LECTURE NOTES IN COMBINATORIAL GAME THEORY, IE619 2025

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1. Where is my ruleset?

Combinatorial Game Theory is famous because of a variety of recreational rulesets that fit under a common umbrella with significant mathematical properties. These are usually two-player games, in the last-move-win convention, which we call *normal play*. The motivation for this field of Mathematics and Computer Science are classical rulesets such as Go, Chess, Checkers, Tic Tac Toe and many more. However, sometimes their properties land slightly outside of a convenient set of axioms, and we study instead those rulesets that have tighter mathematical properties. During this course, students are invited to develop and explore the theoretical properties of their own rulesets, developed during the course.

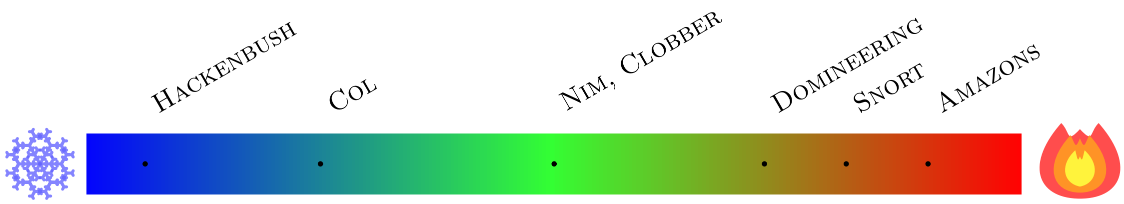


Figure 1. A temperature scale for rulesets.

Combinatorial games are games without chance and with no hidden information. Games such as Poker, Whist and Black Jack are disqualified because they involve hidden cards, and for example Yahtzee, Pachisi and Monopoly are disqualified because play depends on the outcome of a dice.

Let us describe some popular rulesets; see also the intuitive temperature classification Figure 1. We will return to the idea of this concept, but it attempts to capture 'the relative urgency of playing first'. Almost all recreational play combinatorial games have *partizan* rules meaning that the set of move options usually depends on who is to play; otherwise rules are called *impartial* (the move options do not depend on the name of the current player). Here are six partizan rulesets and one impartial:

- Col: This ruleset is played on any finite graph. For example, it can be played on an n by m board, where orthogonal squares are neighbors. A player may place a stone anywhere, except at nodes neighboring any of their own (previously placed) stones.

- SNORT: This ruleset is as COL, but where players instead cannot place stones neighboring opponent stones.
- CLOBBER: an n by m game board; Left plays black pieces and Right plays white pieces. An orthogonal neighbor stone of opposing color can be clobbered and removed from the game board, while your stone takes its position. Starting position: a checker board pattern.
- TOADS&FROGS: a 1 by n game board; Left plays Toads and Right plays Frogs. Toads move to the left and Frogs move to the right. A piece can slide to a neighboring empty cell, or jump one of the opponent's pieces. Starting position: t Toads to the right, and f Frogs to the Left.
- DOMINEERING: an n by m game board; Left places horizontal domino tiles, and Right places vertical Domino tiles. Starting positions: an empty board.
- TOPPLING DOMINOES: Left plays bLue dominoes and Right plays Red dominoes. Both players can topple a green domino. Domino pieces are placed in a sequence. Players can topple any direction.
- NON-ATTACKING QUEENS: an n by m game board; Players place a Queen of Chess in a square, such that they do not attack any previously placed Queen.

COL and SNORT have special interest. Of course a COL graph with two nodes and one edge is a $\mathscr{P}$-position. It is cold in the sense that no player wants to start. On the other hand, SNORT on the same position is hot, in the sense that both players want to start, to secure the neighboring node for any future use, if played in a disjunctive sum of games. In general, the more you play in a COL component, the worse are your future prospects in this component. For example on a $2n \times 1$ strip, if you use up the maximal number of n moves, then the opponent is rewarded a resource of n non-interrupted moves to use at any point in future play. In SNORT, the reverse situation is apparent. If you manage to play uninterrupted n moves on the same strip you can guarantee n free moves. On a complete graph on n nodes, the benefit is even more dramatic, and the differing properties between the rulesets is even more accentuated. Namely, it takes only one single move, in the case of SNORT to gain n moves, whereas in COL the position is simply a $\mathscr{P}$-position independently of the size of n.

The ruleset CLOBBER on the other hand, apart from the set of $\mathscr{P}$-positions, has none of these qualities, but it is a brilliant example of a so-called all-small ruleset, where the final result is a matter of who has a kind of generalized parity advantage.

To practice the concept of a disjunctive sum of games, in the class we play a tournament game such as: $G + H + K + M$, where $G =$ TOADS&FROGS on a 9 by 1 strip, $t = f = 3$, $H =$ DOMINEERING on a 5 by 3 board, $K =$

TOPPLING DOMINOES red blue green red blue blue, and where $M =$ NON-ATTACKING QUEENS on an 8 by 8 board (or a NIM position). Who wins if Left starts? Who wins if Right starts?

Variation PARTIZAN NON-ATTACKING QUEENS: this game is played with Black Queens for Left and White Queens for Right? Left places Black Queens that are not attacked by existing White Queens. Vice versa rules for Right. Same colored Queens do not attack each other.

This course lets students build their own normal-play rulesets, and then study the theory through their properties.[1] Let us include some guidelines that extend the initial axioms from Lecture 2. The ruleset should:

(i) not aim at achieving any 'condition', such as four-in-a-row or similar;
(ii) not have been studied before;
(iii) have a name;
(iv) be scalable;
(v) respect the disjunctive sum operator;
(vi) have a one-line description of what each player can do; a five year old should be able to quickly learn the rules and enjoy playing a game;
(vii) be partizan;
(viii) have "board feel", that is, a potential for various play strategies;
(ix) have some interesting math properties.

Item (i) requires some explanation. Popular rulesets such as "four-in-a-row" could be envisioned as a normal-play game, by defining such a component dead when this condition has been achieved. But this is one layer more complicated then our typical rulesets that simply end when a player cannot play according to the rules.

Games where ending conditions are given by satisfying given conditions are usually called maker-maker, maker-breaker etc. Other examples are graph theory games with rules such as: one of the players attempts to form a triangle, and the other player is trying to finish the game with a triangle free graph. This is a bit hard to envision as a normal-play game, because the player who makes the last move by forming a triangle might be the losing player.

To test if a candidate ruleset satisfies guideline (i), one may ask the question: apart from the axioms, is it required to say anything else about the ending/winning of a game? In a typical combinatorial game such as the above tournament game examples, this is not necessary. Given a game board, it suffices to say how to play the pieces, and the winner is given by the normal-play axiom.

Regarding item (ii); it is not hard to find new rulesets. CGT is a very young subject; most conceivable normal-play rulesets (satisfying the axioms and the guidelines) have yet to be defined/studied. Your ruleset can be brand

[1] These lectures are not accompanied with standard exercises, since students are instead designing their own rulesets to study their values and so on. At the end of every lecture though we have quizzes, some of which are mentioned in these lecture notes.

new, or a variation of an existing ruleset. CG-Suite has already implemented the potential to study variations of for example Clobber, Toads&Frogs, and more, by varying game paratemeters. If you like coding, you may look into further generalizations by rearranging the existing code. We have seen new implementations already, such as Slamming Toads&Frogs, Quadro Count and more. Since your ruleset is original, as item (iii) suggests, you should find a name for it.

Recreational rulesets that motivate our theory often come with a fixed game board size/shape, and perhaps a fixed number of pieces (Chess, Tic-Tac-Toe, Checkers, Fox&Geese and many more). From a mathematical point of view this is not convenient; usually we ask questions both about 'small games' and 'larger games'. Item (iv) says that your rules should be scalable.

Think about item (v), with respect to item (i). If rules say that players should achieve some condition, what happens when we play two such games in a disjunctive sum. Should the composite game be terminated when the (four-in-a-row) has been achieved? If so it is not normal play. Even if one could impose a normal play rule when played on a single component, it is not convenient, when we start adding games. Probably the best way to impose an achievement rule to a normal play disjunctive sum situation is to stipulate that a component dies when the given condition is achieved. But this requires an understanding what it means to be 'a component'. And it removes the original idea of the achievement game, that whoever reaches the goal first is the winner. Hence we avoid such rulesets for the purpose of this course. Any other extra rule that makes disjunctive sum operator require a special treatment is not favorable.

The ruleset should be partizan, because it gives us means to explore the rich theory that has been developed in the field since the 1970s. Finally, it is usually good if a ruleset has "board feel", that is, players should be able to learn reasonable play strategies as they improve their play styles, and part of that usually incorporates easily understandable terminal positions.

Sometimes a ruleset does not have a board feel but it has very nice mathematical properties. Such rulesets are usually referred to as "math-games" (such Nim, Wythoff Nim etc.). If a game has a non-trivial polynomial time solution (in succinct input size), it is a math-game but usually not a play-game (in the spirit that it is too easy to win for any player that studied the basic math properties). If it has good board feel and human relatable strategies, then one migth refer to it as a "play-game". Many CGT rulesets have both qualities, and students are encouraged to look for rulesets where infinite subsets of positions have nice mathematical properties, but with an overall satisfactory board feel; which motivates above item (ix). We can learn mathematics through games, and vice versa. Ultimaltely students are challanged to prove some math properties of infinitely large classes of their rulessets, although this stretches a bit beyond this course; indeed, any interest in this direction would encourage us to build a follow up to this course, where that basics we introduce here have already been covered, and various research topics in the field could be further investigated.

We follow some axioms, as listed:

(i) There is a game board (a set of positions) and some ruleset that determines how given pieces are played;
(ii) there is perfect and complete information;
(iii) there are two players, and one of them is the starting player;
(iv) the players take turns moving;
(v) every game terminates;
(vi) these are win/loss games, and a player who cannot move loses.

Item (vi) is usually called *normal-play*. This convention is based on the goodness of movability. It is never bad to have more move options. The axioms give us a means to predict who is winning a given game in *perfect play*. Namely, we can use a method attributed to Ernst Zermelo [Z1913] (who is also the father of set theory and the axiom of choice etc.) often called *backward induction*. This method will be reviewed in Lecture 4. The mathematics of combinatorial games is very rich, and this study is dubbed Combinatorial Game Theory (CGT).

2. Nim and Wythoff Nim

The first combinatorial game that appears in the literature is Nim [B1902]. A finite number of beans are split into heaps. For example, a starting position could be four heaps of sizes $2, 3, 4$ and 5 beans respectively. Let us denote this position by $(2, 3, 4, 5)$. The current player choses one of the heaps and removes at lest one bean, and at most the whole heap. This is a normal-play game, so the player with the last move wins.

Bouton discovered a method to find a winning move if there is one. The tool is called nim addition, and it is performed as follows. Write the heap sizes row-wise in binary, and add without carry, that is each column adds to 0 if and only if it contains an even number of 1s. Let us compute the nim-sum of our sample game:

$$\begin{array}{cccc} & 1 & 0 & 1 \\ & 1 & 0 & 0 \\ & 0 & 1 & 1 \\ \oplus & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array}$$

The nim-sum is 0. The meaning of this in terms of Nim play is that the player who does not start wins in optimal/perfect play. Every move is losing. Let us say, for example, that the first player removed three beans from the third heap. Then the new position is $(2, 3, 1, 5)$. And, by using nim-addition on that position, we obtain

$$\begin{array}{cccc} & 1 & 0 & 1 \\ & 0 & 0 & 1 \\ & 0 & 1 & 1 \\ \oplus & 0 & 1 & 0 \\ \hline & 1 & 0 & 1 \end{array}$$

Since the nim-sum is non-zero, there is a NIM move to a position such that the nim-sum becomes zero. That is the idea of Bouton's theory. Here, there is only one winning move, namely take all beans from the heap of size five.

Bouton's proof demonstrates that, given any starting position, and given best play by both players, exactly one of the players is able to play to a 0-position in every move (until the game ends). Later, in Theorem 66, we prove this in general.

It is easy to prove this in case of two heaps, and it was discovered in class, namely, if the starting position is (m, n), with say $m < n$, then the winning first move is to (m, m). The next position will be of the form (m, k), for some $0 \leqslant k < m$, which is of the 'same form' as the first position. Namely, the heaps are of different sizes. Exactly one of the players can, by every move, give the two heaps the same size. Note that this implies that the nim-sum is 0, so indeed it is a special case of the above more general idea.

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the non-negative integers.

Later, we will use the common $*$-notation for NIM heaps. That is, $*$ is a NIM heap of size one, $*2$ is a NIM heap of size two, and in general, for $n \in \mathbb{N} = \{1, 2, \ldots\}$, $*n$ is a NIM heap of size n.

The second ruleset that appears in the literature is Wythoff's variation of NIM, which is called WYTHOFF NIM [W1907] or sometimes CORNER THE LADY or CORNER THE QUEEN [B1966]. It is played on two heaps and the rules are as in NIM, or instead, a player may remove the same number of beans from both heaps, at least one from each heap, and and most twice the number of beans of the smaller heap. This game can equivalently be represented by a single Queen of CHESS, which, by each move, must reduce its distance to the lower left square, denoted by $(0, 0)$.

Suppose that the Queen is placed on position $(5, 4)$. The equivalent WYTHOFF NIM position is two heaps, one of size 5 and the other of size 4. The set of options is $\{(5, y), (x, 4), (5-t, 4-t) \mid 0 \leqslant y \leqslant 4, 0 \leqslant x \leqslant 3, 1 \leqslant t \leqslant 4\}$. See Figure 2.

An elegant method of finding the so-called $\mathscr{P}$-positions ($\mathscr{P}$revious player wins) is to recursively paint the $\mathscr{N}$-positions ($\mathscr{N}$ext or Curre$\mathscr{N}$t player wins), and fill in the smallest un-colored cells with $\mathscr{P}$s. Clearly $(0, 0)$ is a $\mathscr{P}$-position. Thus, each position of the form $(x, 0), (0, x)$ and (x, x) for positive integers x will be $\mathscr{N}$-colored. The method is displayed in Figure 3.

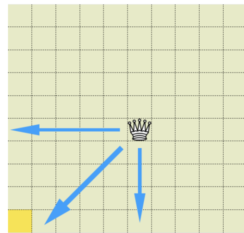


FIGURE 2. The figures illustrate typical move options of CORNER THE QUEEN. The lower left corner represents the terminal position $(0, 0)$.

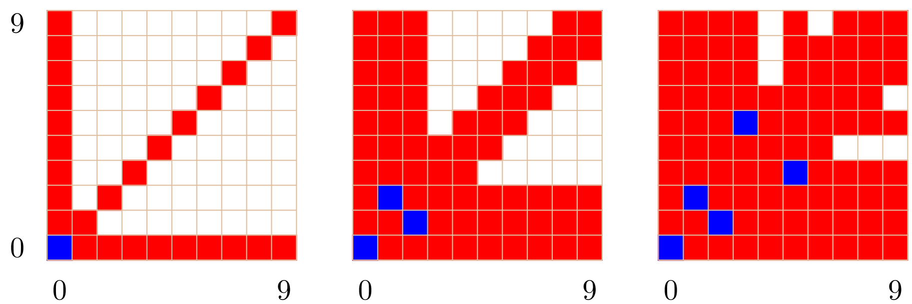


FIGURE 3. A geometric view of the losing positions of WYTHOFF NIM. The $\mathscr{N}$-positions are recursively painted in red, given 'smallest new' $\mathscr{P}$-positions. The terminal position is to the lower left.

This method of painting reveals symmetric $\mathscr{P}$-positons of the form (A_n, B_n) and (B_n, A_n), with the first 8 entries as in Table 1. We note that the classical so-called Fibonacci (or Pingala) numbers, defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, if $n \geqslant 0$, appear in some of the entries, namely $(1, 2), (3, 5), (8, 13), \ldots$.[2] The Golden Section is the irrational number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. It is well known that $\frac{F_n}{F_{n-1}} \to \phi$, as $n \to \infty$. Moreover, we note a possible pattern: for all n, $B_n - A_n = n$. There is a very elegant closed formula expression of the $\mathscr{P}$-positions.

Theorem 1 ([W1907]). *The $\mathscr{P}$-positions of* WYTHOFF NIM *are described by, for all $n \in \mathbb{N}_0$,*

$$(A_n, B_n) = \left(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\right),$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

[2]"The Fibonacci numbers were first described in Indian mathematics, as early as 200 BC in work by Pingala on enumerating possible patterns of Sanskrit poetry formed from syllables of two lengths." Wikipedia.

We prove this classical result later in Section 14.8, where we also include other appealing theorem statements. The main tool will be the so-called Wythoff Properties; see Theorem 67.

TABLE 1. The first 8 $\mathscr{P}$-positions of WYTHOFF NIM (modulo symmetry).

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18

Both these rulesets NIM and WYTHOFF NIM are *impartial*; all options are common for the two players. This is a bit special/limited, as for most recreational/professional games (such as CHESS, GO, CHECKERS, etc) the available moves usually differ for the two players, which is called *partizan*. From a mathematical point of view, the impartial setting has very satisfactory results, for example via Sprague and Grundy's discoveries in the 1930s (here Section 14.3), but there is an even richer and more colorful theory in the general partizan setting, discovered by Berlekamp, Conway and Guy in the 1970-80s. Therefore, in the first part of this course, we will focus on the latter, and students will be encouraged to build and study their own *partizan* rulesets as we move along. Apart from the Sprague-Grundy theory, impartial rulesets (such as WYTHOFF NIM) can have very beautiful and mind-blowing solutions on its own, and we will peek into some of it in Section 14, but on the other hand it completely misses many central CGT topics such as "number of move advantage", "game comparison", "heat", "infinitesimal" and much more. Therefore we postpone the impartial line of play until Section 14.

3. Example rulesets and their properties

The two players of a combinatorial game are usually called Left (female and positive) and Right (male and negative). A *sum* of the combinatorial games G, H, K and M, is defined as the composite game where, at each stage of play, the current player picks one of the components G, H, K or M and makes a move in that component (see Section 4 for a more formal treatment). We write this sum of games as $G + H + K + M$. For example, if player Left starts and plays in the H component, then the next position is $G + H^L + K + M$. Next, player Right picks one of the components and plays his move, for example to $G + H^L + K^R + M$, and so on. The game continues until there is no move in either component. As usual, in normal play, the player who cannot move loses.

The *rulesets* are at the core of combinatorial game theory. A ruleset does not need to come with a starting position, and given a ruleset one can usually envision an infinite number of possible starting positions. When we use the word "game position", or just "position", we usually mean a ruleset together with a starting position. The word "game" can be used freely and the surrounding context explains its local meaning. A ruleset is impartial, if for every position in the ruleset, the move options do not depend on who starts. A ruleset is partizan if there exists a position in the ruleset for which the Left and Right options differ. Partizan rulesets include the impartial ones as a subset, but it is quite unusual that a partizan position has the same Left and Right options.

4. Normal play structures, Zermelo's theorem and the CGT outcome classes

Let us prove Zermelo's theorem in our setting of normal-play with no draw games. This is the first fundamental result of Combinatorial Game Theory (CGT), Theorem 2.

This theorem will be proved in the setting of our recursively defined games.

When we prove results about games by induction, we may assume that a desired property is satisfied by all options of a game G, and then we prove that this implies that the property holds for the game G itself. Observe that if G does not have any option, then the desired property vacuously holds. Hence the base case does not require any further mention (!). (Sometimes this induction method is referred to as "Conway Induction", due to one of the founders of the field.). Let us practice this idea in the proof of the First Fundamental Theorem of CGT.

Theorem 2 (The First Fundamental Theorem of CGT)**.** *Consider any normal-play combinatorial game G. Exactly one of the players can force a win.*

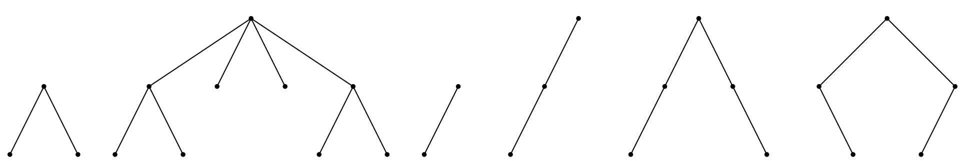
Proof. Suppose, by induction, that the statement holds for all options of a game G. Without loss of generality, suppose player Left starts. If, by induction, Right can force a win from every Left option of G, then Left cannot force a win from G. And hence Right can force a win from G. Note that this holds true also in the case of no Left options of G. Otherwise Left choses an option form which she, by induction, can force a win. □

Observe that this result implies four well defined *outcome classes* in combinatorial games. From now on, we will drop the word "force" in the statement of Zermelo's theorem, and instead think of the players as *perfect*; they both have access to unlimited computational power; withing fractions of a second they are able to find a winning move if there is one, independently of any complexity issues. Let us denote the outcome classes by $\mathscr{L}$ (Left wins independently of who starts), $\mathscr{N}$, $\mathscr{P}$ and $\mathscr{R}$ (Right wins independently of who starts). Thus, we get that every game G belongs to exactly one of these four outcome classes, and we write $G \in \mathscr{P}$ if the current player loses G; $G \in \mathscr{N}$ if this player wins; $G \in \mathscr{L}$ if player Left wins, and $G \in \mathscr{R}$, if player Right wins.

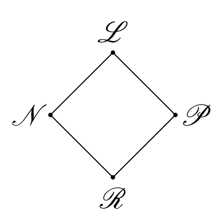
In this lecture we will define the notions of partial order of games, game equivalence and disjunctive sum (addition) of games. Then, in Lecture 6, we prove that the normal-play games, under the disjunctive sum operator, have a group structure. Specifically, in Theorem 8, we prove that every game has an inverse, and we will see that this is a main tool for *constructive game comparison* (Theorem 9). In this spirit, we begin here by defining the notion of game comparison in a non-constructive but exhaustive way.

