.

In particular, if a game is tepid, then its reduced canonical form is a number.

Theorem 70. *For every game G, there exists a unique game H in reduced canonical form such that $G =_{\text{inf}} H$.*

Definition 71. The *reduced canonical form* of a game G, denoted $\text{rcf}(G)$, is the unique reduced canonical form H, such that $\text{rcf}(G) = H$.

Here is an example of a not too complicated game, but where a human eye might not capture the essential information immediately,

$$G = \{1*, 1\downarrow \mid \{1\downarrow \mid \downarrow\}\}.$$

Is this game infinitesimally close to some game with a much simpler form? Yes, perhaps not very surprisingly $\text{rcf}(G) = 1$.

17. Intuitive temperature theory

We have already seen examples of hot games: games with some urgency to play first. Typically one would think of the *switches*, games of the form $\pm x = \{x \mid -x\}$, for some positive number x. For example $G = \pm 10 = \{10 \mid -10\}$ is hot, and it has temperature $T(G) = 10$ (see Figure 16 for its so-called thermograph).

We find the temperature of a hot game, by cooling it, and observing when the cooled game becomes a tepid game. In our example, the smallest number $t \geqslant -1$, for which $G_t := \{10 - t \mid -10 + t\}$ is tepid, is indeed $t = 10$. And this defines $T(G)$. Indeed, $G_{10} = *$, an infinitesimal, and they are all tepid. We are thinking of 'cooling' as if each player is paying a penalty of t. As long as they can keep paying the penalty and benefiting in 'score = number of moves' playing first, the game can still be cooled further. Thus, the hot game $G = \{3 \mid 1\}$ can be cooled by at most 1 until it freezes and becomes the tepid game $2* = 2 + *$, where no player benefits by paying further penalty in exchange for playing first.

To begin with, by using a naïve understanding of heat, how should one play optimally in the disjunctive sum

$$\{10 \mid -10\} + \{3 \mid 1\} + \{1 \mid \{0 \mid -100\}\}?$$

The hottest game component is $\{10 \mid -10\}$, and the first player has a winning move there. It does not need to be the unique winning move though. There is another winning move for one of the players.

The general definition of temperature is recursive in t, starting at the two stops. It includes the possibility of cold games with negative temperature; the coldest game has defined temperature -1, and that holds for any game that is an integer (see Figure 14 for its thermograph). The integers cannot be further cooled.

Suppose that we wish to 'cool' the game $1/2 = \{0 \mid 1\}$ to find its temperature. We have to 'pay' a negative penalty (because the game is already cold). We get $(1/2)_{-1/2} = \{0 + 1/2 \mid 1 - 1/2\} = 1/2 + *$, but if $t < -1/2$, then $(1/2)_t$ is not tepid, but in fact hot. The same idea works for any non-integer number game, and by using that $\frac{m}{2^k} = \left\{ \frac{m-1}{2^k} \mid \frac{m+1}{2^k} \right\}$, we get that the temperature of $\frac{m}{2^k}$ is $-\frac{1}{2^k}$.

The standard definition of temperature found in books is non-constructive, but is seems a bit difficult to avoid this, since we must cool a game G everywhere all at once; we start the cooling at the Left and Right stops of $G_0 = G$, and continue until $\mathrm{Ls}(G_t) = \mathrm{Rs}(G_t) = 0$. We omit this somewhat technical definition (see [S2013] for a rigorous treatment of temperature). An intuitive description suffices for practical purposes, to find the temperatures and the *mean values* of some hot games from your rulesets.

Let us explain this concept with an example. Consider the game $\{3 \mid -2\}$. Is this game 'better' for Left or Right? Well, anyone can see that it depends on who starts, and it seems also that Left has a definite advantage in that she can earn one more move than Right could, if she gets to start. The mean value of a game measures this type of 'advantage'.

The *mean value* of a game G is defined as the number $m(G)$ such that the difference $n \cdot G - n \cdot m(G)$ is bounded by a constant, independently of the size of the positive integer n. The mean value theorem states that such a number m exists, and that is suffices to take either of the stops to 'compute' it.

Theorem 72 ([S2013])**.** *For any given game G, the mean value $m(G)$ exist and equals* $\lim \frac{\mathrm{Ls}(n \cdot G)}{n} = \lim \frac{\mathrm{Rs}(n \cdot G)}{n}$.

A standard tool to find both the temperature and mean of a game is via its *thermograph*. The thermograph of a (hot) game G is drawn, by starting at the Left and Right stops and gradually cooling (by increasing the penalty t), and watching carefully that at every line drawn follows the Left and Right stops of the current G_t. This procedure is most easily understood by drawing some example games. Let us explain via Figures 14 to Figures 18. The first picture represents a cold game, the second a tepid game, and all other are hot games. Take a look at Figures 14 and 15. They look superficially similar, but there is an important difference. The games have different temperatures, so their thermographs should look different, right? And they do, the first game has temperature -1, and the *mast* continues down below the picture, whereas the second picture has temperature 0, because it is a tepid game, and that is illustrated by the fact that the bottom of the mast rests at the horizontal line, at the top of a trivial thermograph. It is easy to see that the location of the mast is the mean value of these two games.