The definition of game comparison (Definition 5) takes into the account game addition (Definition 3), and an inherited partial order of outcomes (see the below "Outcome Diamond"). Moreover it uses a recursively defined bracket notation of a game. We use it in parallel with a standard game tree representation, where Left options are left slanting edges and Right options are right slanting edges. Every game G has a recursive definition as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ represents the set of left and right options of the game G, respectively. If $G^{\mathcal{L}} \neq \varnothing$, a typical Left option is $G^L \in G^{\mathcal{L}}$, and similarly, a typical Right option is $G^R \in G^{\mathcal{R}}$. By the recursive definition, we would write, for example $G^L = \left\{G^{L\mathcal{L}} \mid G^{L\mathcal{R}}\right\}$, and so on.

For example, a Nim heap of size 2 is the game $*2 = \{0, * \mid 0, *\}$, where $* = \{0 \mid 0\}$. The integer games belong to the partizan theory, and they are defined recursively as $0 = \{\,\mid\,\}$, $1 = \{0 \mid\,\}$ and $n = \{n-1 \mid\,\}$, for $n > 0$. Similarly, for all $n \in \mathbb{N}$, $-n = \{\,\mid -n+1\}$. Let us draw the game trees of the games $*, *2, 1, 2, \{1 \mid -1\}$ and $\{-1 \mid 1\}$.



The standard convention is the total order "Left" > "Right", that is, Left is the "maximizer" and Right is the "minimizer". This induces the Outcome Diamond



with $\mathscr{L} > \mathscr{P}, \mathscr{N}, \mathscr{R}$ and $\mathscr{R} < \mathscr{P}, \mathscr{N}, \mathscr{L}$ but $\mathscr{N} \parallel \mathscr{P}$. Here '$\parallel$' denotes '$\not>$' and '$\not<$' and '$\neq$'. That is, the outcomes $\mathscr{N}$ and $\mathscr{P}$ are *confused*, *fuzzy* or *incomparable*. All these three words (and more) appear in the literature.

5. A disjunctive sum of games

So far, we have used the notion of a sum of games in an intuitive way. Now we will present the standard formal way. The disjunctive sum of games is defined in a recursive manner. It is a tradition in CGT to omit the brackets for set union, and instead simply write A, B for two sets of (Left) options.

Definition 3 (Disjunctive Sum)**.** Consider games G and H. Then $G + H = \left\{G + H^{\mathcal{L}}, G^{\mathcal{L}} + H \mid G + H^{\mathcal{R}}, G^{\mathcal{R}} + H\right\}$, where $\mathcal{X} + G = \{X + G : X \in \mathcal{X}\}$, if $\mathcal{X}$ is a set of games.

The outcomes alone do not suffice to understand how to play well a disjunctive sum of games. Table 2 illustrates that.

Suppose that we know the outcomes of the individual games G and H. Now we wish to compute the outcome of the sum of G and H. If one of the outcomes is a $\mathscr{P}$-position, then we know the outcome of the sum; if both outcomes are either $\mathscr{L}$ or $\mathscr{R}$, then we know the outcome of the sum. Otherwise we cannot yet know the outcome of the sum. The notion of outcomes requires a refinement, where alternating play in the separate components is not mandatory.

Table 2. Given the outcomes of G and H, when can we know the outcome of $G + H$?

$G \setminus H$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{R}$
$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	?	?
$\mathscr{P}$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{R}$
$\mathscr{N}$	?	$\mathscr{N}$	?	?
$\mathscr{R}$	?	$\mathscr{R}$	?	$\mathscr{R}$

If both outcomes are $\mathscr{L}$ then Left can obviously follow her winning strategies in both components, individually and independently of who starts, and analogously for Right. If one of the components is a $\mathscr{P}$-position, then the other component will determine the outcome of the sum, namely, if the first player plays in the $\mathscr{P}$-position, then the other player can respond there in a manner that they will get the last move in that component. Hence the first player can be forced to 'open' the other component. A $\mathscr{P}$-position cannot affect the outcome in a disjunctive sum.

The following example explains some question marks in Table 2.

Example 4. Suppose $G, H \in \mathscr{N}$. This holds if, for example $G = H = *$, a single heap of Nim. Then $G + H \in \mathscr{P}$. We could also have $G = *$ and

$H = *2$. Then $G + H \in \mathscr{N}$. Hence the question-mark is motivated in this case.

If $G \in \mathscr{L}$ and $H \in \mathscr{N}$, we could have $H = \{0 \mid -100\}$ and $G = 1$, with $G + H \in \mathscr{N}$. But we could also have $H = *$ and $G = 1$, which would give $G + H \in \mathscr{L}$.

Suppose that $G \in \mathscr{L}$ and $H \in \mathscr{R}$. We could have $G = 1 + *$ and $H = -1$ which gives $G + H \in \mathscr{N}$. On the other hand $G = 1$ and $H = -1$ gives $G + H \in \mathscr{P}$. Similarly $G = 10$ and $H = -1$ gives $G + H \in \mathscr{L}$, while $G = 1$ and $H = -10$ gives $G + H \in \mathscr{R}$. Hence, all outcomes are possible. The other question marks are similar.

Let us define the partial order of games. Sometimes we view the perfect play "outcome" as a function, and we write $o(G) = \mathscr{P}$ if $G \in \mathscr{P}$, and so on.

Definition 5 (Partial Order)**.** Consider games G and H. Then $G \geqslant H$ if, for all games X, $o(G + X) \geqslant o(H + X)$. And $G = H$ if $G \geqslant H$ and $H \geqslant G$.[3]

This is the desired refinement of the partial order of the outcomes. Namely, it assures Left that the game G is no worse for her than the game H, if played in any arbitrary disjunctive sum. However, it might appear that almost all games would remain incomparable with such a strong notion of a partial order. And moreover, the definition is non-constructive, so there is no algorithm that could determine the relation between two games, unless one can find another equivalent way of expressing the partial order. And indeed, that this is possible will be our second fundamental theorem of combinatorial games. The first major tool is that the games constitute a group structure, and we will prove that in the next lecture. The negative of a game will be the game where the players have swapped roles. Let us give the recursive definition here, and prove its consistency in the next lecture.

Definition 6 (Negative)**.** Consider a game G. Then the Negative of G is $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\}$.

Similar to Definition 3, if $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a set of games, then $-\mathcal{X} = \{-X_1, \ldots, -X_n\}$.

6. The second fundamental theorem

In this lecture we first establish that normal-play games form a group structure (Theorem 8), and then we prove the Second Fundamental Theorem of CGT (Theorem 9) and its corollary (Corollary 10).

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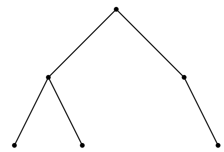
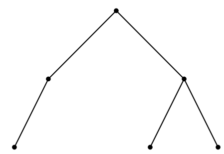
[3]A partial order is a relation that satisfies 1. Reflexivity (aRa); 2. Antisymmetry (if aRb and bRa then a=b); 3. Transitivity (if aRb and bRc then aRc). It is easy to prove that our definition satisfies these axioms. Then it easily follows that '=' is an equivalence relation, which satisfies Reflexivity, Symmetry (aRb implies bRa) and Transitivity. One has to check first that the axioms 1-3 hold for the partial order of outcomes. But this is also easy to check.

Let us begin with an example of a Negative of a game (Definition 6). In terms of game trees, let

$G =$ 

Then

$-G =$ 

In terms of game forms, these games are $G = \{* \mid -1\}$ and $-G = \{1 \mid *\}$, where, as before, $* = \{0 \mid 0\}$, $1 = \{0 \mid \}$ and $-1 = \{ \mid 0\}$. As an exercise, we may add these two games, and expand this sum as one single game form:[4]

$$\begin{aligned} G + (-G) &= \{* - G, 1 + G \mid * + G, -1 - G\} \\ &= \{\{-G, * - 1 \mid -G, * + *\}, \{G, 1 + * \mid 1 - 1\} \mid \cdot\} \\ &= \{\{-G, \{-1 \mid *, -1\} \mid -G, \{* \mid *\}\}, \{G, \{*, 1 \mid 1\} \mid \{-1 \mid 1\}\} \mid \cdot\}, \end{aligned}$$

where we have omitted to expand the Right options since they are symmetric. This game form can, with some patience, and as an exercise, be drawn as a large game tree. But it should be equivalent to 0, as, starting with $G+(-G)$, the previous player can mimic the current player at each stage, until the current player cannot move. This is covered by Theorem 8, which will be our first application of Definition 5. It will establish that the set of all games together with the disjunctive sum operator constitutes a (partially ordered) group structure.

An abelian group, $(\mathbb{G}, +)$, satisfies five properties.

- Neutral Element: There exists an element, $0 \in \mathbb{G}$, such that for all $G \in \mathbb{G}, 0 + G = G$;
- Closure: for all $G, H \in \mathbb{G}$, $G + H \in \mathbb{G}$;
- Negative: for all $G \in \mathbb{G}$, there exist an element '$-G$' such that $G + (-G) = 0$;
- Commutativity: for all $G, H \in \mathbb{G}$, $G + H = H + G$;
- Associativity: for all $G, H, K \in \mathbb{G}$, $(G + H) + K = G + (H + K)$.

Suppose now that $(\mathbb{G}, +)$ is our set of games together with the disjunctive sum operator. All properties, except "Negative" are easy exercises.

[4]This example is admittedly a bit complicated. But its purpose is also to illustrate how easy it is to build complex game trees that are equal to the empty game 0. Take *any* game G and add its inverse, and there you go, a zero-game! If you instead started with a single complex game tree of '$G - G$', it would usually be much harder to see that it equals 0. This is an exercise to do once, and then in a sense 'never again'.

The following decomposition of the outcomes will be useful.

Definition 7 (Partial Outcomes)**.** Let $\mathscr{L} = (\mathrm{L}, \mathrm{L})$, $\mathscr{P} = (\mathrm{R}, \mathrm{L})$, $\mathscr{N} = (\mathrm{L}, \mathrm{R})$ and $\mathscr{R} = (\mathrm{R}, \mathrm{R})$. The first (second) coordinate declares who wins if Left (Right) starts. For a given game G, denote $o(G) = (o_{\mathrm{L}}(G), o_{\mathrm{R}}(G))$, where the partial outcomes $o_{\mathrm{L}}(G), o_{\mathrm{R}}(G) \in \{\mathrm{L}, \mathrm{R}\}$ denotes who wins in perfect play depending on who starts, Left and Right, respectively.

Sometimes we call the partial outcomes $o_{\mathrm{L}}(G)$ and $o_{\mathrm{R}}(G)$ the *result* (who wins in perfect play), when Left and Right starts, respectively.[5] The term "individual outcomes" appers also in the literature, with the same meaning.

Theorem 8 (Negative Game)**.** *For any game G, $G + (-G) = 0$.*

Proof. We have to demonstrate that, for any game G, for all games X, $o(G - G + X) = o(X)$. The proof is by induction on $G - G + X$. If Left cannot play in X, then $o_{\mathrm{L}}(X) = \mathrm{R}$. Similarly, if Left cannot play in X, then $o_{\mathrm{L}}(G - G + X) = \mathrm{R}$, because, if Left can play in $G - G$, then Right can mimic, and so on, which ultimately leads to Right getting the last move in the '$G - G$' component. This 'base case' is analogous for the function o_{R}.

Suppose that the statement holds for all options of G. For example, if G^R is a Right option of G, then $G^R - G^R = 0$. If Left has an option in X, then there are two cases to consider, namely $o_{\mathrm{L}}(X) = \mathrm{L}$ or $o_{\mathrm{L}}(X) = \mathrm{R}$.

Suppose first that $o_{\mathrm{L}}(X) = \mathrm{L}$, and consider the game $G - G + X$. Suppose that Left's winning move in X is to X^L. We claim that $o_{\mathrm{R}}(G - G + X^L) = \mathrm{L}$. Namely, if Right plays to $G - G + X^{LR}$, then Left wins by induction (Right does not have a winning move in X^L). And if Right plays in the '$G - G$' part, then Left can mimic. This would result in $G^R - G^R = 0$ or $G^L - G^L = 0$, by induction.

Suppose next that $o_{\mathrm{L}}(X) = \mathrm{R}$, that is, Left does not have a winning option in X played alone. In the game $G - G + X$, if Left plays her losing move in the X-component, then Right can respond locally to $G - G + X^{LR}$ and win by induction. If Left starts by playing in the '$G - G$' component, then Right can mimic, and the argument is the same as in the previous paragraph.

The proofs for Right playing first are analogous (symmetric). □

The second fundamental theorem of combinatorial games is as follows. We utilize that games form a group structure. In particular, that every game has an inverse (Theorem 8), and the inverse is the defined Negative of the game.

Theorem 9 (Second Fundamental Theorem)**.** *Consider games G and H. Then $G \geqslant H$ if and only if Left wins the game $G - H$ playing second, that is if and only if $o_{\mathrm{R}}(G - H) = \mathrm{L}$.*

[5]The word "result" is of course much more general than "perfect play", and should apply to any situation where for example two non-optimal human players apply their best understanding of the game, or say whenever any intermediate beliefs of a learning agent, leads to a 'result' of a game. Or, who won that particular recreational game at that specific date and time?.

Before the proof, we give two examples of how to use this result.

Let $G = \{* \mid -1\}$ and let $H = *2$. We use Theorem 9 to show that $G \not\geqslant H$. That is, it suffices to show that Left does not win the game $G - H$ playing second. If Right plays to $-1 + *2$, then Left can only respond to -1 or $-1 + *$, and loses either way. How about the reverse inequality? Is $H \geqslant G$? That is, can Left win if Right starts in the game $H - G = *2 + \{1 \mid *\}$? If Right plays to $*2 + *$, then Left can respond to $* + *$; if Right plays to $* + \{1 \mid *\}$ or to $\{1 \mid *\}$, then Left can respond to 1 or $1+^*$ and wins in either case. Altogether this proves that $H > G$.

Let $G = \{0 \mid 1\}$ and $H = *$. It is easy to check that $o_{\mathrm{R}}(G - H) = \mathrm{L}$. Hence $G \geqslant H$. In addition, since $o_{\mathrm{L}}(G - H) = \mathrm{L}$, then in fact $G > H$.

Proof of Theorem 9. By Theorem 8, we may study the game $G - H$. We must prove that $G - H \geqslant 0$ is the same as Left wins $G - H$ playing second. By definition, $G - H \geqslant 0$ means that, for all X, then $o(G - H + X) \geqslant o(X)$.

Suppose first that, for all X, $o(G - H + X) \geqslant o(X)$. This holds in particular for $X = \{\,\mid\,\}$. But then,

$$\mathrm{L} = o_{\mathrm{R}}(\{\,\mid\,\}) \leqslant o_{\mathrm{R}}(G - H + \{\,\mid\,\}) = o_{\mathrm{R}}(G - H),$$

and hence $o_{\mathrm{R}}(G - H) = \mathrm{L}$ (there are only two results, and $\mathrm{L} \geqslant \mathrm{R}$).

For the other direction, suppose that Left wins $G - H$ playing second. We must prove that, if $o_R(G - H) = \mathrm{L}$, then, for all X,

$$o(G - H + X) \geqslant o(X). \tag{1}$$

We analyze the partial outcomes, o_L and o_R.

If $o_{\mathrm{L}}(X) = \mathrm{R}$ then $o(G - H + X) \geqslant o(X)$. Therefore, let us assume that $o_{\mathrm{L}}(X) = \mathrm{L}$. In particular, this means that Left has a winning move in X played alone, to say X^L. She can play this move in the game $G - H + X$, to $G - H + X^L$. If Right responds in the '$G - H$' part, then, by assumption, Left has a 'local winning' response inside that part, to say $(G - H)^{RL}$. And if he plays to $G - H + X^{LR}$, then Left wins by induction (since $o_{\mathrm{L}}(X^{LR}) = \mathrm{L}$). Hence, $o_{\mathrm{L}}(G - H + X) = \mathrm{L}$.

Similarly, assume that $o_{\mathrm{R}}(X) = \mathrm{L}$, and we must prove that $o_{\mathrm{R}}(G - H + X) = \mathrm{L}$, under the assumption that $o_{\mathrm{R}}(G - H) = \mathrm{L}$. If Right starts in the '$G - H$' part, then Left has a winning response, by assumption, and otherwise, if Right starts in the 'X' part, then, by $o_{\mathrm{R}}(X) = \mathrm{L}$, Left can respond locally and win, since by induction $o_{\mathrm{R}}(X^{RL}) = \mathrm{L}$. □

It is convenient to be explicit about all relations in the inherited partial order.

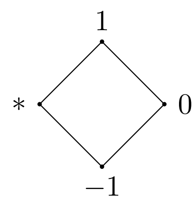
Corollary 10 (Bijection Partial Order and Outcomes)**.** *Consider games G and H. Then*

- *$G = H$ if and only if $G - H \in \mathscr{P}$;*
- *$G > H$ if and only if $G - H \in \mathscr{L}$;*
- *$G < H$ if and only if $G - H \in \mathscr{R}$;*

- *$G \parallel H$ if and only if $G - H \in \mathscr{N}$.*

Proof. For the first item, apply Theorem 9 for $G \geqslant H$ and $H \geqslant G$. The other items are similar, namely apply Theorem 9 for $G \geqslant H$ and $H \not\geqslant G$, $G \not\geqslant H$ and $H \geqslant G$, and $G \not\geqslant H$ and $H \not\geqslant G$ respectively. □

In Corollary 10, it is instructive to revisit the outcome diamond, and instead of outcomes put games *born* by day 1 as representatives of the outcome classes. A game's *birthday* is defined recursively. We will do this before Theorem 14.



7. Game reductions and canonical form

Here we discuss the two reduction theorems on combinatorial games; they concern *domination* and *reversibility*. We will show that together they imply, for any game G, the existence of a unique reduced form, usually referred to as the *canonical form*, the *game value*, or just the *value* of G.[6] We state the results in terms of Left options, and the symmetric statement in terms of Right options has an analogous statement and proof. These two results are nice applications of the Second Fundamental Theorem and its corollary.

Let us start with some examples. If $G = \{1, 2, 3 \mid *\}$, then Left could ignore the Left options 1 and 2, and hence it should hold that $G = \{3 \mid *\}$. This guess can be verified by using Corollary 10 as follows. The previous player wins $\{1, 2, 3 \mid *\} + \{* \mid -3\}$, by mimic strategy, unless Left starts and plays in the first component to $1 - G$ or $2 - G$. Then Right responds to $1 - 3$ or $2 - 3$ and wins.

This is all good, but note that we found a simpler form of G by guess work. Domination is better in that it achieves the same result but without guessing a simpler equivalent form.

Recall that in normal play combinatorial games, a player who can mimic the other player has a winning strategy. When we use the phrase "mimic", then we mean that the next player mirrors the previous players move, such that the composite game becomes equal to $0 \in \mathscr{P}$. We will continue to see many examples of this.

Theorem 11 (Domination)**.** *Consider any game G. If there are Left options $A, B \in G^{\mathcal{L}}$, such that $A \leqslant B$, then $G = \{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$.*

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[6]Another term one might hear with an equivalent meaning is "simplest form".

Proof. Let $H = \{G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}}\}$. Then $-H = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}} \setminus \{-A\}\}$. By Corollary 10, it suffices to prove that $G+(-H) \in \mathscr{P}$. Observe that the Left options of $-H$ are the Negatives of the Right options of G. Hence any play in those options can be mimicked, and then $G^R - G^R = 0 \in \mathscr{P}$ settles those cases. In fact, Right as a starting player has fewer options than Left, and all his moves can be mimicked by Left. Similarly, if Left starts by playing to $G^L + (-H)$, where $G^L \neq A$, then Right can respond in the $-H$ component to $G^L - G^L = 0 \in \mathscr{P}$.

The remaining case is if Left as the starting player plays to $A - H$. Then Right cannot mimick, since A is not a Left option in H. But Right can respond to $A - B \leqslant 0$, and win (playing second in $A - B$). □

By this result, we see immediately that $G = \{1, 2, 3 \mid *\} = \{3 \mid *\}$, because $3 \geqslant 2 \geqslant 1$.

The next result concerns the reduction *reversibility.* We have seen already that the game $G = \{* \mid *\} = 0$, and one can argue directly that it is true because the game is a $\mathscr{P}$-position. The game's simplest reduced form is 0, but it cannot reduce via domination, because there is only one option. We obviously need another tool. Luckily, one can argue by using 'reversibility', that this holds true: If Left plays to $*$, then Right has on option that is no worse than the original game $\{* \mid *\}$, that is, it reverses Left's move. Therefore Left's move is meaningless and should be reduced to whatever remains after Right's 'automatic' response. We prove that this idea holds in general, and give some more examples after the result.

Let us remind the reader, that a Left option in a Negative game, say $-G$, is of the form $-G^R$, and a Right option is of the form $-G^L$. Moreover, sometimes the notation G^L means a generic Left option, while other times, depending on the given context, G^L is a specific option, with a prescribed property. Indeed, we are using this latter meaning in the following statement.

Theorem 12 (Reversibility)**.** *Consider any game G. If there is a Left option G^L with a Right option $G^{LR} \leqslant G$, then $G = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}$.*

Proof. For a fixed Left option G^L as in the statement, let

$$H = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}}\} \mid G^{\mathcal{R}}\}. \tag{2}$$

Observe that, by definition, G and H have the same set of Right options. We prove that $o(G - H) = \mathscr{P}$.

To this purpose, consider the game $G - H$. Similar to the proof of Theorem 11, moves that can be mimicked cannot disprove the result. Hence it suffices to analyze two cases:

(i) Left starts by playing to $G^L - H$, where G^L is as in the statement;
(ii) Right starts by playing to $G - G^{LRL}$, for some $G^{LRL} \in G^{LR\mathcal{L}}$.

In case (i), Left starts by playing to $G^L - H$, and we must prove that Right wins. By assumption, Right can respond to $G^{LR} - H$, where G^{LR} is such that $G^{LR} - G \leqslant 0$. That is, by Theorem 9,

$$o_{\mathrm{L}}(G^{LR} - G) = \mathrm{R}. \tag{3}$$

Since the Negative games $-H$ and $-G$ have the same set of Left options, by (3), Right wins if Left plays in the $-H$ component to, say $G^{LR} - H^R = G^{LR} - G^R$. Thus assume she plays to some $G^{LRL} - H$. But then, by definition of H, Right can respond to $G^{LRL} - G^{LRL} = 0$.

For the second item, we must prove that Left wins the game $G - H$ whenever Right starts by playing to $G - G^{LRL}$ for some $G^{LRL} \in G^{LR\mathcal{L}}$. Recall that $G - G^{LR} \geqslant 0$. This means that $o_{\mathrm{R}}(G - G^{LR}) = \mathrm{L}$, and so, for all Right options $G - G^{LRL}$, Left wins. □

The proof gives us an insight that, if there are several Right options of the form G^{LR}, such that $G^{LR} \leqslant G$, then any replacement set of the form $G^{LR\mathcal{L}}$ suffices to define an equivalent game $H = G$ as in (2).

Let us give two examples of 'similar looking' games, one of which reduces by reversibility but the other does not reduce. The games are $G := \{0, * \mid 0\}$ and $H := \{0, * \mid *\}$. For both games, the only Left option with a Right option is $G^L = H^L = *$, and $H^{LR} = 0 \leqslant H$ but $G^{LR} = 0 \not\leqslant G$, by Theorem 9, namely Left wins H^{LR}, but not G^{LR}, playing second. Observe that $H^{LR\mathcal{L}} = \varnothing$. Hence $H = \{0 \mid *\}$.

Note that we can prove directly, by using Theorem 9, that the game $G \neq \{0 \mid 0\}$; hence, the Left option $*$ cannot be reversible. Namely, Right does not win $G - * = G + *$, playing second. If Left starts by playing to $* + *$, Right loses.

In combinatorial game theory, the rank of a game tree has another traditional terminology, namely *birthday*.

Definition 13 (Formal Birthday)**.** A game is born by day 0 if it has no options. A game is *born by* day $n > 0$ if every option is born by day $n - 1$. A game is *born at* day n if it is born by day n but not by day $n - 1$.

The formal birthday concerns the literal form of a game. We often skip the word "formal". Sometimes there is a risk of misunderstanding, because we often consider the equavialence class of a game, via its simplest form representation. If we want to be explicit, in this case, we may write *canonical form birthday*. We have a very elegant result, that such a respresentation is unique. Consider a game G. Independently of the order of applying the reduction theorems (Domination and Reversibility), the end result, when no more reduction is possible, is a unique simplest form, often known as its *game value*, or the *canonical form* of G. This may be regarded as a third fundamental theorem of (normal play) CGT.

Theorem 14 (Canonical Form/Game Value)**.** *Suppose that domination and reversibility have been applied to a game G until no more reduction is possible, or any further reduction results in the same literal form game. If two literal form games G' and G'' are the end results of such reductions, then they are identical.*

Proof. Suppose, by induction, that this holds true for all games of birthday smaller than n. Let us prove that, then it holds for a game G of birthday n. By the induction hypothesis, we may assume that every option of G is in its unique reduced form.

Claim: We can pair the options of G' and G'', such that for each option G'^L there is a G''^L such that $G'^L = G''^L$ (and similarly for Right).

Proof of Claim: We are using the assumption that $G'' - G' \in \mathscr{P}$ together with the facts that there are no reversible or dominated options.

Suppose that Right starts and plays to $G'' - G'^L$. Since G' and G'' do not have any dominated or reversible options, we get that to win (playing second) Left must play to an option of the form $G''^L - G'^L$. Namely, a Left option of the form $G'' - G'^{LR} \lhd\!\!| \; 0$; we must have "$\lhd\!\!|$" (which is the same as "$\not\geqslant$") since the option is not reversible;[7] i.e. Left cannot win playing second. Thus Left could win playing second if and only if G''^L is such that $G''^L \geqslant G'^L$.

Suppose next that Left starts and plays to $G''^L - G'$. Since G'' and G' are equal, Right must have a winning move. Again, this must be of the first form. Thus, for some Left option G'^{L_1}, we obtain the inequalities $G'^{L_1} \geqslant G''^L \geqslant G'^L$. Since G' does not have any dominated options, all these three games must be equal.

But then, since these options are in reduced forms, by induction, since they are equal they must be identical. Hence, G' and G'' must also be identical. ☐

For example, the game 0 is born by day 0, while $*$, -1 and 1 are born at day 1. Together with 0, they form the same partial order as the above Outcome Diamond. When concerning the canonical form birthday, there are 22 games born by day 2 (See Figure 4). There are 1474 games born by day 3. The number of games born by day 4 is huge, recently estimated between 10^{28} and 10^{185}, by Koki Suetsugu. The number of games born by day 5 is unknown.

Game temperature. There are *cold* games, there are *tepid* games and there are *hot* games. HACKENBUSH is an example of a ruleset for which all games are cold. The game is played with red or blue pieces stacked upon each other in various directions. Right can remove Red pieces and Left can remove bLue pieces. Any piece that ceases to have a connection to the ground falls off and is no longer part of the game. In Figure 5 we list a few examples together with their game values (they will studied later in this section).

CLOBBER is an example of a ruleset with only tepid positions (in fact all positions are so-called *all-small*). TOPPLING DOMINOES is a ruleset with

[7] If $G \lhd\!\!| \; 0$, then Right wins playing first in G.

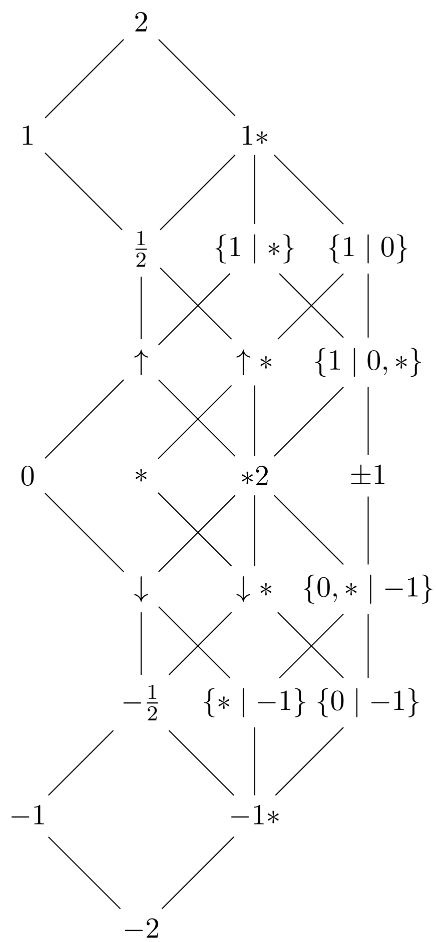


FIGURE 4. There are 22 games born by day 2. The picture shows the partial order of these 22 games, where an edge represents 'upper node > lower node'. The structure is a lattice: every disjoint pair of nodes has a least upper bound (a join) and a greatest lower bound (a meet).

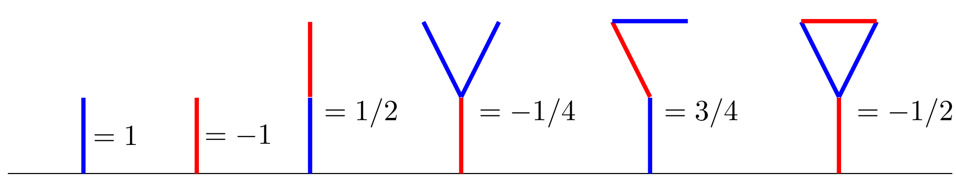


FIGURE 5. Some HACKENBUSH positions and their values.

a variety of positions, in particular, we can build arbitrarily hot positions; imagine a stretch of pieces throughout the room, with same colored pieces to the left and right of the middle respectively. Such positions can be made arbitrarily hot, that is, the first player can gain a huge number of 'free moves'. On the other hand, even without using any theory about numbers, one can show that HACKENBUSH does not have any $\mathscr{N}$-positions (an interesting exercise).

The right most picture in Figure 5 reduces to $-1/2$. This can be proved in a similar fashion to what we usually do, that is by using domination and/or reversibility, or it can be done directly by justifying this guess, by showing that $G-1/2 \in \mathscr{P}$. But here we will discover an easier way that only applies to games that are *numbers*.

8. Zugzwang games are numbers

Let us start by defining the *cold* games; we will later see that they are all *numbers*, defined as the integers (Definition 17) and the dyadics (Definition 19). The most important property of these games is known in Chess and other games as *zugzwangs*. As mathematicians, we require the usual hereditary property.[8]

Definition 15 (Zugzwang)**.** The game G is a zugzwang, if, for all options G^L and G^R, $G^L < G^R$, and all options are zugzwangs.

The proof of the following result is (again!) an elegant application of the principle of induction.

Theorem 16 (Zugzwang)**.** *Consider a zugzwang game G. Then, for all options, $G^L < G < G^R$.*

Proof. We study the game $G-G^L$ and by symmetry the argument for $G-G^R$ is analogous. Note that $o_L(G-G^L) = \mathrm{L}$, because Left can play to $G^L-G^L = 0$; this is true for all games, not just zugzwangs.

Hence it suffices to prove that, if G is a zugzwang, then $o_R(G-G^L) = \mathrm{L}$. If G is a zugzwang and Right plays to G^R-G^L, then Left wins, by Definition 15.

Therefore, assume instead that Right plays to some $G-G^{LL}$. Then, since G^L is a zugzwang whenever G is, induction gives $o_R(G^L-G^{LL}) = \mathrm{L}$. □

We will soon reach a famous result, by Conway, which usually is referred to as the "Simplicity Theorem for Numbers" or just "Number Simplicity" (Theorem 22). With this mind, the notion of "simplicity" requires and explanation: consider two canonical form games G and H. Then, we say that G is *simpler* than H, if it has smaller birthday than H.[9]

After that we will prove that the canonical form zugzwangs have a one-to-one correspondence with the *integers* and *dyadic rationals*; let us define them here.

[8] In graph theory, a property of a graph is hereditary if it applies to all its subgraphs.

[9] One could think of relaxing the requirement of "canonical", in the notion of simplicity, but that will not be interesting here.

Definition 17 (Integer Game)**.** For all $n \in \mathbb{N}$, let the integer game $n = \{n-1 \mid \}$.

We have already defined the negative of a game, so the definition gives also the negative integer games of the form $-n = \{ \mid 1-n\}$, $n \in \mathbb{N}$. It is easy to prove that the integer games behave as we expect under addition of games. For example the disjunctive sum game $m + (-n) := \{m-1 \mid \} + \{ \mid 1-n\}$ equals the game $(m-n) := \{m-n-1 \mid \}$, if $m > n$, and otherwise it equals $(m-n) := \{ \mid m-n+1\}$. Think about the various interpretations of addition and subtraction here. Obviously the disjunctive sum $7 + (-5)$ should equal the game 2. But they are vastly different combinatorial objects. The disjunctive sum has birthday 12, while the game 2 has birthday two. The simplest way to prove this is using *number simplicity*. Let us state it here, in the special case of sums of integer games.

Proposition 18 (Number Simplicity for Integers)**.** *Consider a disjunctive sum of integer games $G = a + b$. Then G equals the simplest integer x such that, for all options G^L and G^R, $G^L < x < G^R$.*

In Proposition 18, let $a = 7$ and $b = -5$. Then $7+(-5) = \{6-5 \mid 7-4\}$, since Left (Right) can only move in the first (second) component. The options are games of smaller birthday than G, so, by induction, we may apply standard arithmetic rules. Hence $7 + (-5) = \{1 \mid 3\}$. The simplest integer game between the games 1 and 3 is the game 2. Suppose next that $a = 7$ and $b = 5$. Then $G = 7 + 5 = \{6 + 5, 7 + 4 \mid \}$. The simplest integer larger than both Left options is 12, which is the desired game. Note that the second inequality is vacously satisfied.

Of course, the standard CGT argument can also be used. Let us prove this statement in general in the first case, when $m > n$. It suffices to prove that the second player wins the disjunctive sum

$$(m + (-n)) + (-(m-n)). \tag{4}$$

Note that we must separate the games with paranthesises, due so the double use of the "+" and "−" operators. We use induction to establish that

$$\{m-1 \mid \} + \{ \mid 1-n\} + \{ \mid n-m-1\} \in \mathscr{P} \tag{5}$$

That is, the second player can revert the game to the same form. Suppose Left starts. Then she must play to

$$\{(m-1)-1 \mid \} + \{ \mid 1-n\} + \{ \mid n-m-1\} \tag{6}$$

Right can play to

$$\{(m-1)-1 \mid \} + \{ \mid 1-(n-1)\} + \{ \mid n-m-1\} \tag{7}$$

But, by induction, $(n-1)-(m-1) = n-m$, and so the game is of the same form.

Let us do the same argument but use Proposition 18 instead of (5).

Definition 19 (Dyadic Rational Game)**.** For all $k \in \mathbb{N}$, let the game $1/2^k = \{0 \mid 1/2^{k-1}\}$. The game $m/2^k$ is m copies of $1/2^k$ in a disjunctive sum.

That is, if $k \geqslant 1$, then $1/2 = \{0 \mid 1\}$, $1/4 = \{0 \mid 1/2\}$, and so on. Similar to the negative integers we also get the negative dyadics beginning with the forms $-1/2^k = \{-1/2^{k-1} \mid 0\}$.

Definition 20 (Number Game)**.** The number games are the integers and the dyadics.

We can prove a couple of nice properties of the dyadic rational games.

Theorem 21 (Dyadic Properties)**.** *For all $k \in \mathbb{N}$,*

(a) $1/2^k > 0$*;*
(b) $1/2^k > 1/2^\ell$*, if* $\ell > k$*;*
(c) $1/2^k + 1/2^k = 1/2^{k-1}$*;*
(d) for all odd m*,* $\frac{m}{2^k} = \left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\}$*, and, if the options are represented in their canonical forms with odd numerators (or possibly as integers), then this game is in canonical form.*

Proof. For (a), by Corollary 10, it suffices to prove that $1/2^k \in \mathscr{L}$ (Left wins). By Definition 19, playing first, she wins in her first move. Playing second, she wins by induction, since Right plays to $1/2^{k-1}$. The base case is the game $1/2$ when Right plays to 1.

For (b), we prove that Left wins $1/2^k - 1/2^\ell$, if $\ell > k$. Left wins playing first to $1/2^k - 1/2^{\ell-1}$, by induction, or by mimic if $\ell - 1 = k$. If Right starts by playing to $1/2^k$, he loses, by (a), but he loses by induction if he plays to $1/2^{k-1} - 1/2^\ell$.

For (c), we prove that $1/2^k + 1/2^k - 1/2^{k-1} \in \mathscr{P}$. If Right starts, he plays either to $1/2^k + 1/2^k > 0$, by (a), or he plays to $1/2^k + 1/2^{k-1} - 1/2^{k-1} = 1/2^k > 0$, by (a). If Left starts, she plays either to $1/2^k - 1/2^{k-1} < 0$ or $1/2^k - 1/2^{k-2} < 0$, both by (b).

For (d), we begin by proving that

$$\left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\} - \frac{m}{2^k}$$

is a $\mathscr{P}$-position.

If Left starts by playing to $\frac{m-1}{2^k} - \frac{m}{2^k}$, then Right can respond to $\frac{m-1}{2^k} - \frac{m-1}{2^k} = 0$, by playing in a $-\frac{1}{2^k}$ component. If Left starts by playing to $\left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\} - \frac{m-1}{2^k} - \frac{1}{2^{k-1}}$, then Right can play to $\frac{m+1}{2^k} - \frac{m-1}{2^k} - \frac{1}{2^{k-1}} = \frac{2}{2^k} - \frac{1}{2^{k-1}} = 0$, by (c).

Similarly, if Right starts by playing to $\frac{m+1}{2^k} - \frac{m}{2^k}$, then Left can play to $\frac{m}{2^k} - \frac{m}{2^k} = 0$, by playing in the $\frac{1}{2^k}$ component. And if Right plays in the second sum-component, then Left responds to $\frac{m-1}{2^k} - \frac{m-1}{2^k} = 0$. Next we prove that this is the canonical form. There is no domination since there is only one option.

To prove that the Left option is reversible, we must since its Right option $\frac{m-2}{2^k}+\frac{1}{2^{k-1}}=\frac{m}{2^k}$, but when we replace it with its set of Left options, we find by induction that gives again the same option. Hence, no further reduction is possible. □

Let us denote the number games by $\mathbb{D}$; we think of them as dyadic rational games, where the integer games are trivial dyadics.

Let us study the nonnegative integers and dyadic rationals recursively via their birthdays as in Figure 6 (the negative ones are analogously defined). We will establish that this construction follows the birthdays of the corresponding number games. We start with $D_0 = \{0\}$, $D_1^+ = \{0, 1\}$ and $D_2^+ = \{0, 1/2, 1, 2\}$. Note that the new dyadics are centered between the old ones. We continue in this fashion, by seeing that $D_3^+ = \{0, 1/4/, 1/2, 3/4, 1, 3/2, 2, 3\}$, and

$$\begin{aligned} D_4^+ &= D_3^+ \cup \{1/8, 3/8, 5/8, 7/8, 5/4, 7/4, 5/2, 4\} \\ &= \{0, 1/8, 1/4/, 3/8, 1/2, 5/8, 3/4, 7/8, 1, 5/4, 3/2, 7/4, 2, 5/2, 3, 4\}. \end{aligned}$$

In general $D_n^+ = D_{n-1}^+ \cup \left\{n, \frac{d_i+d_{i+1}}{2} : d_i, d_{i+1} \in D_{n-1}^+\right\}$. For all n, let $D_n^- = \{-x : x \in D_n^+\}$. If we let $n \to \infty$, then this construction gives all numbers. Let for all n, let $D_n = D_n^+ \cup D_n^-$ and let $D = \bigcup_n D_n$. Then, obviously, $\mathbb{D} = D$.

Observe, that, by construction, if $x, y \in D_n$, with $x < y$, then there is a unique z, such that $x < z < y$, and $z \in D_i$ for some smallest $i < n$. This is the *simplest* dyadic between x and y. If we fill in the edges in the binary tree, then the simplest number between two nodes is their nearest common ancestor. For example $\{11/16 \mid 9/8\} = 1$ and $\{1/16 \mid 7/8\} = 1/2$. As usual, such statements can be verified via the usual P-position argument, by verifying that $\{11/16 \mid 9/8\} - 1 = 0$ and $\{1/16 \mid 7/8\} - 1/2 = 0$, etc. But we have a very general and elegant classical result, with deep consequences to follow.

Theorem 22 (Number Simplicity)**.** *Consider a zugzwang game G. Then G equals the simplest dyadic x such that, for all options G^L and G^R, $G^L < x < G^R$. And x is the canonical form of G.*

Proof. If x is the simplest dyadic such that

$$G^L < x < G^R. \tag{8}$$

Then we must establish that $G - x \in \mathscr{P}$.

By induction, we may assume that all options of G are simplest form number games. Then, we may use domination to single out one Left and one Right option such that $G = \left\{G^L \mid G^R\right\}$. Since G is a zugzwang, by Theorem 16, we have $G^L < G < G^R$.

1) We begin by proving that $o_{\mathrm{L}}(G - x) = \mathrm{R}$. By assumption $G^L - x < 0$, so Right wins if Left starts in G.

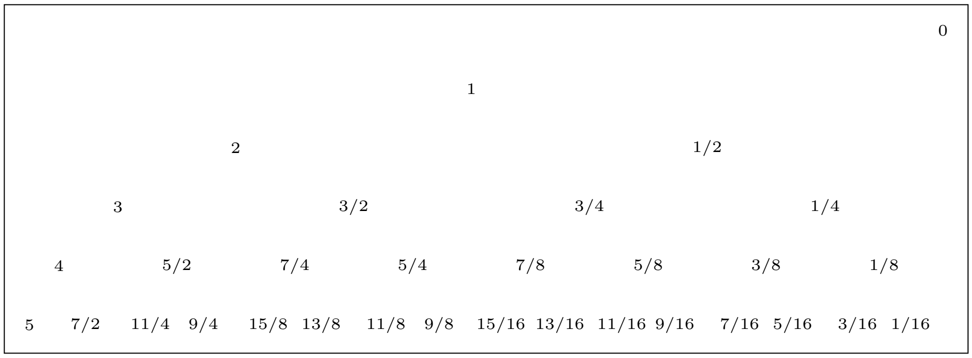


FIGURE 6. The births of the numbers in $\mathbb{D}$ follow a binary tree structure. Each node has exactly two children. For example node 0 has the children -1 and 1, and $3/4$ has children $7/8$ and $5/8$; at the top, we find the birthday zero number, followed by the birthday one numbers, and so on. We display the nonegative number games born by day 5, and the negative ones are symmetrically obtained.

Hence, suppose instead that Left starts by playing to $G - x^R$. Since x^R is simpler than x, then, by (8), either,

(a) $x^R \not> G^L$, or
(b) $x^R \not< G^R$.

Observe that all options are zugzwangs, and by induction numbers, which have a total order, i.e. no two games are fuzzy. Therefore we get instead:

(a) $x^R \leqslant G^L$, or
(b) $x^R \geqslant G^R$.

Suppose that (a) were true. Then, since x is a number, we get $x < x^R \leqslant G^L < x$, where the last inequality is by the assuntion on x, which is impossible. Hence (b) holds, and so Right can respond to $G^R - x^R \leqslant 0$, and win.

2) Symmetrically, we can prove that $o_{\mathrm{R}}(G - x) = \mathrm{L}$. If Right plays to $G^R - x > 0$, he loses. Suppose therefore that Right plays to $G - x^L$. Since x^L is simpler than x, by (8) (and by no two numbers fuzzy), either

(a) $x^L \leqslant G^L < x$, or
(b) $x^L \geqslant G^R > x$.

In case of (a), Left can respond to $G^L - x^L \geqslant 0$, and win. The case (b) contradicts that x is a number.