The next three thermographs are more interesting. Of course, Figure 16 is the thermograph of the game in the first paragraph of this section. And it illustrates nicely the idea of cooling that game until we reach *mast value* 0 and temperature 10. Again, the mast value is the same as the mean value of the game. Check this by playing a large sum of games, where each component is of the form $\{10 \mid -10\}$. The first player's advantage is quickly diminished by the fact that the second player can, at each response cancel

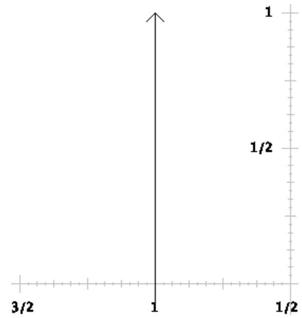


FIGURE 14. The thermograph of the number game 1.

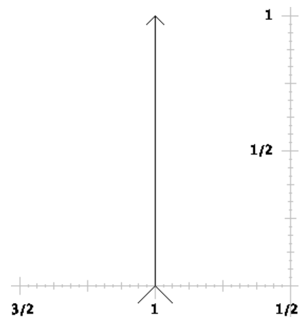


FIGURE 15. The thermograph of the tepid game $\{1 \mid 1\} = 1*$.

the first player's advantage. For switches, it is easy to see that Theorem 72 holds and it is also easy to see that the mast value and the mean value is the same. That this holds in general is another theorem proved in [S2013].

Theorem 73 ([S2013])**.** *For any game, the mast value and the mean value is the same.*

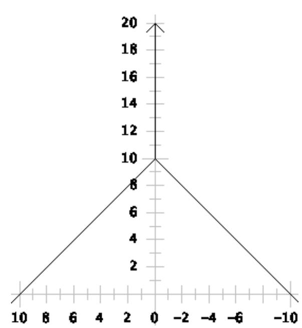


FIGURE 16. The thermograph of the switch $\{10 \mid -10\}$.

In the next picture, we study games of the form $G = \{a \mid \pm b\}$, where $a > b$ are positive numbers. There is a geometric approach to arrive at the vertical border to the right in Figure 17. Namely use the thermograph of the switch $\pm b$, and turn it 45 degrees to the left, by fixing it at the point of the Right stop. The leftmost *wall* of the thermograph of the Right option becomes the rightmost wall of the game G. The slope of the leftmost wall remains the same as in the previous picture (but both the mean value and the temperature change).

Also in this type of games, we can justify Theorems 72 and 73 directly, by inspection. Try this!

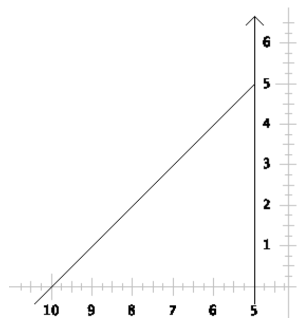


FIGURE 17. The thermograph of the game $\{10 \mid \{5 \mid -5\}\}$.

We will leave the justification of the thermograph in Figure 18 as an exercise. The idea is to raise the horizontal bottom level as one cools the game, and carefully check which option leads to the Right stop at each phase of the cooling. As in the previous picture, the Right options have to be turned 45 degrees to the left, by fixing the Right stop of the cooled game.

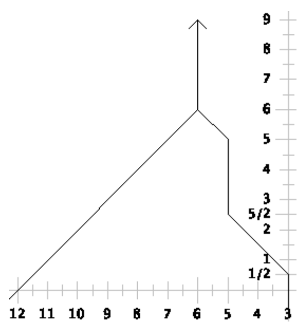


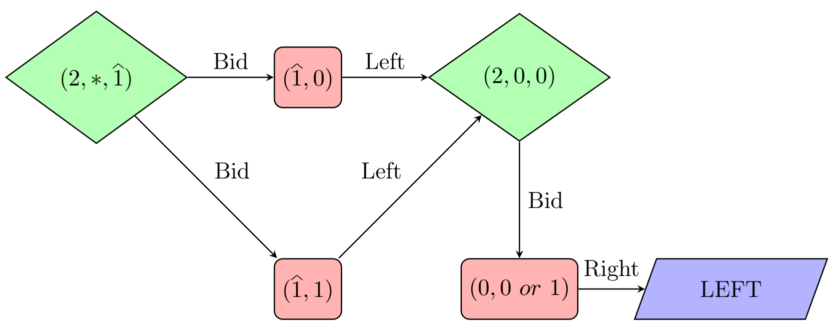
FIGURE 18. The thermograph of the game $\{12 \mid \{5 \mid -5\}, \{3 \mid 2\}\}$.