For the last part, we may assume there is at most one Left- and one Right option; hence there is no domination. By Theorem 21 (d), we may write

$x = \frac{m}{2^k} = \left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\}$, so

$$x^L = \frac{m-1}{2^k} \tag{9}$$

$$= \frac{n}{2^j} \tag{10}$$

$$= \left\{\frac{n-1}{2^j} \mid \frac{n+1}{2^j}\right\}, \tag{11}$$

for n odd, and the corresponding j.

Hence

$$x^{LR} = \frac{n+1}{2^j} \tag{12}$$

$$\geqslant \frac{m}{2^k}, \tag{13}$$

with equality only if $j = k-1$ and $m-1 = n/2$, in which case $x^{LRL} = \frac{m-1}{2^k}$. And otherwise reversibility is not possible. □

Similar to Theorem 69, where we prove that all impartial games equal nimbers, we here will establish that all zugzwang games equal numbers (dyadic rational or integer games).

Theorem 23 (Zugzwangs and Numbers)**.** *Every zugzwang equals a number, and every number equals a zugzwang.*

Proof. The second part is direct from Theorem 21 (d) if the number is a dyadic rational, and if the number is an integer, the statement is vacuously true.

To prove that every zugzwang equals a number, we use induction. Every option of the zugzwang is a zugzwang and hence by induction a number. Numbers have a total order, so, by using domination, we may write $G = \{G^L \mid G^R\}$, for a single Left and Right option, respectively. Now, we use the simplicity theorem, to deduce the simplest form number of G. □

This is a very satisfactory result, and a building block for many applications in this theory. Perhaps one could call it the third fundamental theorem of (normal play) combinatorial games. However, note that there are literal form games that equal numbers, but are not zugzwangs. For example $\{* \mid *\} = 0$, but is no zugzwang.

9. Number translation and avoidance

Number avoidance means that, in a disjunctive sum of games, every player avoids playing in a number component, unless there is nothing else to do. There is a slick symbol for "Left wins playing first", that is "Right does not win playing second": $0 \lhd\!\!\mid G$ has the same meaning as $0 \not\geqslant G$. The following are general lemmas.

Lemma 24. *Let $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, and let $H = \{G^{\mathcal{L}}, A \mid G^{\mathcal{R}}\}$, with $A \lhd\!\!\mid G$. Then $G = H$.*

Proof. A mimic strategy suffices to prove that $G - H \in \mathscr{P}$, unless Right plays to $G - A$. But then the result follows by $G - A \mathrel{|\!\!>} 0$; Left wins playing first. □

Lemma 25. *Consider a game G with at least one Left option. Then, for any Left option G^L, $G^L \mathrel{<\!\!|} G$.*

Proof. The game $G^L - G \mathrel{<\!\!|} 0$, since Right wins playing first to $G^L - G^L$. □

We are aiming to prove a "translation property" for numbers. This is also a strong version of "number avoidance": you do not play in a number unless there is nothing else to do. First we prove a somewhat weaker result, but still insighfull. Suppose we play the composite game $\{\{2 \mid 1/2\} \mid *, *2, \{\downarrow \mid -1/2\}\} + 9/32 + 1/16 - 53/64$. By "weak number avoidance", without computing, Left knows that, if she has a winning move, it is the single option in the first game component (because the other components sum up to a number). So, she knows what to do, even while igonoring the final result of the game!

Theorem 26 (Weak Number Avoidance)**.** *If Left can win $G + x$ playing first, where x is a number and G does not equal a number, then she has a winning move of the form $G^L + x$.*

Proof. It suffices to prove that, if $G + x^L \geqslant 0$, for some x^L, then this implies $G^L + x \geqslant 0$, for some G^L.

By assumption, G does not equal a number, but by Theorem 21, x^L equals a number. Therefore $G + x^L \neq 0$. Hence, the hypothesis becomes $G + x^L > 0$. Thus Left wins playing first in $G + x^L$. Again, since x^L equals a number, by induction, she has a winning move of the form $G^L + x^L \geqslant 0$.

But numbers are zugwangs, and so, by Theorem 16, $x > x^L$, which implies $G^L + x \geqslant 0$. □

It turns out that this result is a consequence of "number translation" (which is also called "strong number avoidance"), but wait, we actually will need Theorem 26 to prove Lemma 28, which is used in the proof of number translation. We must be careful not to run into cycles. Let us state the result that we are aiming for.

Theorem (Number Translation - Strong Number Avoidance). Suppose that x is a number game, and G is a game that is not a number. Then, $G + x = \left\{G^{\mathcal{L}} + x \mid G^{\mathcal{R}} + x\right\}$.

For example, $\{1 \mid 1\} = 1 + * = 1*$, and $\{1 \mid 0\} = \pm 1/2 + 1/2$, and $\{\{1 \mid 1\} \mid 1\} = 1 + \downarrow$.

Note that, by the definition of a disjunctive sum, for G any game and x a number,

$$G + x = \left\{G^{\mathcal{L}} + x, G + x^{\mathcal{L}} \mid G^{\mathcal{R}} + x, G + x^{\mathcal{R}}\right\}.$$

Hence, it suffices to prove that the options where the players move in the number component are all dominated. However, it turns out that it is not possible to prove this result by using only what we learnt so far. Let us develop some more theory to prove this result, and we will restate it as Theorem 32.

Combinatorial game theory has recursively built min/max type functions that are also common in classic game theory. Here they are called the Left- and Right stops respectively, by the idea: “stop when you reach a number”.

Definition 27 (Stops)**.** The Left- and Right stops are,

$$\mathrm{Ls}(G) = \begin{cases} x, & \text{if } G = x \ \text{ equals a number,} \\ \max \mathrm{Rs}(G^L), & \text{otherwise;} \end{cases}$$

$$\mathrm{Rs}(G) = \begin{cases} x, & \text{if } G = x \ \text{ equals a number,} \\ \min \mathrm{Ls}(G^R), & \text{otherwise.} \end{cases}$$

Here max and min ranges over all the Left and Right options, respectively.

Two simple but instructive examples are the literal form games $G = \{1 \mid -1\}$ and $H = \{-1 \mid 1\}$, with the stops $\mathrm{Ls}(G) = 1$, $\mathrm{Rs}(G) = -1$, but $\mathrm{Ls}(H) = \mathrm{Rs}(H) = 0$, because H equals the number zero.

These functions are useful in many ways to analyze our games. In particular, we get a very slick proof of the Number Translation Theorem, Theorem 32. It will depend on some more lemmas though. Note that stops follow a total order (since they are numbers).

Lemma 28. *For any game G,* $\mathrm{Ls}(G) \geqslant \mathrm{Rs}(G)$.

Proof. If G is a number, then $\mathrm{Ls}(G) = \mathrm{Rs}(G)$, so we are done. Hence, suppose G is not a number. The proof is by way of contradiction.

Suppose $\mathrm{Ls}(G) < \mathrm{Rs}(G)$. Then there is a number x such that $\mathrm{Ls}(G) < x < \mathrm{Rs}(G)$.[10] Here we use weak number avoidance. If Left starts in the game $G - x$, then, if she has a winning move, there is one of the form $G^L - x$. If G^L is a number, we stop, and observe that $G^L - x = \mathrm{Ls}(G) - x < 0$. Hence Right wins. If G^L is not a number, then, by weak number avoidance, if Right can win he has a winning move in G^L to say G^{LR}. If G^{LR} is a number, we stop, and observe that $G^{LR} - x = \mathrm{Ls}(G) - x < 0$. The argument does not depend on when we stop: we conclude, if Left starts then Right wins. The analogous argument (using weak avoidance) gives that Left wins $G - x$ when Right starts. Altogether we get that G equals x, which contradicts the assumption that G is not a number. □

Lemma 29. *For any games G and H,* $\mathrm{Rs}(G + H) \geqslant \mathrm{Rs}(G) + \mathrm{Rs}(H)$.

Proof. In the game $G+H$, if Right starts by playing to G^R+H, then, unless G^R is a number, Left can respond to $G^{RL}+H$, and if Right starts by playing to $G+H^R$, then Left can respond to $G+H^{RL}$. Then use induction. Let us fill in some details to this argument. Suppose without loss of generality that

[10] The dyadics are dense on the real number line.

Right's minimizing move in $G+H$ is in the G component, to say G^R+H. Then, by definition of the stops, $\mathrm{Rs}(G+H)=\mathrm{Ls}(G^R+H)$. Now, Left who is the maximizer may have many options, but (if G^R is not a number) she has in particular a move to $G^{RL}+H$, where G^{RL} maximizes her response in G^R played alone. Therefore $\mathrm{Rs}(G^{RL}+H)\leqslant \mathrm{Ls}(G^R+H)=\mathrm{Rs}(G+H)$. Now use induction to conclude that $\mathrm{Rs}(G^{RL})+\mathrm{Rs}(H)\leqslant \mathrm{Rs}(G^{RL}+H)\leqslant \mathrm{Ls}(G^R+H)=\mathrm{Rs}(G+H)$. Finally, the option G^R might not have been the minimizing option for Right, playing in G alone, so to summarize, we get

$$\mathrm{Rs}(G)+\mathrm{Rs}(H)\leqslant \mathrm{Rs}(G^{RL})+\mathrm{Rs}(H) \tag{14}$$
$$\leqslant \mathrm{Rs}(G^{RL}+H) \tag{15}$$
$$\leqslant \mathrm{Ls}(G^{R}+H) \tag{16}$$
$$=\mathrm{Rs}(G+H). \tag{17}$$

If $G^R=x$ is a number, then $\mathrm{Rs}(G+H)=x+\mathrm{Ls}(H)\geqslant \mathrm{Rs}(G)+\mathrm{Rs}(H)$, by applying Lemma 28 on the game H. □

The following two lemmas are pivotal in the proof of number translation. As we will see, together with the definition of a disjunctive sum of games and domination, they comprise the full proof.

Lemma 30. *For any game G that is not a number, there is a Left option G^L such that $\mathrm{Rs}(G^L-G)\geqslant 0$.*

Proof. We get

$$\mathrm{Rs}(G^L-G)\geqslant \mathrm{Rs}(G^L)+\mathrm{Rs}(-G) \tag{18}$$
$$=\mathrm{Rs}(G^L)-\mathrm{Ls}(G) \tag{19}$$
$$=\mathrm{Ls}(G)-\mathrm{Ls}(G)=0, \tag{20}$$

where (18) is by Lemma 29, (19) is by symmetry, and (20) is by assuming that G^L is such that $\mathrm{Ls}(G)=\mathrm{Rs}(G^L)$. □

Note that, if $G^{\mathcal{L}}=\varnothing$, then G equals a number. This follows because as we have seen, all numbers are zugzwangs.

Lemma 31. *Suppose that G is such that $\mathrm{Rs}(G)\geqslant 0$. Then $G+x>0$, if x is a positive number.*

Proof. Clearly this holds if G is a number.

Otherwise, suppose Right starts by playing in the G component, to say G^R+x. Then, unless G^R is a number, Left has a response G^{RL} such that $\mathrm{Rs}(G^{RL})\geqslant 0$. And we may use induction. If G^R is a number, then this number is no smaller than 0, by the definition of Right stop, and so Left wins $G+x$, by the assumption on x.

If Right starts by playing in the x component, to say $G+x^R$, then we can use induction, since $x^R>x$.

If Left starts, observe that Lemma 28 implies that $\mathrm{Ls}(G) \geqslant \mathrm{Rs}(G) \geqslant 0$. That is, she has a move such that $\mathrm{Rs}(G^L) \geqslant 0$. Then, $G^L + x > 0$, by induction. Altogether, we get $G + x > 0$ □

Let us restate the main result of this section.

Theorem 32 (Number Translation - Strong Number Avoidance)**.** *Suppose that x is a number game, and G does not equal a number. Then, $G + x = \left\{G^{\mathcal{L}} + x \mid G^{\mathcal{R}} + x\right\}$.*

Proof. By definition of disjunctive sum,

$$G + x = \left\{G^{\mathcal{L}} + x, G + x^{\mathcal{L}} \mid G^{\mathcal{R}} + x, G + x^{\mathcal{R}}\right\}.$$

Thus, by domination, it suffices to prove that there exists a G^L such that $G^L + x \geqslant G + x^L$, for any x^L. We combine Lemma 30 with Lemma 31; together they establish that there exists G^L such that $G^L - G + x - x^L > 0$. □

10. In a very small world

There is a multitude of games that are smaller than any positive number. As an example, let us start with the first we came to think of. If x is a positive number, then $x > *$.

Lemma 33. *For any number game $x > 0$, then $x > *$.*

Proof. Consider the game $x + *$, where x is a positive number. Left playing first to x wins. If Right starts by playing to x, then Left wins. If Right plays to $x^R + *$, then Left wins by induction, since $x^R > x$ equals a number. □

Definition 34. The game g is an infinitesimal, with respect to numbers, if for all positive numbers x, $-x < g < x$.

Mostly we omit the part "with respect to numbers", but as we will see later there is an infinite hierarchy of games that are infinitesimals with respect to each other. We have already observed that the the game $*$ is an infinitesimal. However $*$ is confused with 0. The following two results are perhaps more remarkable. There are also positive games so small that, however many copies you add, you will not reach one move for Left, the game 1. One such game is $\uparrow = \{0 \mid *\} > 0$. A related game is $\uparrow* = \uparrow + * = \{0, * \mid 0\}$. However this game is fuzzy that is, incomparable with 0. But if we add another "up", then again we reach a positive game, namely $\Uparrow* > 0$, and so on. One can find many games that satisfies the following result, that is sometimes known as "non-Archimedean".[11]

Theorem 35. *There are positive infinitesimals, with respect to numbers.*

[11] The Archimedean property states that, for all positive x, and all positive y, there is a natural number n such that $nx > y$. Note that our real number system has this property, so combinatorial games are much richer, in this sense.

Proof. The game $\uparrow$ is positive. We will demonstrate that, for all $n \in \mathbb{N}$, $n \cdot \uparrow < 1$. Suppose that Left starts in the game $1 + n \cdot \downarrow$. She plays to $1 + * + (n-1) \cdot \downarrow$. If Right responds to $1 + (n-1) \cdot \downarrow$, then she wins by induction, and if he plays to $1 + * + (n-2) \cdot \downarrow$, then Left can play to $1 + (n-2) \cdot \downarrow$, and win by induction. Suppose next that Right starts in the game $1 + n \cdot \downarrow$. Then he plays to in the game $1 + (n-1) \cdot \downarrow$, and Left wins by induction. □

In one lecture we mentioned briefly another choice, the class of so-called *uptimals*, a sequence of the form $\uparrow > \uparrow^2 > \uparrow^3 > \ldots > 0$. They build infinite hierarchies of yet smaller positive games. By definition, the game $\uparrow^n = \left\{0 \mid * - \uparrow - \uparrow^2 - \cdots - \uparrow^{n-1}\right\}$. For example $\uparrow^2 = \{0 \mid \downarrow *\}$. Moreover, the uptimals are infinitely small with respect to each other.

Theorem 36. *Fix a positive integer n. Then, for all positive integers m $\uparrow^n > m \cdot \uparrow^{n+1}$.*

Proof. Exercise! □

This result motivates a special notation for uptimals. We use a standard positional numeration system, but where the digit denotes the number of the given uptimal, respectively. For example $0.10\overline{2}3 = \uparrow + 2 \cdot \downarrow^3 + 3 \cdot \uparrow^4$ (here we use $\overline{x}$ to denote the negative of a positive uptimal x).

The next result concerns even smaller games, called Tiny(1),

$$+_1 = \{0 \mid \{0 \mid -1\}\}$$

and Miny(1),

$$-_1 = \{\{1 \mid 0\} \mid 0\}\,.$$

Theorem 37. *There are positive infinitesimals, with respect to any uptimal $\uparrow^n$.*

Proof. Exercise! (Try with Tiny.) □

This is the content of the next lecture. There is some overlap.

10.1. **More on Infinitesimals - a** TOPPLING DOMINOES **approach, by Anjali.**

Definition 38. A short game G is infinitesimal if $x > G > -x$ for every positive number x.

Consider the combinatorial game of TOPPLING DOMINOES. There are three types of dominoes in the game; red which can be toppled by Left, blue which can be toppled by Right and green which can be toppled by both. Take the following game with two red dominoes as in Figure 7. This game has the value of 2.

Now, let's add a blue domino in between the two red domino in figure 8. Let this game be G_1. This has the value

$$G_1 = \{0, * \mid 1\}$$

The second left option is reversible, and hence

$$G_1 = \{0 \mid 1\}$$
$$= \frac{1}{2}$$

Adding one more blue domino in the middle gives us the game in Figure 9. Let this game be G_2. This game has the form

$$G_2 = \{0, \{0| -1\} \mid 1, *\}$$

The Left option $\{0| -1\}$ is reversible and $1 < *$. Thus,

$$G_2 = \{0 \mid *\}$$
$$= \uparrow$$

We can compare this game with any number, say 1. We find that $1 > \uparrow$. We want to find out how many $\uparrow$s it might take for the value to be more than 1, if possible at all. Actually, by induction, we can prove that, $\forall n \geqslant 1$,

$$1 > n \cdot \uparrow.$$

The game $\uparrow$ is an infinitesimal. It has a very small game value, infinitesimally small with respect to the dyadic rationals. Consider the sequence $1, 1/2, 1/4, 1/8, \ldots$. It rapidly tends to 0, right? Well, in the amazing world of combinatorial games, there is some space between this infinite sequence that 'converges to 0', and the game 0. And now we will demonstrate what this means.

Now, let's add one more blue domino in the middle so that, now there are three blue dominoes in between two red domino. See figure 10. Let this game be G_3.

$$G_3 = \{0, \{0| -2\} \mid \{0| -1\}, *\}$$

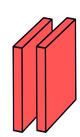


FIGURE 7. $G = 2$

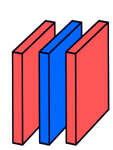


FIGURE 8. $G_1 = \frac{1}{2}$

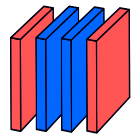


FIGURE 9. $G_2 = \uparrow$

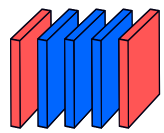


FIGURE 10. $G_3 = ⧾_1$

The second Left option is reversible and the second Right option is dominated by $\{0| - 1\}$.

$$G_3 = \{0 \mid \{0| - 1\}\}$$
$$= ⧾_1$$

This type of games are called *Tiny* and it is a type of infinitesimal. This particular game is Tiny(1) written as $⧾_1$.

Definition 39. For all $G > 0$, the games $⧾_G$ and $⧿_G$ are defined by

$$⧾_G = \{0 \mid \{0 \mid -G\}\} \text{ and } -(⧾_G) = ⧿_G = \{\{G \mid 0\} \mid 0\}.$$

We can prove that $\uparrow > ⧾_1$. Moreover, we have the following theorem.

Theorem 40. *The game* $⧾_1$ *is infinitesimally small with respect to* $\uparrow$*. That is,* $\forall n \geqslant 1$*,*

$$\uparrow > n \cdot ⧾_1.$$

Proof. We want to find the outcome of $\uparrow + n \cdot ⧿_1$ where $⧿_1 = \{\{1 \mid 0\} \mid 0\}$.

Case 1: Left starts. If Left moves in $\uparrow$ to go to $0 + ⧿_1$ then Right would win the game since, $⧿_1 < 0$. Hence, Left moves in one of the $⧿_1$ to go to $\uparrow + \{1 \mid 0\} + (n-1) \cdot ⧿_1$. Then, Right moves to $* + \{1 \mid 0\} + (n-1) \cdot ⧿_1$. By induction we prove that $\uparrow + n \cdot ⧿_1$ is won Left when Left starts.

Case 2: Right starts. Right has advantage in $⧿_1$ as $⧿_1 < 0$ so Right moves in $\uparrow$ to go to $* + n \cdot ⧿_1$. This is a $\mathcal{N} - position$ since Right can undo all Left moves in $⧿_1$. Left will not move in $*$ unless necessary and win the game. If at certain point Right moves in $*$ then Left will gain an advantage by going to $1 + m \cdot ⧿_1$, where $m \leqslant n$ and Left would win similarly.

Therefore, the game $\uparrow + n \cdot ⧿_1$ is an $\mathcal{L} - position$ which implies $\uparrow > n \cdot ⧾_1, \forall n \geqslant 1$. ∎

We can now easily make the game $\boldsymbol{+}_2$ by adding one more blue domino in the middle, making it 4 instead of 3 blue dominoes in Figure 10, and so on. We have the following result.

Lemma 41. *The game* $\boldsymbol{+}_2$ *is infinitesimally small with respect to* $\boldsymbol{+}_1$. *That is,* $\forall n \geqslant 1$,

$$\boldsymbol{+}_1 > n \cdot \boldsymbol{+}_2.$$

Proof. To prove this, we first show that $\boldsymbol{+}_1 + \boldsymbol{-}_2 > 0$ which implies $\boldsymbol{+}_1 > \boldsymbol{+}_2$.

$$\begin{aligned}\boldsymbol{+}_1 &= \{0 \mid\mid 0 \mid -1\}\\ \boldsymbol{+}_2 &= \{0 \mid\mid 0 \mid -2\}\\ \boldsymbol{-}_2 &= \{2 \mid 0 \mid\mid 0\}\end{aligned}$$

Case 1: Left moves first

Left makes a move to go to the game $\boldsymbol{+}_1 + \{2 \mid 0\}$ since $\boldsymbol{+}_1$ is positive game and it is advantage for the Left. Next, the Right's best move to replicate Left's move in $\boldsymbol{+}_1$ and the game becomes $\{0 \mid -1\} + \{2 \mid 0\}$. Now, the Left can move to the game $\{0 \mid -1\} + 1$ which gives advantage of free move to the Left. This game has only one move for the Right so the game becomes 1 and thus, Left wins.

Case 2: Right moves first

Right best move is disrupt the positive part of the game by moving to the game $\{0 \mid -1\} + \boldsymbol{-}_2$. Next, Left replicates the last move by Right and the game becomes $\{0 \mid -1\} + \{2 \mid 0\}$. Now, Right's best move to have a free move by going to $-1 + \{2 \mid 0\}$. Left now moves to, remove the all the Right domino that Left is able to, in order to force Right to let go of the free move. Thus, the Left moves to $-1 + 1$. Now, Right has only one move left and the game becomes 1. Thus, Left wins.

Therefore, we see that Left can force a win regardless of who moves first. Hence, we have

$$\begin{aligned}&\boldsymbol{+}_1 + \boldsymbol{-}_2 > 0\\ \implies &\boldsymbol{+}_1 > \boldsymbol{+}_2\end{aligned}$$

Let $\boldsymbol{+}_1 + k \cdot \boldsymbol{-}_2 > 0$ be true for some k. Now, we need to show that this is true for $k + 1$.

Case 1: Left moves first

Left's best move is to get rid of all the negative parts of the game, i.e., all the $\boldsymbol{-}_2$(s). This is done by turning these into $\{2 \mid 0\}$. Thus, the game becomes $\boldsymbol{+}_1 + \{2 \mid 0\} + k\boldsymbol{-}_2$. Now, the Right best move to get rid of the positive part of the game, i.e., $\boldsymbol{+}_1$, or make a move in $\{2 \mid 0\}$.

Suppose Right decides to make in $\boldsymbol{+}_1$ then the game becomes $\{0 \mid -1\} + \{2 \mid 0\} + k \cdot \boldsymbol{-}_2$. Left's best move is to keep getting rid of negative parts. This way the game eventually becomes $\{0 \mid -1\} + \{2 \mid 0\}$ on Right's turn. Left can go to $\{0 \mid -1\} + 2$ and Left wins.

Now, suppose Right decides to make a move in $\{2 \mid 0\}$ then the game becomes $\{0 \mid -1\} + k \cdot ⧿_2$. Again, Left's best move is to get rid of negative parts, this can happen k times, and the games end up becoming $\{0 \mid -1\} + \{2 \mid 0\}$ or $-1 + \{2 \mid 0\}$ with Right's turn to make a move. Again, this is clearly won by Left.

Case 2: Right moves first

Right's best move is to get rid of the positive part of the game and keep the negative parts of the game intact. Now, the game becomes $\{0 \mid -1\} + (k+1)⧿_2$. Now, this will follow like the above case, leading to Left winning.

Therefore, by induction $⧾_1 + k \cdot ⧿_2 > 0$ is true for natural number k. This implies that

$$⧾_1 > k \cdot ⧿_2, \forall k > 0$$

Thus, proving the lemma. □

This lemma gives the following theorem.

Theorem 42. *Let G and H be two game such that $G > H \geqslant 0$. Then $⧾_H$ is infinitesimally small with respect to $⧾_G$ i.e.*

$$⧾_G > n \cdot ⧾_H, \ \forall n \geqslant 1.$$

Proof. The proof follows through induction using the previous lemma. □

11. An overview of atomic weight theory

In this section you will learn "the raison d'être" for *atomic weight theory*, a tremendous tool for all-small games, developed first in Winning Ways [BCG1982]. We will review some of the interesting proofs from the lierature, but not all; the main references are [S2013, ANW2007].

In an all-small combinatorial game, for all subgames, either both players can move or neither can. Let us begin with some ruleset example, a disjunctive sum of a FLOWER GARDEN with a single strip of BIPASS. The ruleset FLOWER GARDEN [BCG1982] is a subset of GREEN-BLUE-RED HACKENBUSH; it has a green stalk of integer length, and on top a flower, with blue and red petal leaves. See Figure 13 to the right.

BIPASS is a much more recent ruleset [LN2023]. A bi-collective of one-directional micro organisms, consisting of a red tribe and a blue tribe, live in close proximity, and they take turns moving. The red tribe moves by letting one of its members crawl rightwards across a number of blue amoebae, while settling in the spot of a blue amoeba, and thus pushing each bypassed amoeba one step to the left, whereas the blue tribe moves by letting one of its members crawl leftwards across a bunch of red amoebae, while shifting the position of each bypassed amoeba one step to the right. Amoebae cannot bypass their own kind. When an amoeba reaches end of line, it cannot be played, and thus dies (of boredom). See Figure 11. The exception is if no more moves are possible in the full collective; in this case the last moving tribe wins, and is rewarded eternal life, as in Figure 12. This ruleset is called BIPASS.

Who wins the game in Figure 13, and why? To understand this, let us dwell a bit on the theory of atomic weights (from the books).

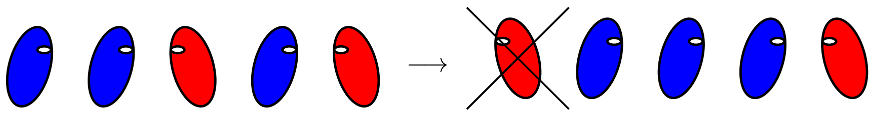


FIGURE 11. The middle amoeba crawled to the left end. When an amoeba does not face any opponent, even at a far distance, it gets removed, because it cannot be used in the game by either player.

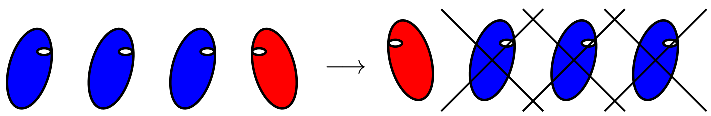


FIGURE 12. By moving, the single red amoeba bypassed all remaining amoebae, and will be celebrated as a hero by its resurrected tribe.

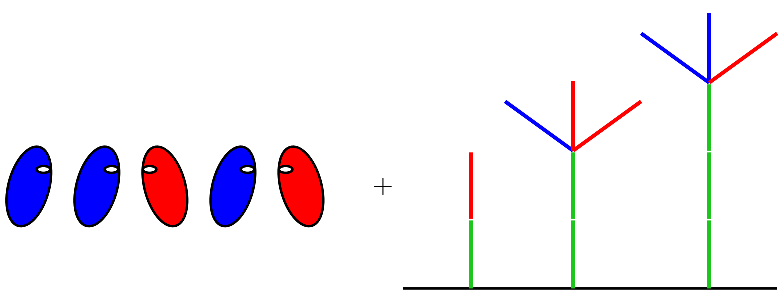


FIGURE 13. A BIPASS strip in disjunctive sum with a FLOWER GARDEN.

Definition 43 (Far Star)**.** The *far star*, denoted ☆, is an arbitrarily large NIM heap; that is both players can move to any NIM heap from ☆.[12]

By this definition, it follows that ☆ is absorbing, with respect to other nim heaps. That is, for all n, $\text{☆} + *n = \text{☆}$. Equivalence modulo ☆ is obtained as follows.

[12]In the books it is usually stated that it has the additional property that $\text{☆} + \text{☆} = \text{☆}$, but we are not going to use that here. Instead, any technical manipulation that involves far star will be justified by applying sufficiantly large nim heaps.

Definition 44 (Equivalence Modulo $\star$)**.** Let G, H be normal play games. Then $G \geqslant_{\star} H$, if, for all games X, and for all sufficiently large m, n, $o(G + X + *m) \geqslant o(H + X + *n)$, and $G =_{\star} H$ if $G \geqslant_{\star} H$ and $H \geqslant_{\star} G$.

By Definition 43, the 'game' $\star$ should be treated as a standard combinatorial game, since, for each game G, for any sufficiently large n (depending only on the followers of G), $o(G + *n)$ is constant. For any G, we may take n greater than the birthday of G. Sometimes we abuse notation, and instead of for all sufficiently large m, n write $o(G + X + \star) \geqslant o(H + X + \star)$ in Definition 44.

Theorem 45 (Constructive $\star$-equivalence, [S2013])**.** *Let G, H be normal play games. Then $G \geqslant_{\star} H$ if and only if $G - H > \downarrow + \star$, and $G =_{\star} H$ if and only if $\downarrow + \star < G - H < \uparrow + \star$.*

Proof. It suffices to study the inequality. Let us begin by proving that $G \geqslant_{\star} H$ implies $G - H > \downarrow + \star$. Take $X = \uparrow - H$ in Definition 44. Thus, we are assuming $o(G - H + \uparrow + \star) \geqslant o(\uparrow + \star)$. But it is easy to prove that

$$\uparrow + \star > 0. \tag{21}$$

And so, $G - H + \uparrow + \star > 0$, which, by regarding $\star$ as a large nim heap, implies $G - H > \downarrow + \star$. To prove the claim (21), playing first, Left can play to $\uparrow + *2$, and win. Playing second, if Right starts by playing to $* + \star$, then Left can respond to $* + * = 0$. Similarly, if Right plays in the $\star$ component, then Left wins (if he moves to $\uparrow + 0$, Left responds to 0, and otherwise Left moves to $\uparrow$).

For the other direction, we must prove that, if $G - H > \downarrow + \star$, then for any X,

$$o(G + X + \star) \geqslant o(H + X + \star). \tag{22}$$

Our intuition does not help us here, since, at a first glance it looks as if the "$\downarrow$" might break the proposed inequality of the outcomes. But the far star will come in handy, just when we thought all hope is lost. We will review two completely different proofs from the literature [ANW2007, S2013], perhaps neither will give us a tangible explanation but even so may be appreciated as one of those 'pure math proof' that we must rely on, when all intuition goes wrong.

Let us start with the proof idea from "Lessons in Play" [ANW2007]. The proof is by way of contradiction, and we (as Right) will play three games at once against an oracle (as Left), who claims that they can win all three games. They play first in one of them. And we play first in the other two games. We keep track of standard alternating play in each game. The three games are as follows, where Left plays first in (iii):

(i) $-(G + X) + \star$;
(ii) $(G + X) - (H + X) + \uparrow + \star$;
(iii) $(H + X) + \star$.

Note that we added and subtracted X in the second game. The X-component is the same in all three games and it originates from the outcome identity that we set out to try and contradict, (22). We will use that the far-star in each component can be chosen as arbitrary sufficiantly large nimbers, to altogether reach a desired contradiction. A contradiction can be stated as "Left wins all three games" (check this!). We can let the oracle as Left start in (3), and Right (as us) start in the other two games. If the oracle starts in the $(H + X)$ component in (3), then we 'mimic' this move in the $-(H + X)$ component in (2). Next, the oracle might play in the $(G + X)$ component in (2), and we can mimic in the $-(G + X)$ component in (1), and so on. For any 'non-far-star-move' and 'non-up-move', this typ of mimic procedure can continue, until we get the last move in all such components.

Observe that at each stage of such play, after the oracle moved, we will be the starting player in all three games; hence we are free to play in either. And, if we get the last move in any game (1), (2) or (3) then we have reached the desired contradiction.

Next let us study oracle moves in a far-star-component, to say $*k$. We will respond in game (2) in the up-component, and then we will 'finitize' the remaining far stars into very large nim heaps $*m$ and $*n$ such that $*k \oplus *m \oplus *n \oplus * = 0$. If the oracle plays in the up-component in game (2), then we instead 'finitize' the far-star-components such that $*k \oplus *m \oplus *n = 0$. In this way, we can assure that we get the last move in one of the three games, and hence Right cannot win them all, as stipulated.

By this contradiction, the statement holds.

Next we study the idea from "Combinatorial Game Theory" [S2013]. To come...

□

For example, we have $G =_{\star} 0$ if and only if Left wins $G + \uparrow + *n$ and Right wins $G + \downarrow + *n$ for some sufficiently large n, and n is sufficiently large if $*n$ does not equal any follower of $G + \uparrow$ or $G + \downarrow$. This motivates the naming as a constructive $\star$-equivalence. This method is useful also in proofs, for example whenever we have a good guess of one of the games G or H, then we use this result to verify whether our candidate games are equal. See Theorem 50 for how it applies to atomic weight theory.

Let X be a set. Then $X + y = \{x + y : x \in X\}$. If X is a set of all small games, let $\mathrm{aw}(X) = \{\mathrm{aw}(x) : x \in X\}$.

The product of a game G and $\uparrow$ is: $0 \cdot \uparrow = 0$; $n \cdot \uparrow = \uparrow + (n-1) \cdot \uparrow$, in case $G = n$ is an integer. Otherwise $G \cdot \uparrow = \{G^{\mathcal{L}} + \Uparrow * \mid G^{\mathcal{R}} + \Downarrow *\}$.

Lemma 46 ([S2013])**.** *Consider any normal play game G.*

(i) If $G \cdot \uparrow \geqslant_{\star} 0$ then $G \geqslant 0$.
(ii) If g is all-small then there is a unique G such that $g =_{\star} G \cdot \uparrow$.

In a proof of this lemma, the first item implies the second, and we use the uniqueness to define atomic weight.

Definition 47 (Atomic Weight)**.** The atomic weight of an all-small game g is the unique game $G = \mathrm{aw}(g)$ such that $G \cdot \uparrow =_{\star} g$.

Example 48. Let $g = *n$. By Theorem 45, $\mathrm{aw}(*n) = 0$. Namely, we claim that Left wins $*n + \uparrow + \star$ (and symmetrically Right wins $*n + \downarrow + \star$). If Left starts, she can play to $\uparrow > 0$. If Right starts by playing to $*m + \uparrow + \star$, then Left wins (by induction). If he plays to $*(n+1) + \star$, then Left can respond to $*(n+1) + *(n+1) = 0$. If he plays to $*n + \uparrow + *m$, then Left wins playing first (Right can win playing first if and only if $*n + *m = *$).

Example 49. Let $g = \uparrow$ and let $h = \uparrow*$. By Theorem 45, $\mathrm{aw}(g) = \mathrm{aw}(h) = 1$. Namely $\star < \uparrow$ and $\star < \uparrow*$. By using also symmetry, the verification is similar to the one in Example 48.

In this example we had to guess the atomic weight 1 and then verify. There is a constructive/recursive method for computing atomic weights.

Theorem 50 (Constructive Atomic Weight)**.** *Let g be an all-small game, and let*

$$G = \left\{ \mathrm{aw}(g^{\mathcal{L}}) - 2 \mid \mathrm{aw}(g^{\mathcal{R}}) + 2 \right\}.$$

Then $\mathrm{aw}(g) = G$, unless G is an integer. In this case, compare g with the far star. If

(i) $g \,\|\, \star$, *then* $\mathrm{aw}(g) = 0$*;*
(ii) $g < \star$, *then* $\mathrm{aw}(g) = \min\{n \in \mathbb{Z} : n \,|\!\!\triangleright\, G^L\}$*;*
(iii) $g > \star$, *then* $\mathrm{aw}(g) = \max\{n \in \mathbb{Z} : n \,\triangleleft\!\!|\, G^R\}$*.*

Theorem 51 (Atomic Weight Properties)**.** *Let g and h be all-small games. Then*

(i) $\mathrm{aw}(g + h) = \mathrm{aw}(g) + \mathrm{aw}(h)$*;*
(ii) $\mathrm{aw}(-g) = -\mathrm{aw}(g)$*;*
(iii) if $\mathrm{aw}(g) \geqslant 1$*, then* $g \,|\!\!\triangleright\, 0$ *(Left wins playing first);*
(iv) if $\mathrm{aw}(g) \geqslant 2$*, then* $g > 0$ *(Left wins).*

As usual '$|\!\!\triangleright$' denotes greater than or confused with. In particular (iv) is the raison d'être for atomic weight, and it is popularly called "the two-ahead-rule". We will have plenty use for it.

For example, we argue that the game in Figure 13 is an $\mathscr{R}$-position. Namely, the chosen rulesets satisfy very elegant properties with respect to the atomic weight theory. If a flowers in the garden has more bLue (Left plays bLue) than Red petal leaves, then this flower has atomic weight one. And since atomic weights are additive, we can simply compute the number of red flowers minus the number of blue flowers to get the atomic weights of a Flower Garden (see for example [BCG1982] or [S2013]). Bipass has the inverse property in a sense: the more pieces of the opponent the better. Let $\Delta(g)$ be the number of bLue amoebae minus the number of Red amoebae in a Bipass strip. Then $\mathrm{aw}(g) = \Delta(g)$. See [LN2023] for a proof of this result. By these two results, we can compute the atomic weight of the disjunctive sum game in Figure 13, and indeed, it is two, so the two ahead rule applies, Right is two atomic units ahead of Left, and so he will win independently of who starts. (How?)

12. An overview of reduced canonical form, and a bit about temperatures and hotstrat

Recall the definitions of Left- and Right stops, Ls and Rs respectively, from Definition 27. A game G is *hot* if $\mathrm{Ls}(G) > \mathrm{Rs}(G)$. If $\mathrm{Ls}(G) = \mathrm{Rs}(G)$, then G is *cold*, if a number, and *tepid* otherwise. And G is an *infinitesimal* if $\mathrm{Ls}(G) = \mathrm{Rs}(G) = 0$.[13]

Example 52. For example $G = \{\{2 \mid 1\} \mid -1\}$ is hot, because $1 = \mathrm{Ls}(G) > \mathrm{Rs}(G) = -1$. The number game $G = 1$, of course satisfies $\mathrm{Ls}(1) = \mathrm{Rs}(1) = 1$. On the other hand, $H = \{\{2 \mid 1\} \mid 1\} = 1 + \{\{1 \mid 0\} \mid 0\} = 1 + ⧿_1$ is tepid, with $\mathrm{Ls}(H) = \mathrm{Rs}(H) = 1$. Moreover, $g = \{\{1 \mid \uparrow\} \mid \downarrow\}$ is infinitesimal, because $\mathrm{Ls}(g) = \mathrm{Rs}(\{1 \mid \uparrow\}) = \mathrm{Ls}(\uparrow) = 0$ and $\mathrm{Ls}(g) = \mathrm{Rs}(\downarrow) = 0$.

In a sense, the "one move for Left", which is hidden to the left of the Left option g^L (in the last game in Example 52), will mostly be irrelevant in play. Similar to Miny and Tiny though, the option does not reverse out. The hinge here of course is the word "mostly". In an environment with similar infinitesimals, there are situations where the hidden "one move for Left" can become alive. But those situations are rare, and the concept of a Reduced Canonical Form (rcf) removes the appearance of infinitesimals for all subgames. It becomes an important tool to analyze games that otherwise seem intractable.

In fact, there are two approximation techniques related to these concepts, namely Temperature Theory (see Section 13), which outputs a temperature and a mean value, and Reduced Canonical Form (rcf), which outputs a coarsening of the usual canonical form (value) of a game. By taking the reduced canonical form approximation of a hot game, the temperature and mean value remain the same. Hence, one may view Temperature Theory as a last resort, if both the game value and the rcf are intangible for a human eye. As usual, we are looking for information of how to play well games in a disjunctive sum. There is a classical strategy called *hotstrat*, which says: play in the hottest component. This is often the best strategy, but not always. Take, for example, the disjunctive sum game

$$G = \{1/2 \mid -100\} + \{100 \mid 1/2\} + \{0 \mid \{-1 \mid -101\}\}$$

[13]A special case of infinitesimal games are the all small games. But there are many infinitesimals that are not all small. For example, we have already seen $⧾_1$ and $⧿_1$.

Hotstrat fails here. The game is an $\mathscr{N}$-position, but if Left starts in the hottest component, she loses.[14] She needs to first eliminate Right's threat in the coolest component.

A brief introduction to Temperature Theory is the topic of the next section. In fact G is problematic also from a point of view of reduced canonical form: in that case G remains the same, and its canonical form appears more complicated for a human eye than the displayed disjunctive sum. We will see other examples when rcf is more helpful.

The reduced canonical forms for the games in Example 52 are $\mathrm{rcf}(G) = \{\{2 \mid 1\} \mid -1\}$, $\mathrm{rcf}(H) = 1$, and $\mathrm{rcf}(g) = 0$. We will study the meaning of these statements in what comes, and we will define all concepts accordingly.

First we define the equivalence classes modulo inf, and state a main theorem with tools of efficient computation of inf-equivalence. Then, we define the reductions modulo inf, and at last we state a result of uniqueness of the end result of these reductions.

There are many hot games for which $\mathrm{rcf}(G) \neq G$. For example $G' = \{\{2 \mid 1*\} \mid -1\}$ is hot and in canonical form, but $\mathrm{rcf}(G') = \{\{2 \mid 1\} \mid -1\}$.

The equivalence relation modulo infinitesimals is defined as follows.

Definition 53 (Equivalence Mod Inf)**.** Consider game G and H. Then $G \geqslant_{\mathrm{inf}} H$, if, for all positive numbers x, $G - H > -x$. And $G =_{\mathrm{inf}} H$ if $G \geqslant_{\mathrm{inf}} H$ and $H \geqslant_{\mathrm{inf}} G$.

That is, $G =_{\mathrm{inf}} H$ if $-x < G - H < x$ for all positive numbers x. The games G and H are *infinitesimally close* if $G =_{\mathrm{inf}} H$, and this is also called "equivalent modulo infinitesimals" or simply "inf-equivalent". The relation $\geqslant_{\mathrm{inf}}$ is a partial order, the partial order modulo infinitesimals. We may use terms such as "numberish" if a game is infinitesimally close to a number, and if $G =_{\mathrm{inf}} H$, then we say that H is G-ish and vice versa.

The first result of this section simplifies game comparison modulo infinitesimals.

Theorem 54 (Constructive Inf-inequality)**.** *Consider two games G and H. The following are equivalent.*

(i) $G \geqslant_{\mathrm{inf}} H$*;*
(ii) $\mathrm{Rs}(G - H) \geqslant 0$*;*
(iii) $G - H \geqslant \epsilon$ *for some infinitesimal* ϵ*.*

Proof. Since $\mathrm{Rs}(G-H)$ is a number, item (ii) follows from item (i) by letting $x > 0$ be arbitraty small. It requires and argument that (ii) implies that it suffices to take $\epsilon = n \cdot \downarrow$, for some large n; this is Theorem 4.9 in [S2013]. Then (iii) follows. And (iii) implies (i), by the definition of an infinitesimal (with respect to numbers). □

[14]Other instructive examples of when it fails uses the concept of over heating (a type of 'inverse' of cooling), but we do not have the time to cover that topic in this course; see [S2013].

The second and third items in Theorem 54 are efficient tools, and they often appear in proofs and examples.