18. BIDDING COMBINATORIAL GAMES, BY PREM

Consider Left and Right playing a game G . Instead of the conventional alternating play, here the move order is determined through a Discrete Richman bidding. The total budget TB, available for them is fixed. Left's budget is p and Right's is q such that $p+q = \text{TB}$. A player who wins the bid, moves in G and shifts the winning bid to the other player. Additionally, there is a tiebreaking marker, that is initially held by one of the players and can be included in the bid. The tiebreaking marker have no value but in case of a tie, the player who is currently holding the tiebreaking marker will be the winner of the bid and the winning bid together with the tiebreaking marker will get shifted to other player. There is a final auction at the empty game. The player who moves last, wins the game. Moreover, similar to alternating play, in this bidding set up: "last move wins" is same as "cannot move loses"

Formally, given a total budget TB, let us define $\mathcal{B} = \{0, \dots, \mathrm{TB}, \hat{0}, \dots, \hat{\mathrm{TB}}\}$, the set of all feasible player budgets. Here a "feasible budget" includes the information of the marker holder. A game is a triple $(\mathrm{TB}, G, \tilde{p})$, where Left's part of the budget is $\tilde{p} \in \mathcal{B}$. If TB is understood, we write $(G, \tilde{p})$.

For example, $(2, *, \widehat{1})$ means the game is $* = \{0 \mid 0\}$ with total budget 2 in which Left player has budget 1 and the tiebreaker marker. In $(2, *, \widehat{1})$, Left can bid 0 or $\widehat{0}$ or 1 or $\widehat{1}$, however Right can bid 0 or 1. Let's consider Left is bidding the amount $\widehat{1}$. Then for both choices of Right, Left will win the bid and make a move in the game $*$. The current bidding game is $(2, 0, 0)$. Now Left will bid 0 and for all choices of Right, Right will win the bid and have to make a move in the game 0. Since, Right cannot move, therefore Left wins the game.



The following result ensures that by the introduction of bidding, there will not be any mixed strategy equilibrium.

Theorem 74 (First Fundamental Theorem)**.** [KLRU2022] *Consider the bidding convention where the tie-breaking marker may be included in a bid. For any game $(TB, G, \tilde{p})$ there is a pure strategy subgame perfect equilibrium, computed by standard backward induction.*

Observe that in case of a tie, the marker is transferred. Therefore, by this automatic rule, the special case $\mathrm{TB} = 0$ corresponds to alternating normal play rules.

Theorem 75. *Consider* $\mathrm{TB} = 0$. *Then bidding play is identical to alternating play. The current player is the player who holds the marker.*

Next, we define the *outcome* of a bidding game.

Definition 76 (Outcome)**.** The *outcome* of the game (TB, G) is $o(G)$, defined via the $2(\mathrm{TB}+1)$ tuple of partial outcomes as

$$o(G) = (o(G, \widehat{\mathrm{TB}}), \ldots, o(G, \hat{0}), o(G, \mathrm{TB}), \ldots, o(G, 0)).$$

Here the first half of the outcome corresponds to when Left holds the marker and the rest corresponds to when Right holds the marker. The length of the outcome is $2(\mathrm{TB}+1)$.

Since this notation can be quite lengthy, we instead adopt word notation. For example instead of (R, R, L, L) we simply write RRLL.

Definition 77 (Outcome Relation)**.** Consider a fixed TB and the set of all budgets $\mathcal{B}$. Then for any games G and H, $o(G) \geqslant o(H)$ if, $\forall\, \tilde{p} \in \mathcal{B}, o(G, \tilde{p}) \geqslant o(H, \tilde{p})$.[12]

Feasibility of outcome. A careful observation shows that for $\mathrm{TB} = 1$, an outcome such as RLRL would be rare, since Right wins without either money or marker, but loses if he is given a dollar. Next, we state (the proof is straightforward) that such outcomes are impossible; outcomes are *monotone*.

Theorem 78 (Outcome Monotonicity)**.** *Consider $\tilde{p} \in \mathcal{B}$, with $p < \mathrm{TB}$. Then $o(G, \tilde{p}) \leqslant o(G, \widetilde{p+1})$ (same marker holder in both games).*

Another careful observation shows that for $\mathrm{TB} = 1$, an outcome such as LLRR is monotone but for this outcome, Left loses with a dollar budget, but wins with the marker alone. This is also not possible. The next results shows that marker can not be worth more than a dollar.

Theorem 79 (Marker Worth)**.** *Consider* $\mathrm{TB} \in \mathbb{N}$. *Then* $o(G, \hat{p}) \leqslant o(G, p+1)$.

We can view an outcome as a string of L's and R's. From Outcome Monotonicity and Marker Worth, Theorems 78 and 79, we see that not all such strings can appear as an outcome of a game. Thus, let us define the notion of a *feasible* outcome.

Definition 80 (Feasible Outcome)**.** An outcome is feasible if it satisfies Outcome Monotonicity (Theorem 78) and Marker Worth (Theorem 79). For a given TB, the set of all feasible outcomes is $\mathcal{O} = \mathcal{O}_{\mathrm{TB}}$.

The next result shows that corresponding to every feasible outcome, there is a bidding game.

Theorem 81 (Main Theorem)**.** *Consider any total budget* $\mathrm{TB} \in \mathbb{N}_0$. *An outcome, say ω, is feasible if and only if there is a game G such that $o(G) = \omega$.*

For more details see [KLRU2022] and [KLRU2023].

[12] It is easy to check that the outcome relation is reflexive, antisymmetric and transitive. Hence the set of all outcomes together with this relation is a poset.

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