For example,

$$1 \geqslant_{\text{inf}} \{1 \mid 0\}, \tag{23}$$

by (iii), since $1* \geqslant \{1 \mid 0\}$. Indeed, Left wins $\{1 \mid 1\} + \{0 \mid -1\}$ playing second. Recall, however that $1 \parallel \{1 \mid 0\}$, since Next player wins $1 + \{0 \mid -1\}$. The small shift in parity shifted the comparison in Left's favor.

And the inequality (23) holds by (ii), since $\text{Rs}(1 + \{0 \mid -1\}) = 0 \geqslant 0$.

Of course, similar arguments show that, for any numbers a, b with $a > b$, then $a \geqslant_{\text{inf}} \{a \mid b\}$, while $a \parallel \{a \mid b\}$.

It is worthwhile rewording Theorem 54 in terms of inf-equivalence.

Corollary 55 (Constructive Inf-equivalence)**.** *The following are equivalent.*

(i) $G =_{\text{inf}} H$*;*
(ii) $\text{Rs}(G - H) = \text{Ls}(G - H) = 0$*;*
(iii) $\epsilon \leqslant G - H \leqslant \epsilon'$ *for some infinitesimals* ϵ, ϵ'*.*

Proof. Use Theorem 54. □

For example, $\uparrow =_{\text{inf}} \downarrow$, by (ii), since $\text{Rs}(\Uparrow) = \text{Ls}(\Uparrow) = 0$.

For example, $\{1 \mid 1\} =_{\text{inf}} 1$, by (iii), since $\{1 \mid 1\} = 1*$ and $* \leqslant 1* - 1 \leqslant *$.

Definition 56 (Inf-reduction)**.** Let G be a game.

(i) Suppose $A \in G^{\mathcal{L}}$. Then A is inf-replaceable by B, if $A =_{\text{inf}} B$.
(ii) Suppose $A, B \in G^{\mathcal{L}}$. Then A inf-dominates B, if $A \geqslant_{\text{inf}} B$.
(iii) The Left option G^L is inf-reversible (through G^{LR}), if $G \geqslant_{\text{inf}} G^{LR}$ for some Right option G^{LR}.

Observe that Definition 56 (ii) and (iii) resemble the standard reduction techniques, that together produce a canonical form game value. But (i) is new in this setting. As we will see, altogether they produce a unique simplest form game, the 'canonical form modulo infinitesimals'.

If the game G in Definition 56 already equals a number, then inf-reduction is not interesting. The theory requires that we omit such cases in our inf-reduction techniques.

Theorem 57 (Inf-Replaceable)**.** *Consider a game* $G \notin \mathbb{D}$*, and suppose* $A \in G^{\mathcal{L}}$*. If* A *is inf-replaceable by* B*, then* $G =_{\text{inf}} \left\{G^{\mathcal{L}} \setminus \{A\}, B \mid G^{\mathcal{R}}\right\}$*.*

Proof. Let $H = \left\{G^{\mathcal{L}} \setminus \{A\}, B \mid G^{\mathcal{R}}\right\}$. It suffices to prove that,

(i) if $A \geqslant_{\text{inf}} B$, then $G \geqslant_{\text{inf}} H$;
(ii) if $B \geqslant_{\text{inf}} A$, then $H \geqslant_{\text{inf}} G$.

We prove (i), and (ii) is similar. Suppose $A \geqslant_{\text{inf}} B$. By definition, it suffices to prove that Left wins $G - H + x$ playing second, for any positive number x. Since $x \in \mathbb{D}$ and $G \notin \mathbb{D}$, Number Avoidance implies that Right will not play in the x-component.[15] But all options expect Right playing to $G - B + x$,

[15] Siegel [S2013]proves a slightly stronger version of Number Avoidance.

have mimic responses, implying that Left wins playing second since $x > 0$. However, Left can play to $A - B + x$, which is positive, by the assumption $A \geqslant_{\text{inf}} B$. □

Let us make an observation about the illusiveness of this result if we try to apply it to a game that equals a number. Consider the game $G = \{0 \mid 1*\}$. According to Definition 56, $1*$ is inf-replacable by 1. Now consider the number game $H = \{0 \mid 1\} = 1/2$. One might expect that adding an infinitesimal to the Right option of H, to instead obtain our game G, might not change the situation significantly. However, $G = \{0 \mid 1 + *\} = \{0 \mid \{1 \mid 1\}\} = 1 = \{0 \mid \varnothing\}$ (!), by standard reversibilty. Clearly, inf-equivalence cannot hold for games that differ by one half. The lesson to be learned is that, if in doubt, one should first reduce a game to its usual canonical form, and if this does not reveal a number, then freely apply the inf-replacement technique.

Now, let us do a similar experiment, by analysing inf-reduction instead on the hot game $G = \{1* \mid 0\}$. Indeed $\text{Ls}(G) = 1 > \text{Rs}(G) = 0$. Let $H = 1/2 \pm 1/2$ be the game where we have inf-replaced $1*$ by 1. We use Corollary 55 (ii) to verify inf-equivalence. Indeed $\text{Ls}(G - H) = \text{Ls}(\{1* \mid 0\} - \text{Ls}(\{1 \mid 0\})) = 1 - 1 = 0$, and $\text{Rs}(G - H) = \text{Rs}(\{1* \mid 0\} - \text{Rs}(\{1 \mid 0\})) = 0 - 0 = 0$.

Theorem 58 (Domination Mod Inf, [S2013])**.** *Consider a game $G \notin \mathbb{D}$. If G' is obtained by removing some inf-dominated option (either Left or Right). Then $G =_{\text{inf}} G'$.*

Example: the game $\{1, 1* \mid 0\} =_{\text{inf}} \{1 \mid 0\}$, since 1 inf-dominates $1*$, and G does not equal a number.

Theorem 59 (Reversibility Mod Inf, [S2013])**.** *Consider a game G. Suppose that G^L is inf-reversible through G^{LR}, and let*

$$G' = \{G^{\mathcal{L}} \setminus \{G^L\}, G^{LR\mathcal{L}} \mid G^{\mathcal{R}}\}.$$

(i) If $G' \notin \mathbb{D}$, then $G =_{\text{inf}} G'$;
(ii) If G is hot, then $G =_{\text{inf}} G'$.

It is important that G' in Theorem 59 is not a number. For example, with $H = \{\{2 \mid 1\} \mid 1\}$, as in Example 52, we get that $H^{LR} \leqslant_{\text{inf}} H$, since, by Theorem 32, $H^{LR} = 1 \leqslant_{\text{inf}} 1 + ⧿_1 = H$. And so, if the result would apply to numbers, since $H^{LR\mathcal{L}} = 0$, then $H' = \{0 \mid 1\} = 1/2 \neq_{\text{inf}} 1$.

Luckily, we have other tools, and the Number Translation Theorem (Theorem 32) tells us that $H = 1 + ⧿ =_{\text{inf}} 1$, by using also Theorem 54 (3). Or, we can use directly Theorem 54 (2), by noting that $\text{Ls}(H) = \text{Rs}(H) = 1$.

Let us give an example, where we can use inf-reversibilty. Let $G = \{1, \{100 \mid \uparrow\} \mid \uparrow\}$. It can be verified that this is a canonical form game. For example, $1 \parallel \{100 \mid \uparrow\}$. Consider the Left option with $G^{LR} = \uparrow$. If we can prove that

$$\uparrow \leqslant_{\text{inf}} G, \tag{24}$$

then $G^{LR\mathcal{L}} = *$, which will be dominated by the Left option 1, and it would follow that $G =_{\inf} \{1 \mid 0\}$.

To prove inequality (24), note that $\mathrm{Rs}(G+\downarrow) = \mathrm{Rs}(\{1, \{100 \mid \uparrow\} \mid \uparrow\}+\downarrow) = \min\{\mathrm{Ls}(\uparrow + \downarrow), \mathrm{Ls}(G)\} = 0$, and then use Theorem 54 (ii).

Definition 60. A game G is in *reduced canonical form*, if, for every subgame H of G, either H is in canonical form and is a number, or H is hot and G does not contain any inf-dominated or inf-reversible options.

In particular, if a game is tepid, then its reduced canonical form is a number.

Theorem 61 ([S2013])**.** *For every game G, there exists a unique game H in reduced canonical form such that $G =_{\inf} H$.*

Definition 62. The *reduced canonical form* of a game G, denoted $\mathrm{rcf}(G)$, is the unique reduced canonical form H, such that $\mathrm{rcf}(G) = H$.

Here is an example of a not too complicated canonical game, but where a human eye might not capture the essential information immediately,

$$G = \{1*, 1\downarrow \mid \{1\downarrow \mid \downarrow\}\}.$$

Is this game infinitesimally close to some game with a much simpler form? Yes, perhaps not very surprisingly $\mathrm{rcf}(G) = 1$.

Example 63. Let us give an example of a full ruleset to see how to apply rcf. Consider an instance of PARTIZAN SUBTRACTION, where Left can subtract any odd number and Right can subtract any even number. Let $\langle h\rangle$ denote the canonical form of a heap of size h under this ruleset. Then obviously $\langle 0\rangle = 0$ and $\langle 1\rangle = 1$. Note the large number of options from large heap sizes. However it turns out that most of them will be irrelevant, and in particular when we apply rcf. Note that this ruleset is not all-small, and hence we might expect cold and/or hot games. And in fact, we will get a bit of both.

One can prove that the canonical form of a game is the dyadic $\langle h\rangle = 1/2^{(h-1)/2}$, if the heap size h is odd, and otherwise the canonical form satisfies the recurrence $\langle h\rangle = \{1 \mid 0, \langle h-2\rangle\}$. (This part is left as an exercise.)

The reduced canonical form is the same as the canonical form if h is odd, but if h is even, then $\mathrm{rcf}(\langle h\rangle) = \{1 \mid 0\}$. To verify this, let us compute the stops of $\langle h\rangle + \{0 \mid -1\}$.

We have $\mathrm{Rs}(\{1 \mid 0, \langle h-2\rangle\} + \{0 \mid -1\}) \leqslant 0$, since (for example) Right can start by playing to $\{0 \mid -1\}$. On the other hand, $\mathrm{Ls}(\{1 \mid 0, \langle h-2\rangle\} + \{0 \mid -1\}) \geqslant 0$, since Left can play to $1 + \{0 \mid -1\}$. We may use induction to conclude the statement.

Usually, but not always, the reduced canonical form is a lot simpler than the canonical form. In case the rcf still seems intractable for a human eye, then temperature theory might be a last resort to estimate a reasonable 'play-value' of a game.

13. Intuitive temperature theory

We have already seen examples of hot games: games with some urgency to play first. Typically one would think of the *switches*, games of the form $\pm x = \{x \mid -x\}$, for some positive number x. For example $G = \pm 10 = \{10 \mid -10\}$ is hot, and it has temperature $T(G) = 10$ (see Figure 16 for its so-called thermograph).

We find the temperature of a hot game, by cooling it, and observing when the cooled game becomes a tepid game. In our example, the smallest number $t \geqslant -1$, for which $G_t := \{10 - t \mid -10 + t\}$ is tepid, is indeed $t = 10$. And this defines $T(G)$. Indeed, $G_{10} = *$, an infinitesimal, and they are all tepid. We are thinking of 'cooling' as if each player is paying a penalty of t. As long as they can keep paying the penalty and benefiting in 'score = number of moves' playing first, the game can still be cooled further. Thus, the hot game $G = \{3 \mid 1\}$ can be cooled by at most 1 until it freezes and becomes the tepid game $2* = 2 + *$, where no player benefits by paying further penalty in exchange for playing first.

The idea of temperature theory can easily be invisioned by playing a disjunctive sum of switches. For example, if the game is $\pm 10 + \pm 5 + \pm 1$, then the first player will win if and only if they play in the hottest game component "± 10"; If Left starts and plays to $10 + \pm 5 + \pm 1$, Right has no defence.

To begin with, by using this naïve understanding of heat, how should one play optimally in the disjunctive sum

$$\{10 \mid -10\} + \{3 \mid 1\} + \{1 \mid \{0 \mid -100\}\}?$$

The hottest game component is $\{10 \mid -10\}$, and the first player has a winning move there. It does not need to be the unique winning move though. There is another winning move for one of the players.

Let us give another simpler example, where Right, playing first, benefits by giving a threat, rather than playing in the hottest component. The game is: $\{10 \mid -10\} + \{10 \mid \{9 \mid -11\}\}$. In fact, he wins if and only if he plays in the coldest component.

The general definition of temperature is recursive in t, starting at the two stops. It includes the possibility of cold games with negative temperature; the coldest game has defined temperature -1, and that holds for any game that is an integer (see Figure 14 for its thermograph). The integers cannot be further cooled.

Suppose that we wish to 'cool' the game $1/2 = \{0 \mid 1\}$ to find its temperature. We have to 'pay' a negative penalty (because the game is already cold). We get $(1/2)_{-1/2} = \{0 + 1/2 \mid 1 - 1/2\} = 1/2 + *$, but if $t < -1/2$, then $(1/2)_t$ is not tepid, but in fact hot. The same idea works for any non-integer number game, and by using that $\frac{m}{2^k} = \left\{\frac{m-1}{2^k} \mid \frac{m+1}{2^k}\right\}$, we get that the temperature of $\frac{m}{2^k}$ is $-\frac{1}{2^k}$.

The standard definition of temperature found in books is non-constructive, but is seems a bit difficult to avoid this, since we must cool a game G everywhere all at once; we start the cooling at the Left and Right stops of $G_0 = G$, and continue until $\mathrm{Ls}(G_t) = \mathrm{Rs}(G_t) = 0$. We omit this somewhat technical definition (see [S2013] for a rigorous treatment of temperature). An intuitive description suffices for practical purposes, to find the temperatures and the *mean values* of some hot games from your rulesets.

Let us explain this concept with an example. Consider the game $\{3 \mid -2\}$. Is this game 'better' for Left or Right? Well, anyone can see that it depends on who starts, and it seems also that Left has a definite advantage in that she can earn one more move than Right could, if she gets to start. The mean value of a game measures this type of 'advantage'.

The *mean value* of a game G is defined as the number $m(G)$ such that the difference $n \cdot G - n \cdot m(G)$ is bounded by a constant, independently of the size of the positive integer n. The mean value theorem states that such a number m exists, and that is suffices to take either of the stops to 'compute' it.

Theorem 64 ([S2013])**.** *For any given game G, the mean value $m(G)$ exist and equals* $\lim \frac{\mathrm{Ls}(n \cdot G)}{n} = \lim \frac{\mathrm{Rs}(n \cdot G)}{n}$.

A standard tool to find both the temperature and mean of a game is via its *thermograph*. The thermograph of a (hot) game G is drawn, by starting at the Left and Right stops and gradually cooling (by increasing the penalty t), and watching carefully that at every line drawn follows the Left and Right stops of the current G_t. This procedure is most easily understood by drawing some example games. Let us explain via Figures 14 to Figures 18. The first picture represents a cold game, the second a tepid game, and all other are hot games. Take a look at Figures 14 and 15. They look superficially similar, but there is an important difference. The games have different temperatures, so their thermographs should look different, right? And they do, the first game has temperature -1, and the *mast* continues down below the picture, whereas the second picture has temperature 0, because it is a tepid game, and that is illustrated by the fact that the bottom of the mast rests at the horizontal line, at the top of a trivial thermograph. It is easy to see that the location of the mast is the mean value of these two games.

The next three thermographs are more interesting. Of course, Figure 16 is the thermograph of the game in the first paragraph of this section. And it illustrates nicely the idea of cooling that game until we reach *mast value* 0 and temperature 10. Again, the mast value is the same as the mean value of the game. Check this by playing a large sum of games, where each component is of the form $\{10 \mid -10\}$. The first player's advantage is quickly diminished by the fact that the second player can, at each response cancel the first player's advantage. For switches, it is easy to see that Theorem 64 holds and it is also easy to see that the mast value and the mean value is the same. That this holds in general is another theorem proved in [S2013].

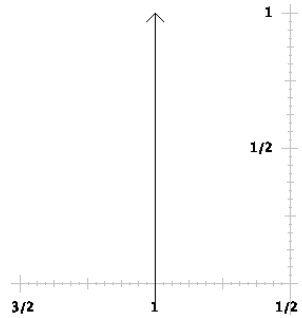


FIGURE 14. The thermograph of the number game 1.

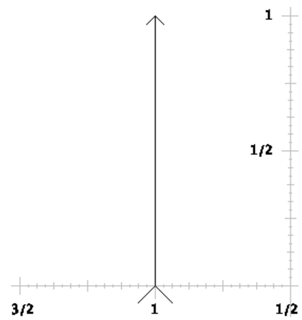


FIGURE 15. The thermograph of the tepid game $\{1 \mid 1\} = 1*$.

Theorem 65 ([S2013])**.** *For any game, the mast value and the mean value is the same.*

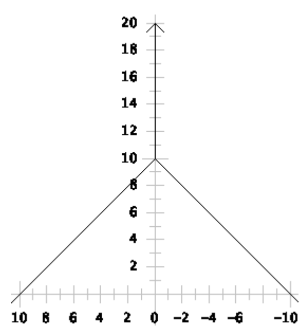


FIGURE 16. The thermograph of the switch $\{10 \mid -10\}$.

In the next picture, we study games of the form $G = \{a \mid \pm b\}$, where $a > b$ are positive numbers. There is a geometric approach to arrive at the vertical border to the right in Figure 17. Namely use the thermograph of the switch $\pm b$, and turn it 45 degrees to the left, by fixing it at the point of the Right stop. The leftmost *wall* of the thermograph of the Right option becomes the rightmost wall of the game G. The slope of the leftmost wall remains the same as in the previous picture (but both the mean value and the temperature change).

Also in this type of games, we can justify Theorems 64 and 65 directly, by inspection. Try this!

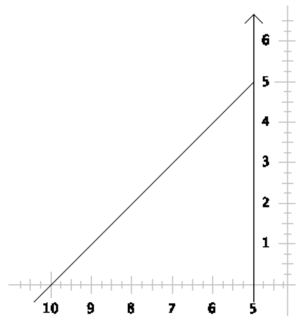


FIGURE 17. The thermograph of the game $\{10 \mid \{5 \mid -5\}\}$.

We will leave the justification of the thermograph in Figure 18 as an exercise. The idea is to raise the horizontal bottom level as one cools the game, and carefully check which option leads to the Right stop at each phase of the cooling. As in the previous picture, the Right options have to be tilted 45 degrees to the left, by fixing the Right stop of the cooled game.

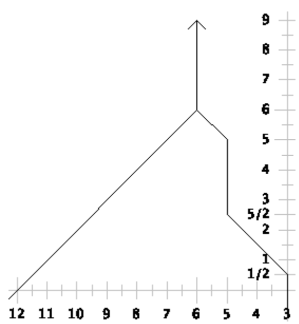


FIGURE 18. The thermograph of the game $\{12 \mid \{5 \mid -5\}, \{3 \mid 2\}\}$.

Note that a cold game might have hot options, and the described procedures do not apply, as one should conveniently reduce a game to its canonical form, before studying its temperature and mean value. For example the literal form game $\{\{1 \mid -1\} \mid 0\}$ has a hot Left option, but the game equals a number since $\mathrm{Ls}(-1) < \mathrm{Rs}(0)$. Indeed if Left moves, then Right can reerse out the move to obtain -1; Altogether, the game must equal -1, with temperature -1.

14. Impartial theory

Suppose that we are given a candidate set of $\mathscr{P}$-positions of an impartial ruleset. To verify this set, we prove that there is no move from one candidate $\mathscr{P}$-position to another candidate $\mathscr{P}$-position. And we abbreviate this property as "$\mathscr{P} \not\to \mathscr{P}$". Moreover, we must prove that each candidate $\mathscr{N}$-position has an option in the set of candidate $\mathscr{P}$-positions. This we write as "$\mathscr{N} \to \mathscr{P}$". This way of thinking of course bases on the idea of induction, as is often the case in CGT.

14.1. **Rigorous Nim.** Using this property of the impartial outcomes, we will next prove two results. First we prove Bouton's theorem on the game of Nim, and in the next subsection we discuss the so-called Wythoff Properties. Recall the nim-sum definition "$\oplus$" from Lecture 2.

Theorem 66 ([B1902])**.** *The impartial theory is more or less independent of the partizan ditto. Of course the properties of $\mathscr{P}$ and $\mathscr{N}$ positions transfer without change, but obviously there is no relevance in the notion of game comparison: all games are incomparable, since players have the same options for all positions.*

Let h_i denote the heap sizes of a game of Nim *on n heaps, written in binary power of two expansion, and where $i \in \{1, \ldots, n\}$. The outcome is a $\mathscr{P}$-position if and only if $\bigoplus h_i = 0$.*

Proof. For the property "$\mathscr{P} \not\to \mathscr{P}$", suppose first that

$$\bigoplus h_i = 0. \tag{25}$$

We must prove that every option has non-zero nim sum. Observe that (25) means that there is an even number of 1s in each column of the disjunctive sum, where the rows are the heap sizes h_i written in binary (as in Lecture 2). If there is no option, we are done. Otherwise, a Nim option reduces exactly one heap, corresponding to a row in our binary representation of the heap sizes. Thus there must be a change of parity of the number of ones in at least one column of (25). Thus the new nim-sum is non-zero.

For the property "$\mathscr{N} \to \mathscr{P}$", suppose next that

$$\bigoplus h_i \neq 0, \tag{26}$$

and let $x = \bigoplus h_i$ be the result of the nim addition in (26). We must prove that there is a heap h_k that can be reduced such that the new nim-sum is zero.

Write the nim sum x in its binary representation, as $x = \sum 2^i x_i$, where, for all i, $x_i \in \{0, 1\}$. By (26), there is a largest index j such that $x_j = 1$. Thus, there must be an odd number of heaps that contain the jth power of 2, 2^j. We claim that either one of these heaps, say heap h_k, suffices.

Similar to the base 2 expansion of x, write $h_k = \sum 2^i h_{k,i}$. Then, by definition of j and h_k, $h_{k,j} = 1$. For all $i < j$ such that $x_i = 1$, let $h'_{k,i} = \overline{x}_i$, where $\bar{\cdot}$ is the binary complement (that is $\overline{0} = 1$ and $\overline{1} = 0$), and otherwise, let $h'_{k,i} = h_i$. For all $i \geqslant j$, let $h'_{k,i} = 0$. A winning move is to reduce h_k to $h'_k = \sum 2^i h'_{k,i}$. That this is a reduction of the heap size h_j follows from the fact that in base two expansion, for all j, $2^j > \sum_{i<j} 2^i$.[16] □

Probably the greatest challenge in this proof is to understand the technical part of the last paragraph. Of course, this part can be expanded by using more sentences in English, similar to the earlier parts of the proof. However, it is also important sometimes to practice reading more 'pure logic' parts of proofs. Why is the "binary complement" introduced in this last paragraph? Is it the best way to do it, or can you find a way to say the same thing without that definition? (There is no ultimate answer to such a question; this question is more meant as a challenge to think of how we write proofs, and why we do what we do in a proof.)

14.2. **The Wythoff Properties.** Next, we will prove that the following properties define the $\mathscr{P}$-positions of WYTHOFF NIM. Consider two sequences of integers $(a_n), (b_n)$, $n \in \mathbb{N}_0$. They satisfy the Wythoff Properties if:

(i) $(a_0, b_0) = (0, 0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

Property (ii) is called "*increasing*". Property (iii) we may call a "*shift*". Together with (ii), the properties (iv) and (v) are usually called "*complementarity*".[17] The following result establishes that, in fact, these properties define a unique pair of sequences.

Theorem 67. *The set $W = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}_0\}$ given by the Wythoff Properties is unique, and it is the set of $\mathscr{P}$-positions of* WYTHOFF NIM.

Proof. Observe that it suffices to prove that the set W is the set of $\mathscr{P}$ - positions of WYTHOFF NIM. It then follows that the properties define a unique pair of sequences. Observe that by (ii) and (iii), both $a = (a_n)$ and $b = (b_n)$ are strictly increasing sequences. Clearly $(a_0, b_0) = (0, 0)$ is the terminal $\mathscr{P}$-position. We must prove that every candidate $\mathscr{P}$-position has no $\mathscr{P}$-position as an option, and we must prove that every candidate $\mathscr{N}$-position has a $\mathscr{P}$-position as an option.

"$\mathscr{P} \not\to \mathscr{P}$": We prove that, for any $n > 0$, (a_n, b_n) does not have an option of the same form. We use (iii), (iv) and (v) to exhaust all possibilities. Suppose that a player removes from a single heap to say (a_i, b_n), with $i < n$.

[16] This property is of course true in any base n expansion if $n \geqslant 2$ is an integer.

[17] Since we are assuming (ii) and (iii), then (iv) and (v) suffice to define complementary sequences; without (ii), in addition, we should require, for all $n \neq m$, $a_n \neq a_m$.

Then, since b is increasing, $b_i \neq b_n$. If they remove from a single heap to (b_i, b_n), then (iv) contradicts that this be of the desired form. If they remove from a single heap to say (a_n, b_i), then again, since b is increasing, $b_i \neq b_n$. If they remove from a single heap to (a_n, a_i), then by (iv) this option is not of the same form. If they remove the same number m from both heaps, then by (iii) the position cannot be of the form (a_i, b_i). Namely $b_n - m - a_n + m = n > i = b_i - a_i$.

"$\mathscr{N} \to \mathscr{P}$": We prove that, if a position is not of the form (a_n, b_n), then it has an option of this form. Consider first (x, b_n), with $x > a_n$. Then remove $x - a_n > 0$ from the first heap. If $x < a_n$, then, by (v), there are two cases.

(a) $x = a_i$, for some $i < n$;
(b) $x = b_i$ for some $i < n$.

In case (a), since b is increasing, there is a move to (a_i, b_i). In case (b), there is a move to (b_i, a_i), since, by (iii) and b increasing, $a_i < b_i < b_n$.

Consider next a position of the form (a_n, x), $x \neq b_n$. If $x > b_n$, then (a_n, b_n) is an option. Hence assume $x < b_n$. Then, by (v) $x = b_i$ or $x = a_i$, for some i. In the first case $i < n$ and so (a_i, b_i) is an option, by (ii). In the second case, the position is (a_n, a_i). We have three cases:

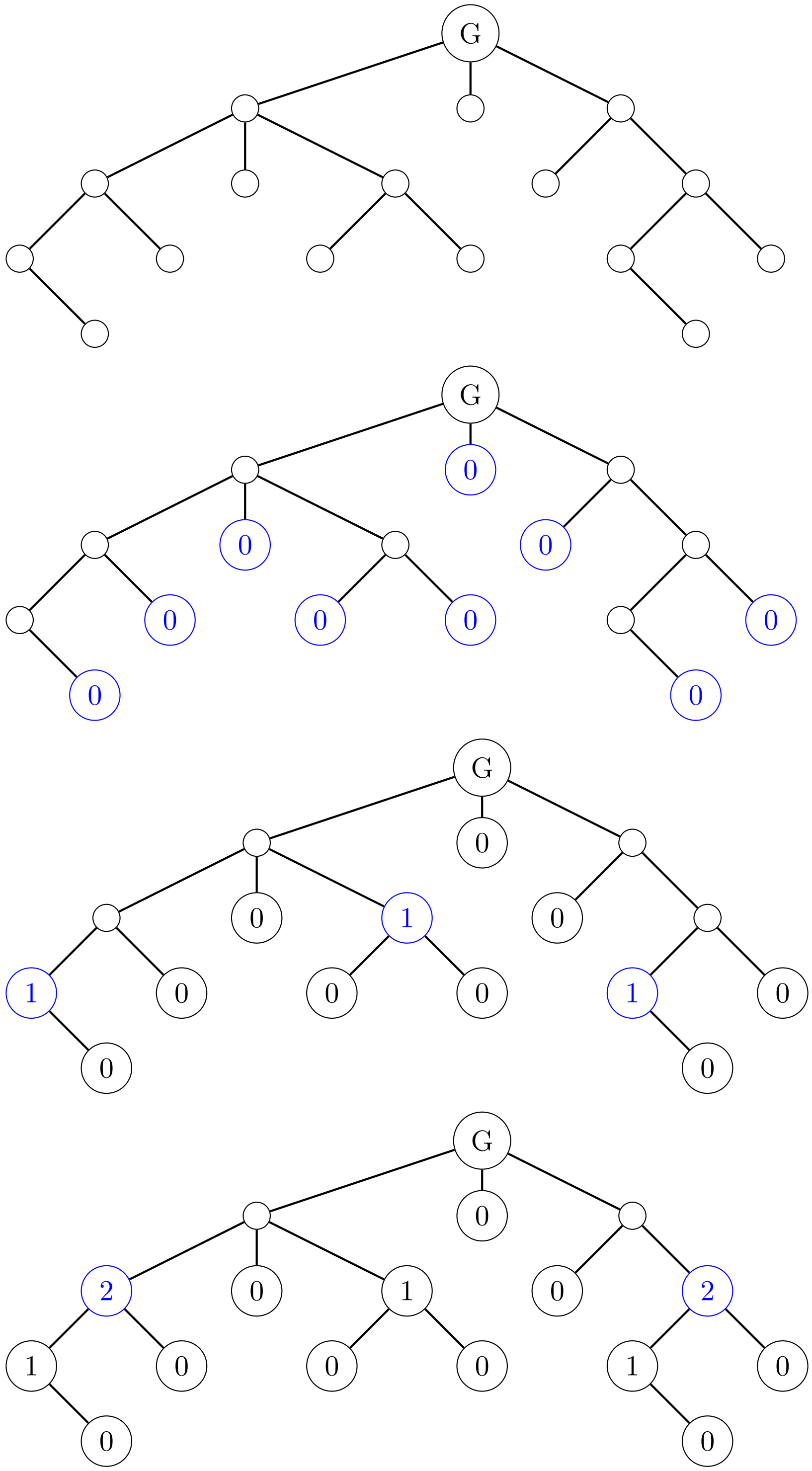
(a) $a_n < a_i < b_n$;
(b) $a_i < a_n < b_i$;
(c) $a_i < b_i < a_n$.

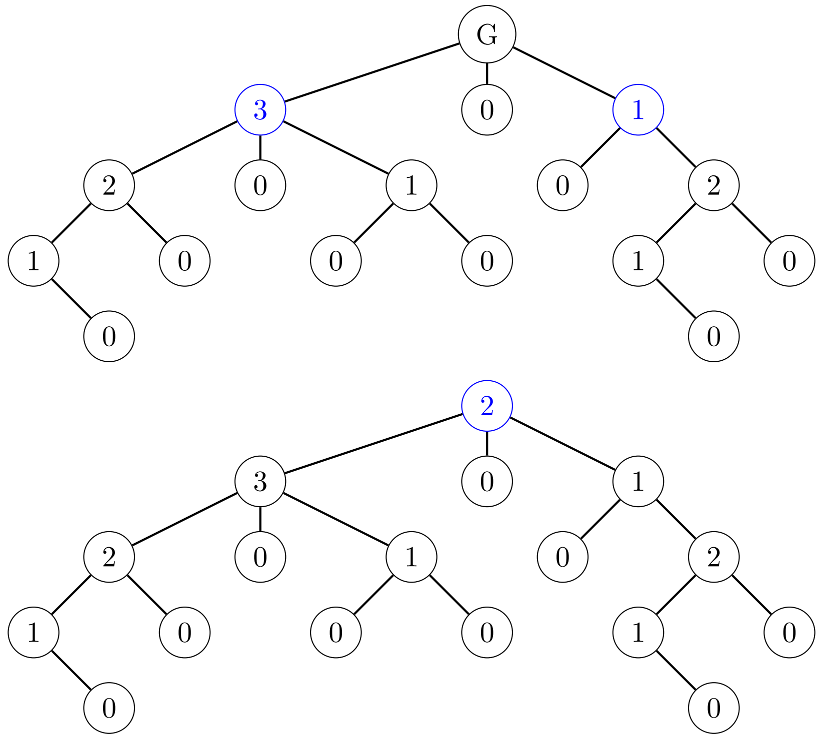
For case (a), observe that $0 < a_i - a_n < b_n - a_n = n$. Therefore there exists $j < n$ such that $a_i - a_n = b_j - a_j = j$. Hence, by (iii), (a_j, b_j) is an option. For case (b), observe that $0 < a_n - a_i < b_i - a_i = i$. Therefore there exists $j < i$ such that $a_n - a_i = b_j - a_j = j$. Hence (a_j, b_j) is an option. For case (c), (a_i, b_i) is an option. □

We will use this result in a later lecture to prove Theorem 1, together with some other representations of WYTHOFF NIM's $\mathscr{P}$-positions. One more component, called Beatty/Lord Rayleigh's Theorem, will be required.

Before we move on, to study more impartial rulesets and their properties, let us illustrate and prove the famous Sprague and Grundy theorem.

14.3. **Sprague and Grundy's contribution.** We review the famous Sprague-Grundy Theory [S1935, G1939]. It says that, for any impartial normal play game G, there is a nim-heap h such that played together $G + h \in \mathscr{P}$. Moreover, the proof provides a constructive way to find that nim heap. The *minimal exclusive* function, abbreviated 'mex' finds the nim-value of a given game. The mex-function is defined as follows. Let $X \subset \mathbb{N}_0$ be a strict subset of the non-negative integers. Then $\text{mex}(X) = \mathbb{N}_0 \setminus X$. For example $\text{mex}\{0, 1, 3, 5, 17\} = 2$ and $\text{mex}\{1\} = 0$. As a motivation, before the proof, let us draw a game tree and compute its equivalent nim-value via the mex-algorithm. Recursively, it computes the nim-values on every sub-position, via the mex-rule, until it finds the root, and assigns its nimber. This is an impartial game so the directions of the slopes do not matter.





Definition 68 (Subgame)**.** Consider a game G. Then H is a subgame of G if there is a sequence of moves from G to H, not necessarily alternating and perhaps empty.

In the literature, subgame is often called "subposition" or "follower" with exactly the same meaning.

Theorem 69. *Every impartial normal play game is equivalent to a nimber.*

Proof. Consider any impartial normal play game G. The statement holds if G does not have any options, so assume that G has options. Assign each sub position of G without any option nim-value 0. Suppose that the statement is true for all games of birthday less than G. That means in particular that each option of G, say G_i, equals a nim-heap, say $*h_i$. We will demonstrate that G equals the nim-heap $*\text{mex}\{h_i\}$, where i ranges over the options of G. To this purpose we play the game $G + *\text{mex}\{h_i\}$, and demonstrate that it is a $\mathscr{P}$-position. Suppose that the first player plays to $*h_j + G$, for some $h_j < \text{mex}\{h_i\}$ (this is possible by the definition of a nim-heap). Then the second player can respond to $*h_j + G_j = *h_j + *h_j \in \mathscr{P}$. Suppose next that the first player plays in the G component to $G_j + *\text{mex}\{h_i\}$. There are two possibilities:

(i) $h_j > \text{mex}\{h_i\}$;
(ii) $h_j < \text{mex}\{h_i\}$.

In case (i), the second player can respond to $*\text{mex}\{h_i\} + *\text{mex}\{h_i\}$, and win. In case (ii), the second player can respond to $*h_j + *h_j$ and win. □

14.4. **A chocolate bar game.** CHOMP is an impartial game played with an m by n chocolate bar (see Figure 19). The lower left piece is poisoned, and the player who chomps it loses (it is a normal play game: think that the poisoned piece is not present). The game is played as follows: point at a remaining piece and chomp off everything above and to the right of that piece. A classical *strategy stealing* argument shows that the first player has a winning strategy for CHOMP played on a rectangular grid. However, nobody fully understands optimal play, unless the grid is a square.

Theorem 70. CHOMP *on a rectangular grid is a first player win.*

Proof. If the grid is a square, then point at position $(1, 1)$, and mimic the rest of play. Otherwise, suppose that the second player has a winning strategy. Take the upper right piece. If that is a winning move we are done. Otherwise, wait and see what the second player does. If the first move is not winning, then he has a winning strategy. But the first player could have played that move in the first move. Hence she has a winning strategy. □

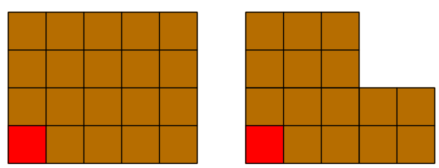


FIGURE 19. Two CHOMP positions. The red piece in the lower left is poisoned and cannot be eaten. The first player pointed at $(3, 2)$ and chomped off four pieces.

14.5. **Subtraction games.** SUBTRACTION is a generalization of NIM defined by a subtraction set $S \subset \mathbb{N}$. The move options from a heap of size x is the set $\{x - s \mid s \in S, x - s \geqslant 0\}$. For example, if $S = \{1, 3, 4, 7\}$, then the first few outcomes and nimbers are as follows:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$o(x)$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$
$\text{nim}(x)$	0	1	0	1	2	3	2	3	0	1	0	1	2	3	2	3

We conclude that both the outcomes and the nimbers are periodic, with period length 8, and starting at the heap of size 0. How do we know this? It follows by observing a repetition of the content of 7 consecutive squares in either row. Why this ‘window’ size of 7? This is because this is the maximum size of a subtraction. This idea assists us in the proof that any subtraction game on a finite subtraction set has *eventually periodic* outcomes.[18]

Theorem 71. *Every subtraction game S has eventually periodic outcomes.*

Proof. There are $2^{\max S}$ possible combinations of outcomes $\mathscr{P}$ or $\mathscr{N}$ in a 'window' of size $\max S$. Since this is a finite number, there must exist a smallest heap size x' for which the outcome window $(o(x'), \ldots, o(x'+\max S-1))$ is the same as $(o(y'), \ldots, o(y'+\max S-1))$, for some $y' > x'$. This defines the period. □

The preperiod of the game S, in Theorem 71, is the finite outcome sequence $o(0), \ldots, o(x'-1)$. Similarly we can prove that the nim-sequence is eventually periodic. The argument is the same, just replace the 2 in the number of outcomes for $|S|+1$ as the number of possible nimbers.

Corollary 72. *Every subtraction game S has eventually periodic nimbers.*

Proof. By the mex rule, since a ruleset S has (at most) $|S|$ options, the largest nimber that can occur is $*(|S|+1)$. The rest of the proof is analogous to that of Theorem 71. □

Many 'small' rulesets have period length equal to the sum of two of the possible subtractions. In our example: $7+1=8$. But there are exceptions. For example, the ruleset $S=\{2,5,7\}$ has period length 22. Moreover many games in SUBTRACTION have a preperiod before the eventual behavior settles. An early example is $S=\{2,4,7\}$. What is its preperiod and period?

14.6. **Fibonacci Nim.** FIBONACCI NIM is played on one heap of pebbles. The first player can remove any positive number of pebbles, except the whole heap. Any other move is restricted by taking at most twice the number of pebbles that the previous player removed.

Recall the Pingala (Fibonacci) numbers, defined by $F_0=0, F_1=1$, and if $n \geqslant 2, F_{n+2}=F_{n+1}+F_n$. Thus, the sequence is $0,1,1,2,3,5,8,13,21,\ldots$.

Theorem 73. *The first player loses* FIBONACCI NIM *if and only if the starting heap size is a Pingala (Fibonacci) number.*

The proof uses a classical result on Pingala numbers, namely: every positive integer decomposes uniquely as a sum of non-consecutive Pingala numbers (see Section 14.10 for a proof). For example $11=8+3$, $23=21+2$, $30=21+8+1$. We write this unique Pingala representation as a binary word $\zeta(x)=\zeta_n\cdots\zeta_1$ where $x=\sum_{i\geqslant 1}\zeta_i F_i$, $\zeta_i \in \{0,1\}$, and we call this representation ZOL, because it was independently discovered by Ostrowski (in 1922), Lekkerkerker (in 1952), and Zeckendorf (in 1972). Note that F_1 is not used in this representation. In our examples, using a binary word notation, thus

[18]Consider a sequence $s=(s_i)$ indexed by the non-negative integers. Then s is 'eventually periodic' if one can decompose the sequence in a finite part $(s_i)_{i\leqslant k}$, the *preperiod*, and an infinite part $(s_i)_{i>k}$, the *periodic* part, for which there is a p such that for all $i>k, s_i=s_{i+p}$. If $k=0$ we say that the sequence is *purely periodic*, or just *periodic*.

$\zeta(11) = 10100$, $\zeta(21) = 1000010$ and $\zeta(30) = 1010001$. The representation is obtained by greedily at each step including the largest Pingala number. Thus, the definition of the Pingala numbers gives that there cannot be two consecutive 1s in ζ. We leave it as an exercise to prove the uniqueness.

One more basic property of ZOL-numeration is that, by adding 1 to a word of alternating 1s and 0s of length k results in the word of length $k+1$ of the form $10\ldots0$. For example $101 + 1 = 1000$ and $1010 + 1 = 10000$. That is, in this numeration, we have $(10)^{(k-1)/2} + 1 = 10^k$, if k is odd, and $(10)^{k/2} + 1 = 10^k$, if k is even. Let us call this property *ZOL-carry*. Note that in the first case, we go from a word ending in an even number of 0s to a word ending in an odd number of 0s, and vice versa in the second case. Note also that, since ZOL-numeration never uses consequtive 1s, for any word, it ends in an alternating sequence of 0s and 1s (perhaps trivial), and before that perhaps there is a double entry of 0s etc.

Theorem 74. *The first player wins if and only if they can remove the smallest number in the ZOL-decomposition of the heap size.*

Notice that this statement includes the previous one (Theorem 73), since the starting player is not allowed to remove the whole heap. In the example $11 = 8 + 3$, the first player removes 3, and then the second player has to move from the Fibonacci number 8. They can remove $1 \leqslant r \leqslant 6$. If they remove 3 or more they lose in the next move. Otherwise the next player can play to 5, again, a Fibonacci number. In their next move they can either win directly, or, if the other player removed 1, they can take 1 from 4 and reach 3. Now, because the other player can only reach 1 or 2, they win in their next move.

We write $(x, 2r)$ for a heap of size x, where the previous player removed r pebbles, and so the current player is allowed to remove at most $2r$ pebbles. (The exception is the starting position, which is of the form $(x, x-1)$, where $x - 1$ may be odd.)

Proof. Consider a previous position $(x, 2r)$. Denote by $Z(x)$ the set of Pingala numbers in the ZOL-representation of x. We must justify the following statements.

(i) If $m = \min Z(x) \leqslant 2r$, then $\min Z(x - m) > 2m$, or $x - m = 0$.
(ii) If $\min Z(x - m) > 2m$, then $\min Z(x) \leqslant 2r$.

Observe that (i) states that if the previous player can remove the smallest ZOL-component of x, then the current player cannot. And (ii) is the opposite statement, if the current player cannot remove the smallest ZOL-component, then the previous player could.

Item (i) is almost automatic by the definition of ZOL-decomposition. Namely, when the smallest ZOL-component of a number has been removed, then, the second smallest becomes the smallest, but it is distanced by at least one Fibonacci number. Formally, by the definition of the Fibonacci numbers, for all $n \geqslant 2$, with the removal of $F_n = \min Z(x)$, then

$$\begin{aligned} 2F_n &< F_n + F_{n+1} \\ &= F_{n+2} \\ &\leqslant \min Z(x - F_n), \end{aligned}$$

by definition of ZOL-numeration, unless $x - m = 0$.

For (ii), by the assumption $\min Z(x-m) > 2m$, we can express the ZOL-decomposition of x as $Z(x) = Z(x-m) \cup Z(m)$. But, then $\min Z(x) = \min Z(m) \leqslant m \leqslant 2r$, where the second inequality is by the rules of play. □

14.7. **The amazing world of Wythoff.** Let us return to Wythoff Nim. We start with a recap that includes a to do list. There are several representations of the $\mathscr{P}$-positions of Wythoff Nim. Let us list a few.

(i) Geometric Approach: A recursive painting of $\mathscr{N}$-positions illuminates smallest missing $\mathscr{P}$-positions (see the introduction).
(ii) Wythoff Properties: We listed five properties that uniquely define the Wythoff sequences, and a proof was included in Section 14.2.
(iii) A Mex-Algorithm: The A-sequence can be recursively computed using the mex-algorithm that we introduced while studying the Sprague-Grundy theory.
(iv) Golden Section: We have stated the result in Theorem 1, but not yet proved it.
(v) ZOL-numeration: The $\mathscr{P}$-positions have a nice interpretation in the ZOL-numeration mentioned in the discussion of Fibonacci Nim.
(vi) A Morphism on Words: Section 14.8.

Let us do (5). First we construct a small table of the first positive integers in ZOL-numeration. We use the binary notation with $F_2 = 1$ as the smallest 'digit', so for example $7 = 5+2 = F_5 + F_3 = 1010$ and $13 = F_7 = 100000$.

n	ZOL(n)
1	1
2	10
3	100
4	101
5	1000
6	1001
7	1010
8	10000
9	10001
10	10010

The bold numbers are those that end in an even number of 0s. That is, $1, 3, 4, 6, 8, 9, \ldots$. We note that those coincide with those in the A-sequence of Wythoff Nim's $\mathscr{P}$-positions. The B-sequence is obtained by adjoining a '0' to the right of the numbers in the A-sequence. And indeed, this is our next theorem.

Theorem 75. *In ZOL-numeration A_n (B_n) is the n^{th} number that ends with an even (odd) number of 0s, and, in this numeration, for all n, $\zeta(B_n) = \zeta(A_n)0$.*

Proof. Recall the Wythoff properties:

(i) $(a_0, b_0) = (0, 0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

By Theorem 67, it suffices to justify each item. But all items except (iii) are immediate by definition. It remains to prove that for all n, in ZOL-numeration, $A_n0 = A_n + n$. By using ZOL-enumeration, as defined on page 56, let us describe A_n as a function of n.

Claim: if the ZOL-numeration $\zeta(n)$ ends in an odd number of 0s, then $\zeta(A_n) = \zeta(n)0$, and otherwise, $\zeta(A_n) = \zeta(n-1)1$.

The corresponding table begins like this:

n	$\zeta(n)$	A_n	$\zeta(A_n)$
1	1	1	1
2	10	3	100
3	100	4	101
4	101	6	1001
5	1000	8	10000
6	1001	9	10001

Before we prove this claim, let us see that it suffices to prove the result. Let us view two examples of this claim:

$5 = \text{"1000"}$, $A_5 = 8 = \text{"10000"}, B_5 = \text{"100000"}$;
$3 = \text{"100"}$, $A_4 = 6 = \text{"1001"}, B_4 = 10 = \text{"10010"}$.

In the first "odd" case we get, by using the definition of Pingala recurrence in the third equality:

$$\begin{aligned}\zeta(B_n) &= \zeta(A_n) + \zeta(n)\\ &= \zeta(n)0 + \zeta(n)\\ &= \zeta(n)00\\ &= \zeta(A_n)0.\end{aligned}$$

And notice that $\zeta(B_n)$ ends in an odd number of 0s, because $\zeta(n)$ does so.

The second "even" case is similar, but it requires a small trick, namely

$$\begin{aligned}\zeta(B_n) &= \zeta(A_n) + \zeta(n)\\ &= \zeta(n-1)1 + \zeta(n)\\ &= \zeta(n-1)1 + \zeta(n-1) + 1\\ &= \zeta(n-1)0 + \zeta(n-1) + 2\\ &= \zeta(n-1)00 + 2\\ &= \zeta(n-1)10\\ &= \zeta(A_n)0.\end{aligned}$$

Here it is important to observe that each line is a valid ZOL-representation. In particular, $\zeta(n-1)10$ is valid, by the following implication: if $\zeta(n)$ ends in an even number of 0s, then $\zeta(n-1)$ does not end in a "1". Namely, if $\zeta(n)$ ends in a "1" then remove it to obtain $\zeta(n-1)$. Otherwise argue by ZOL-carry as explained in Section 14.6.

Proof of Claim: Let us restate the claim here: "If $\zeta(n)$ ends in an odd number of 0s, then $\zeta(A_n) = \zeta(n)0$, and otherwise, $\zeta(A_n) = \zeta(n-1)1$."

Observe that in this statement, all positive integers that end in an even number of 0s in the ZOL-representation are represented; namely the first part gives all those that end in two or more zeros while the second part gives those that end in zero 0s. Therefore it suffices to establish that going from $n-1$ to n implies that the claimed ZOL-representation of A_n is increasing. If $\zeta(n-1)$ and $\zeta(n)$ end in the same parity of 0s, there is nothing to prove. Thus there are two cases to check:

(a) $\zeta(n-1)$ 'odd' and $\zeta(n)$ 'even';
(b) $\zeta(n-1)$ 'even' and $\zeta(n)$ 'odd'.

For item (a) it is immediate by the statement that $A_n - A_{n-1} = 1$; namely the second part concatenates a "1" to a binary word that ends in an odd number of zeroes. For item (b), going from 'even' to 'odd' when $n-1 \to n$, it must be the case that $\zeta(n-1)$ ends in a "1", that is the rules of ZOL-carry apply. Similar to the second case above one can see that in this case $A_n - A_{n-1} = 2$. □

14.8. **More solutions of** Wythoff Nim. Let us continue the Wythoff story, by using the mex-algorithm approach.

Theorem 76. *Let the A and B be the increasing sequences that define* Wythoff Nim*'s $\mathscr{P}$-positons. For all $n \in \mathbb{N}_0$, let*

$$\begin{cases} a_n &= \text{mex}\{a_i, b_i \mid 0 \leqslant i < n\};\\ b_n &= a_n + n.\end{cases}$$

Then, for all n, $a_n = A_n$ and $b_n = B_n$.

Proof. Let us verify that the Wythoff Properties are all satisfied. Item (i) is immediate. That the a-sequence is increasing as in (ii) follows by the definition of mex. Item (iii) is obvious. Item (iv) can be verified by an inductive argument. Namely, suppose that b_{n-1} is the largest element in $\{a_i, b_i \mid 0 \leqslant i < n\}$. Then $b_n = a_n + n > b_{n-1}$, by using also (ii). Hence there can be no collision. Item (v) follows by the definition of mex. □

The Fibonacci Morphism $\varphi : \{0,1\}^* \to \{0,1\}^*$ is defined by,[19]

$$\begin{cases} \varphi(0) & = 01; \\ \varphi(1) & = 0. \end{cases}$$

If the initial seed is 0, then φ generates the infinite *Fibonacci word*, ω. This word is generated recursively as follows:

$$\varphi(0) = 01;$$
$$\varphi(01) = 010;$$
$$\varphi(010) = 01001;$$
$$\varphi(01001) = 01001010,$$

and so on. Note that $\varphi(010) = \varphi(0)\varphi(01)$, and so on. Define

$$\omega = \lim_{n\to\infty} \varphi^n(0),$$

where, for all $n > 0$, for all $x \in \{0,1\}^*$, $\varphi^n(x) = \varphi^{n-1}(\varphi(x))$, where $\varphi^0(x) = x$. We index the letters in ω by $\mathbb{N}$. We get $\omega_1 = 0, \omega_2 = 1, \omega_3 = 0, \omega_4 = 0, \omega_5 = 1$, and so on. The morphism φ has many interesting properties. For example, for all $n \in \mathbb{N}_0$, the lengths of the words $\varphi^n(0)$ correspond to the Fibonacci numbers F_{n+2}.

The following result relates the occurrences of the 0s and 1s in this word with the $\mathscr{P}$-positions of WYTHOFF NIM.

Similar to Theorems 75, the interesting Wythoff Property is the "shift property" (iii). Here, (i) in the Wythoff Properties $A_0 = B_0 = 0$ does not apply.

Theorem 77 ([S1976]). *For all $n \in \mathbb{N}$, A_n equals the index of the n^{th} "0" in ω, and for all n, B_n equals the index of the n^{th} "1" in ω.*

Proof. Clearly A is increasing, so (ii) holds. Similarly complementarity, that is (iv) and (v) hold by definition. It remains to justify (iii), the shift property. But this follows by definition of φ; namely, the nth "1" is written when we read the nth "0", and it is shifted n steps, since, at each translation, we write "01" (instead of just "1"). □

[19] The set of possible finite words on a finite set of letters X is denoted by X^*. A function $f : X^* \to X^*$ is a morphism, under concatenation, if for all $v, w \in X^*$, $f(vw) = f(v)f(w)$.

Let us prove that the sequences from WYTHOFF NIM, $(\lfloor n\phi \rfloor)$ and $(\lfloor n\phi^2 \rfloor)$, are complementary. That is, every positive integer appears in exactly one of these sequences. This can be done in full generality, by instead proving that the sequences $(\lfloor n\alpha \rfloor)$ and $(\lfloor n\beta \rfloor)$ are complementary whenever α and β are irrationals with $1/\alpha + 1/\beta = 1$.

Theorem 78 (Beatty's/lord Rayleigh's Theorem)**.** *Let α and β be irrational numbers with*

$$1/\alpha + 1/\beta = 1. \tag{27}$$

Then the sequences $(\lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n\in\mathbb{N}}$ are complementary.

Proof. Let us call by "collision" the property that the sequences coincide at some integer x. Let us call by "anticollision" the property that both sequences skip some integer x. We will disprove both.

Collision: Since α and β are irrationals, if $\lfloor n\alpha \rfloor = \lfloor m\beta \rfloor = x$ then $x < n\alpha < x+1$ and $x < m\beta < x+1$. Thus $x/\alpha + x/\beta < n+m < (x+1)/\alpha + (x+1)/\beta$. But then, by (27), $x < n+m < x+1$, where all expressions are integers, which is impossible.

Anticollision: Suppose that, for some integer x, both $\lfloor n\alpha \rfloor < x < \lfloor (n+1)\alpha \rfloor$ and $\lfloor m\beta \rfloor < x < \lfloor (m+1)\beta \rfloor$. But since α and β are irrationals, and x is an integer, we get $n\alpha < x < (n+1)\alpha - 1$ and $m\beta < x < (m+1)\beta - 1$. Thus, by dividing with α and β respectively, and adding 'columnwise', we get $n+m < x/\alpha + x/\beta < n+m+2-1/\alpha-1/\beta$. That is, by (27), $n+m < x < n+m+1$, where all entries are integers, which is impossible. □

We are now prepared to prove Theorem 1 by using the Wythoff Properties from page 50. Let us recall them here:

(i) $(a_0, b_0) = (0,0)$;
(ii) for all n, $a_{n+1} > a_n$;
(iii) for all n, $b_n - a_n = n$;
(iv) for all $n, m > 0$, $a_n \neq b_m$;
(v) for all $x \in \mathbb{N}$, there exists an n such that $a_n = x$ or $b_n = x$.

Proof of Theorem 1. By Theorem 67, it suffices to justify that, for all non-negative integers n, $(a_n, b_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$. Item (i) is immediate, and so is item (ii), because $1 < \phi$. And (iii) follows from $\phi^2 = \phi + 1$. Items (iv) and (v) follow from Theorem 78. □

14.9. **Euclid's Game.** The ruleset EUCLID is played on two non-empty heaps of pebbles. A player must remove a multiple of the size of the smaller heap from the larger heap. We represent a position by a pair of positive integers (x, y), where say $0 < x \leqslant y$. Note that if $0 < x = y$, then the position is terminal. Example play: $(2,7) \to (2,3) \to (1,2) \to (1,1)$. Since we put the requirement that (both) heaps remain non-empty, then no more move is possible. Note that the losing move, in this sample play, is forced. As we will see, this is a general property of EUCLID. And more is true; optimal play reduces to minimizing the relative distance of the heaps. Recall the golden section, $\phi = \frac{1+\sqrt{5}}{2}$.

Theorem 79 ([CD1969])**.** *A player wins* EUCLID *if and only if they can remove a multiple of the smaller heap such that the ratio of the heap sizes* (x, y)*, satisfies* $1 \leqslant y/x < \phi$.

Proof. $\mathscr{N} \to \mathscr{P}$: Suppose that the current player is in a position of the form (a, b) with $b/a > \phi$. We must prove that they can find a move to a position (c, d) of the form

$$1 \leqslant d/c < \phi. \tag{28}$$

We claim that there is a positive integer k such that either

- $(c, d) = (b - ka, a)$, or
- $(c, d) = (a, b - (k - 1)a)$

satisfies the desired inequality (28). If $1 \leqslant \frac{a}{b-ka} < \phi$, with (c, d) as in the first item, we are done, so suppose that

$$\frac{a}{b - ka} > \phi. \tag{29}$$

Then $\frac{b-ka}{a} < \phi^{-1}$. And so

$$\begin{aligned} \frac{b-(k-1)a}{a} &= \frac{b-ka}{a} + 1 \\ &< \phi^{-1} + 1 \\ &= \phi. \end{aligned}$$

$\mathscr{P} \not\to \mathscr{P}$: Suppose next that the current player is in a position of the form (a, b), with $1 \leqslant b/a < \phi$. Then there is only one move option, namely $(b - a, a)$, and it follows that

$$\begin{aligned} \frac{b-a}{a} &= \frac{b}{a} - 1 \\ &< \phi - 1 \\ &= \phi^{-1}. \end{aligned}$$

And hence, $\frac{a}{b-a} > \phi$. □

14.10. **Proof of the ZOL-Theorem.** Given a positive integer N, in Section 14.6, we introduced a greedy algorithm to find a claimed unique representation in terms of non-consecutive Pingala numbers. Some question marks remain. Let us here provide a step by step analysis of these claims. We use interchangeably the binary word representation of Pingala number memberships. For example $33 = 21 + 8 + 3 + 1$ is the word 10101 or equivalently the set $\{F_8, F_6, F_4, F_2\}$.

Claim 1. The greedy algorithm is correct.

Proof of Claim 1. The algorithm picks the largest Pingala number, say F_n, smaller than or equal to N, and includes it. Then it includes the largest Pingala number smaller than or equal to $N - F_n$, and so on. It is clear that the chosen Pingala numbers will be non-consecutive, because if it would chose both F_i and F_{i-1}, then it would instead already have chosen F_{i+1}. We have to verify that the resulting sum equals N. Suppose that it is true for all smaller positive integers, and in particular for $N-1$. Recall ZOL-carry from Section 14.6. By adding one, we shift the rightmost alternating word of zeros and ones in the binary word representation $\zeta(N-1)$ to a word of the form $10\cdots0$. This verifies that greedy produces the correct word $\zeta(N)$.

Claim 2. Every ZOL-representation is unique, and obtained by the greedy algorithm.

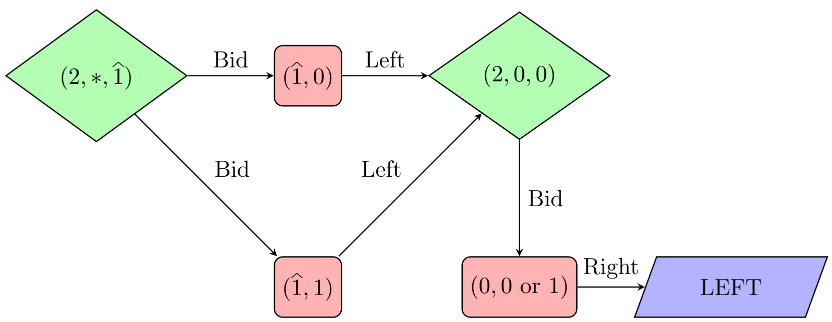
Proof of Claim 2. Suppose we have two distinct ZOL-representations of a given number N. Then there is a largest Pingala number, say F_n, that is not in both representations. The representation that does not contain F_n has to compensate for the loss by adding up smaller non-consecutive Pingala numbers to the same amount. We will prove that this is impossible. The largest possible number one can obtain within these constraints is of the form $1010\cdots01(0)$, where the most significant "1" represents the number F_{n-1}, and where the least significant "1" represents either F_3 or F_2, depending on whether n is even or odd, respectively. Now recursively replace F_n with smaller Pingala representatives, that is $F_n = F_{n-1} + Fn - 2$, and instead of F_{n-2} we take $F_{n-2} = F_{n-3} + F_{n-4}$, and so on. This process terminates by using two of the smallest consecutive Pingala numbers, $F_3 + F_2$ or $F_2 + F_1$, depending on whether n is even or odd, respectively. But the representation cannot use consecutive Pingala numbers, and hence in the first case $F_2 = 1$ must be excludeed, and in the second case, we do not use F_1 anyway. Thus, the largest number we can obtain is $F_n - 1$, which together with Claim 1 proves Claim 2.

15. Bidding Combinatorial games, by Prem

Consider Left and Right playing a normal play game G. Instead of the conventional alternating play, here the move order is determined through a Discrete Richman bidding. The total budget TB, available for them is fixed. Left's budget is p and Right's is q such that $p + q =$ TB. A player who wins the bid, moves in G and shifts the winning bid to the other player. Additionally, there is a tiebreaking marker, that is initially held by one of the players and can be included in the bid. The tiebreaking marker has no value, but in case of a tie, the player who is currently holding the tiebreaking marker will be the winner of the bid and the winning bid together with the tiebreaking marker will get shifted to other player. There is a final auction at the empty game. The player who moves last, wins the game. Moreover, similar to alternating play, in this bidding set up: "last move wins" is the same as "cannot move loses"

Formally, given a total budget TB, let us define $\mathcal{B} = \{0, \ldots, \mathrm{TB}, \hat{0}, \ldots, \hat{\mathrm{TB}}\}$, the set of all feasible player budgets. Here a "feasible budget" includes the information of the marker holder. A game is a triple $(\mathrm{TB}, G, \tilde{p})$, where Left's part of the budget is $\tilde{p} \in \mathcal{B}$. If TB is understood, we write $(G, \tilde{p})$.

For example, $(2, *, \widehat{1})$ means the game is $* = \{0 \mid 0\}$ with total budget 2 in which player Left has budget 1 and the tiebreaker marker. In $(2, *, \widehat{1})$, Left can bid 0, $\widehat{0}$, 1 or $\widehat{1}$, however Right can only bid 0 or 1. Let's consider Left is bidding the amount $\widehat{1}$. Then for both choices of Right, Left will win the bid and make a move in the game $*$. The current bidding game is $(2, 0, 0)$. Now Left will bid 0 and for all choices of Right, Right will win the bid and have to make a move in the game 0. Since, Right cannot move, Left wins the game.



The following result ensures that by the introduction of bidding, there will not be any mixed strategy equilibrium.

Theorem 80 (First Fundamental Theorem)**.** [KLRU2022] *Consider the bidding convention where the tie-breaking marker may be included in a bid. For any game $(TB, G, \tilde{p})$ there is a pure strategy subgame perfect equilibrium, computed by standard backward induction.*

Observe that in case of a tie, the marker is transferred. Therefore, by this automatic rule, the special case $\mathrm{TB} = 0$ corresponds to alternating normal play rules.

Theorem 81. *Consider* $\mathrm{TB} = 0$*. Then bidding play is identical to alternating play. The current player is the player who holds the marker.*

Next, we define the *outcome* of a bidding game.

Definition 82 (Outcome)**.** The *outcome* of the game (TB, G) is $o(G)$, defined via the $2(\mathrm{TB} + 1)$ tuple of partial outcomes as

$$o(G) = (o(G, \widehat{\mathrm{TB}}), \ldots, o(G, \hat{0}), o(G, \mathrm{TB}), \ldots, o(G, 0)).$$

Here the first half of the outcome corresponds to when Left holds the marker and the rest corresponds to when Right holds the marker. The length of the outcome is $2(\mathrm{TB}+1)$.

Since this notation can be quite lengthy, we instead adopt word notation. For example instead of (R, R, L, L) we simply write RRLL.

Definition 83 (Outcome Relation)**.** Consider a fixed TB and the set of all budgets $\mathcal{B}$. Then for any games G and H, $o(G) \geqslant o(H)$ if, $\forall\, \tilde{p} \in \mathcal{B}, o(G,\tilde{p}) \geqslant o(H,\tilde{p})$.[20]

Feasibility of outcome. A careful observation shows that for TB = 1, an outcome such as RLRL would be rare, since Right wins without either money or marker, but loses if he is given a dollar. Next, we state (the proof is straightforward) that such outcomes are impossible; outcomes are *monotone*.

Theorem 84 (Outcome Monotonicity)**.** *Consider $\tilde{p} \in \mathcal{B}$, with $p < \mathrm{TB}$. Then $o(G,\tilde{p}) \leqslant o(G,\widetilde{p+1})$ (same marker holder in both games).*

Another careful observation shows that for TB = 1, an outcome such as LLRR is monotone but for this outcome, Left loses with a dollar budget, but wins with the marker alone. This is also not possible. The next result shows that the marker cannot be worth more than a dollar.

Theorem 85 (Marker Worth)**.** *Consider $\mathrm{TB} \in \mathbb{N}$. Then, for any game G, $o(G,\hat{p}) \leqslant o(G,p+1)$.*

We can view an outcome as a string of L's and R's. From Outcome Monotonicity and Marker Worth, Theorems 84 and 85, we see that not all such strings can appear as an outcome of a game. Thus, let us define the notion of a *feasible* outcome.

Definition 86 (Feasible Outcome)**.** An outcome is feasible if it satisfies Outcome Monotonicity (Theorem 84) and Marker Worth (Theorem 85). For a given TB, the set of all feasible outcomes is $\mathcal{O} = \mathcal{O}_{\mathrm{TB}}$.

The next result shows that corresponding to every feasible outcome, there is a bidding game.

Theorem 87 (Main Theorem)**.** *Consider any total budget $\mathrm{TB} \in \mathbb{N}_0$. An outcome, say ω, is feasible if and only if there is a game G such that $o(G) = \omega$.*

For more details see [KLRU2022] and [KLRU2023].

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